

Resonance and Double Negative Behavior in Metamaterials

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Abstract

A generic class of metamaterials is introduced and is shown to exhibit frequency dependent double negative effective properties. We develop a rigorous method for calculating the frequency intervals where either double negative or double positive effective properties appear and show how these intervals imply the existence of propagating Bloch waves inside sub-wavelength structures. The branches of the dispersion relation associated with Bloch modes are shown to be explicitly determined by the Dirichlet spectrum of the high dielectric phase and the generalized electrostatic spectra of the complement.

1. Introduction

Metamaterials are new class of engineered materials that exhibit electromagnetic properties not readily found in nature. The novelty is that unconventional electromagnetic properties can be created by carefully chosen sub-wavelength configurations of conventional materials. The distinctive properties of metamaterials are derived from geometrically induced resonances localized to specific frequencies. These resonances are used to control propagating modes with wavelengths longer than the characteristic length scale of the material. Metamaterials are envisaged for several application areas ranging from telecommunication and solar energy harvesting to the electromagnetic cloaking of material objects.

A generic metamaterial comes most often in the form of a crystal made from a periodic array of scatterers embedded within a host medium. The physical notions of frequency dependent effective magnetic permeability and dielectric permittivity are used to describe the behavior of propagating modes at wavelengths larger than the length scale of the metamaterial crystal. The past decade has witnessed the development and identification of new sub-wavelength geometries for novel metamaterial properties. These include the simultaneous appearance of negative

effective dielectric permittivity and magnetic permeability. Such “left-handed media” are predicted to exhibit negative group velocity, inverse Doppler effect, and an inverted Snell’s law [41]. The first metamaterial configurations imparted electromagnetic properties consistent with the appearance of a negative bulk dielectric constant [37]. Subsequently electromagnetic behavior associated with negative effective magnetic permeability at microwave frequencies were derived from periodic arrays of non-magnetic metallic split ring resonators [36]. Double negative or left-handed metamaterials with simultaneous negative bulk permeability and permittivity at microwave frequencies have been verified for arrays of metallic posts and split ring resonators [46]. Subsequent work has delivered several new designs using different configurations of metallic resonators for double negative behavior [15, 18, 26, 45, 50, 51, 53].

Current state of the art metallic resonators do not perform well at optical frequencies and alternate strategies are contemplated employing the use of both metals and dielectric materials for optical frequencies [44]. For higher frequencies in the infrared and optical range new strategies for generating double negative response rely on Mie resonances generated inside coated rods consisting of a high dielectric core coated with a dielectric exhibiting plasmonic or Drude type frequency response at optical frequencies [47–49]. A second strategy for generating double negative response employs dielectric resonances associated with small rods or particles made from dielectric materials with large permittivity [27, 38, 42]. Alternate strategies for generating negative permeability at infrared and optical frequencies use special configurations of plasmonic nanoparticles [1, 31, 39]. The list of metamaterial systems is rapidly growing and comprehensive reviews of the subject can be found in [43, 44].

Despite the large number of physically based strategies for generating unconventional properties the theory lacks mathematical frameworks that:

1. Provide the explicit relationship connecting the leading order influence of double negative behavior to the existence of Bloch wave modes inside metamaterials.
2. Provide a systematic identification of the underlying spectral problems related to the crystal geometry that control the location of stop bands and propagation bands for metamaterial crystals.

In this article we provide such a framework for a generic class of metamaterial crystals made from non-magnetic constituents. The crystal is given by a periodic array of two aligned non-magnetic rods, one of which possesses a large frequency independent dielectric constant while the other is characterized by a frequency dependent dielectric response. In this treatment the frequency dependent dielectric response ε_P is associated with plasmonic or Drude behavior at optical frequencies given by [47, 48]

$$\varepsilon_P(\omega^2) = 1 - \frac{\omega_P^2}{\omega^2}, \quad (1)$$

where ω is the frequency and ω_P is the plasma frequency [6]. Here we develop a rigorous method for calculating the frequency intervals where either double negative or double positive bulk properties appear and show how these intervals imply

the existence of Bloch wave modes in the dynamic regime away from the quasi-static limit (see Theorems 5 and 6). It is shown that these frequency intervals are explicitly determined by two distinct spectra. These are the Dirichlet spectrum of the Laplacian associated with the high dielectric rod and the electrostatic spectrum of a three phase high contrast medium obtained by sending the dielectric constant inside the high dielectric rod to ∞ . The electrostatic spectra is introduced in Theorem 1 and discussed in Section 5. The methods are illustrated for ε_P given by (1); however, they apply to dielectrics characterized by single oscillator or multiple oscillator models that include dissipation and are of the form

$$\varepsilon_P(\omega) = 1 + \sum_{j=1}^N \frac{\omega_p^2}{\omega_j^2 - \omega^2 - i\gamma_j\omega}, \quad (2)$$

where ω_j are resonant frequencies, and γ_j are damping factors.

We start with a metamaterial crystal characterized by a period cell containing two parallel, infinitely long cylindrical rods. The rods are parallel to the x_3 axis and are periodically arranged within a square lattice over the transverse $\mathbf{x} = (x_1, x_2)$ plane. The period of the lattice is denoted by d . There is no constraint placed on the shape of the rod cross sections other than they have smooth boundaries and are simply connected. The objective is to characterize the branches of the dispersion relation for H -polarized Bloch waves inside the crystal. For this case the magnetic field is aligned with the rods and the electric field lies in the transverse plane. The direction of propagation is described by the unit vector $\hat{\mathbf{k}} = (\kappa_1, \kappa_2)$ and $k = 2\pi/\lambda$ is the wave number for a wave of length λ and the fields are of the form

$$H_3 = H_3(\mathbf{x})e^{i(k\hat{\mathbf{k}}\cdot\mathbf{x}-t\omega)}, \quad E_1 = E_1(\mathbf{x})e^{i(k\hat{\mathbf{k}}\cdot\mathbf{x}-t\omega)}, \quad E_2 = E_2(\mathbf{x})e^{i(k\hat{\mathbf{k}}\cdot\mathbf{x}-t\omega)} \quad (3)$$

where $H_3(\mathbf{x})$, $E_1(\mathbf{x})$, and $E_2(\mathbf{x})$ are d -periodic for \mathbf{x} in \mathbb{R}^2 . In the sequel c will denote the speed of light in free space. We denote the unit vector pointing along the x_3 direction by \mathbf{e}_3 , and the periodic dielectric permittivity and magnetic permeability are denoted by a_d and μ , respectively. The electric field component $\mathbf{E} = (E_1, E_2)$ of the wave is determined by

$$\mathbf{E} = -\frac{ic}{\omega a_d} \mathbf{e}_3 \times \nabla H_3.$$

The materials are assumed non-magnetic; hence, the magnetic permeability μ is set to unity inside the rods and host. The oscillating dielectric permittivity for the crystal is a d periodic function in the transverse plane and is described by $a_d = a_d(\mathbf{y}/d)$ where $a_d(\mathbf{y})$ is the unit periodic dielectric function taking the values

$$a_d(\mathbf{y}) = \begin{cases} \varepsilon_H & \text{in the host material,} \\ \varepsilon_P = \varepsilon_P(\omega) & \text{in the frequency dependent "plasmonic" rod,} \\ \varepsilon_R = \varepsilon_r/d^2 & \text{in the high dielectric rod.} \end{cases} \quad (4)$$

This choice of high dielectric constant ε_R follows that of [9] where ε_r has dimensions of area. Setting $h^d(\mathbf{x}) = H_3(\mathbf{x})e^{i(k\hat{\mathbf{k}}\cdot\mathbf{x})}$ the Maxwell equations take the

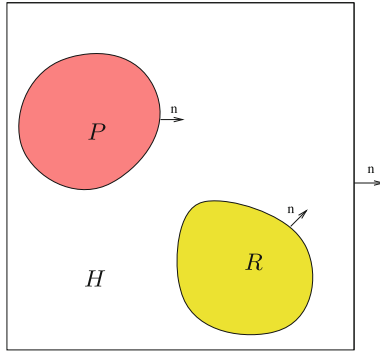


Fig. 1. Cross section of unit cell

form of the Helmholtz equation given by

$$-\nabla_{\mathbf{x}} \cdot \left(a_d^{-1} \left(\frac{\mathbf{x}}{d} \right) \nabla_{\mathbf{x}} h^d(\mathbf{x}) \right) = \frac{\omega^2}{c^2} h^d \quad \text{in } \mathbb{R}^2. \tag{5}$$

The band structure is given by the Bloch eigenvalues $\frac{\omega^2}{c^2}$ which is a subset of the parameter space $\{\frac{\omega^2}{c^2}, -2\pi \leq k_1 \leq 2\pi, -2\pi \leq k_2 \leq 2\pi\}$, with $k = (k_1^2 + k_2^2)^{1/2}$ and $\hat{k}_i = k_i/k$. This constitutes the first Brillouin Zone for this problem.

We set $\mathbf{x} = d\mathbf{y}$ for \mathbf{y} inside the unit period $Y = [0, 1]^2$, put $\beta = dk\hat{k}$ and write $u(\mathbf{y}) = H_3(d\mathbf{y})$. The dependent variable is written $u^d(\mathbf{y}) = h^d(d\mathbf{y}) = u(\mathbf{y}) \exp^{i\beta \cdot \mathbf{y}}$, and we recover the equivalent problem over the unit period cell given by

$$-\nabla_{\mathbf{y}} \cdot \left(a_d^{-1}(\mathbf{y}) \nabla_{\mathbf{y}} u^d \right) = \frac{d^2 \omega^2}{c^2} u^d \quad \text{in } Y. \tag{6}$$

To proceed we work with the dimensionless ratio $\rho = d/\sqrt{\varepsilon_r}$, wave number $\tau = \sqrt{\varepsilon_r}k$ and square frequency $\xi = \varepsilon_r \frac{\omega^2}{c^2}$. The dimensionless parameter measuring the departure away from the quasistatic regime is given by the ratio of period size to wavelength $\eta = dk = \rho\tau \geq 0$. The regime $\eta > 0$ describes dynamic wave propagation while the infinite wavelength or quasistatic limit is recovered for $\eta = 0$. Metamaterials by definition are structured materials operating in the sub-wavelength regime $0 < \eta < 1$ away from the quasistatic limit [36]. For these parameters the dielectric permittivity takes the values $\varepsilon_P = 1 - \frac{\varepsilon_r \omega_p^2 / c^2}{\xi}$, $\varepsilon_R = \frac{1}{\rho^2}$, $\varepsilon_H = 1$, and is denoted by $a_\rho(\mathbf{y})$ for \mathbf{y} in Y and (6) is given by

$$-\nabla_{\mathbf{y}} \cdot \left(a_\rho^{-1}(\mathbf{y}) \nabla_{\mathbf{y}} u^d(\mathbf{y}) \right) = \rho^2 \xi u^d(\mathbf{y}) \quad \text{in } Y. \tag{7}$$

The unit period cell for the generic metamaterial system is represented in Fig. 1. In what follows R represents the rod cross section containing high dielectric material, P the cross section containing the plasmonic material and H denotes the connected host material region.

It is shown in Theorem 4 that the band structure for the metamaterial is characterized by a power series in η and is governed by two distinct types of spectra determined by the shape and configuration of the rods inside the period cells. The series delivers the explicit relationship connecting the leading order influence of quasistatic behavior, as mediated by *effective magnetic permeability and dielectric permittivity*, to the propagation of Bloch wave solutions inside metamaterial crystals made from sub-wavelength $1 > \eta > 0$ structures (see Theorems 4–6).

The relevant spectra for this problem is found to be given by the Dirichlet spectra for the Laplacian over the rod cross sections R together with a generalized electrostatic spectra associated with the infinite connected region exterior to the rods. These spectra provide two distinct criteria that taken together are sufficient for the existence of power series solutions see, Theorem 5. In what follows the power series is developed in terms of a hierarchy of boundary value problems posed separately over the domain R and the domain exterior to the high dielectric rod $Y \setminus R$. The existence of solutions for the boundary value problems inside the high dielectric rod R is controlled by the Dirichlet spectra see, Section 3. Existence of solutions for boundary value problems exterior to R are determined by the generalized electrostatic spectrum see, Theorems 1 and 2. The electrostatic spectra is identified and is shown to be given by the eigenvalues of a compact operator acting on an appropriate Sobolev space of periodic functions see, the discussion in Section 5 and Theorem 7.

The generic class of double negative metamaterials introduced here appears to be new. The motivation behind their construction draws from earlier investigations. The influence of electrostatic resonances on the effective dielectric tensor associated with crystals made from a single frequency dependent dielectric inclusion is developed in the pioneering work of [5, 32, 34] see also the more recent work [39] in the context of metamaterials. These resonances are responsible for negative effective dielectric permittivity. In this context we point out that the sub-wavelength geometry introduced here is a three phase medium and the categorization of all possible electrostatic resonances requires a different approach see Section 5. On the other hand Dirichlet resonances generate negative effective permeability inside high contrast non-dispersive dielectric inclusions, this phenomena is discovered in [9, 10, 16, 17, 35], see also the mathematically related investigations of [12, 25, 29, 52, 35]. Motivated by these observations, we have constructed a hybrid composite crystal that combines both high dielectric inclusions and frequency dependent inclusions for generating double negative response from non-magnetic materials.

The power series approach to sub-wavelength $\eta < 1$ analysis has been developed in [23] for characterizing the dynamic dispersion relations for Bloch waves inside plasmonic crystals. It has also been applied to assess the influence of effective negative permeability on the propagation of Bloch waves inside high contrast dielectrics [24], the generation of negative permeability inside metallic-dielectric resonators [40], and for concentric coated cylinder assemblages generating a double negative media [11].

We conclude noting that earlier related work introduces the use of high contrast structures for opening band gaps in photonic crystals; this is developed in [19–21]. For two phase high contrast media integral equation methods are applied to recover

dispersion relations about frequencies corresponding to Dirichlet eigenvalues [3,4]. The connection between high contrast interfaces and negative effective magnetic permeability for time harmonic waves is made in [30]. More recently two-scale homogenization theory has been developed for three dimensional split ring structures that deliver negative effective magnetic permeability [8,13]. For periodic arrays made from metal fibers a homogenization theory delivering negative effective dielectric constant [7] is established. A novel method for creating metamaterials with prescribed effective dielectric permittivity and effective magnetic permeability at a fixed frequency is developed in [33].

2. Background, Basic Theory, and Generalized Electrostatic Resonances

In this section we outline the steps in the power series development of Bloch waves and establish the mathematical foundations for establishing existence of power series solutions. We introduce the function space $H^1_{\text{per}}(Y)$ defined to be all Y -periodic, complex valued square integrable functions with square integrable derivatives with the usual inner product and norm given by

$$(u, v)_Y = \int_Y (\nabla u \cdot \nabla \bar{v} + u\bar{v}) \, dy \quad \text{and} \quad \|u\|_Y = (u, u)^{1/2}_Y. \tag{8}$$

We also introduce the space $H^1_{\text{per}}(Y \setminus R)$ given by all Y -periodic, complex valued square integrable functions with square integrable derivatives and the inner product and seminorm

$$(u, v) = \int_{Y \setminus R} \nabla u \cdot \nabla \bar{v} \, dy \quad \text{and} \quad \|u\| = (u, u)^{1/2}. \tag{9}$$

The variational form of (7) is given by

$$\int_Y a_\rho^{-1} \nabla u^d \cdot \nabla \bar{v} = \int_Y \frac{\rho^2 \xi}{c^2} u^d \bar{v} \tag{10}$$

for any $\tilde{v} = v(\mathbf{y})e^{i\hat{k} \cdot \tau \rho \mathbf{y}}$, where $v \in H^1_{\text{per}}(Y)$. On writing $u^d = u(\mathbf{y})e^{i\hat{k} \cdot \tau \rho \mathbf{y}}$ and setting $\eta = \tau \rho$ we transform (10) into

$$\begin{aligned} & \int_H \tau^2 \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) (\nabla + i\eta\hat{k})u \cdot \overline{(\nabla + i\eta\hat{k})v} + \int_P \tau^2 \xi (\nabla + i\eta\hat{k})u \cdot \overline{(\nabla + i\eta\hat{k})v} \\ & + \int_R \eta^2 \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) (\nabla + i\eta\hat{k})u \cdot \overline{(\nabla + i\eta\hat{k})v} = \int_Y \eta^2 \xi \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) u \bar{v} \end{aligned} \tag{11}$$

We introduce the power series

$$u = \sum_{m=0}^{\infty} \eta^m u_m \tag{12}$$

$$\xi = \sum_{m=0}^{\infty} \eta^m \xi_m \tag{13}$$

where u_m belongs to $H^1_{\text{per}}(Y)$. In view of the algebra it is convenient to write $u_m = i^m \underline{u}_0 \psi_m$ where \underline{u}_0 is an arbitrary constant factor.

We now describe the underlying variational structure associated with the power series solution. Set

$$z = \varepsilon_p^{-1}(\xi_0) = \left(1 - \frac{\varepsilon_r \omega_p^2 / c^2}{\xi_0} \right)^{-1}$$

and for u, v belonging to $H^1_{\text{per}}(Y)$ we introduce the sesquilinear form

$$B_z(u, v) = \int_H \nabla u \cdot \nabla \bar{v} \, dy + \int_P z \nabla u \cdot \nabla \bar{v} \, dy. \tag{14}$$

Here $Y \setminus R = H \cup P$ and the form $B_z(u, v)$ is well defined for functions in $H^1_{\text{per}}(Y \setminus R)$. Substitution of the series into the system (11) and equating like powers of η , produces the infinite set of coupled equations for $m = 0, 1, 2 \dots$ given by

$$\begin{aligned} & \tau^2 B_z(\psi_m, v) + \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \tau^2 \int_{Y \setminus R} \left[\sum_{l=1}^{m-1} (-i)^l \xi_l \nabla \psi_{m-l} \cdot \nabla \bar{v} \right. \\ & \left. + \hat{\kappa} \cdot \sum_{l=0}^{m-1} (-i)^l \xi_l (\psi_{m-1-l} \nabla \bar{v} - \nabla \psi_{m-1-l} \bar{v}) - \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \bar{v} \right] \\ & - \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \tau^2 \varepsilon_r \frac{\omega_p^2}{c^2} \int_H [\hat{\kappa} \cdot (\psi_{m-1} \nabla \bar{v} - \nabla \psi_{m-1} \bar{v}) - \psi_{m-2} \bar{v}] \\ & - \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \int_R \left[\sum_{l=0}^{m-2} (-i)^l \xi_l \nabla \psi_{m-2-l} \cdot \nabla \bar{v} \right. \\ & \left. + \hat{\kappa} \sum_{l=0}^{m-3} (-i)^l \xi_l (\psi_{m-3-l} \nabla \bar{v} - \nabla \psi_{m-3-l} \bar{v}) - \sum_{l=0}^{m-4} (-i)^l \xi_l \psi_{m-4-l} \bar{v} \right] \\ & + \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \int_R \varepsilon_r \frac{\omega_p^2}{c^2} [\nabla \psi_{m-2} \cdot \nabla \bar{v} + \hat{\kappa} (\psi_{m-3} \nabla \bar{v} - \nabla \psi_{m-3} \bar{v}) + \psi_{m-4} \bar{v}] \\ & - \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \int_Y \left[\sum_{l=0}^{m-2} \sum_{n=0}^l \xi_{m-2-l} \xi_n \psi_{l-n} i^{l-n-m} \bar{v} \right. \\ & \left. + \varepsilon_r \frac{\omega_p^2}{c^2} \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \bar{v} \right] = 0, \quad \text{for all } v \text{ in } H^1_{\text{per}}(Y). \tag{15} \end{aligned}$$

Here the convention is $\psi_m = 0$ for $m < 0$.

The determination of $\{\psi_m\}_{m=0}^\infty$ proceeds iteratively. We start by determining ψ_0 on $Y \setminus R$, this function is used as boundary data to determine ψ_0 in R from which we determine ψ_1 on $Y \setminus R$ and the full sequence is determined on iterating this cycle. The elements ξ_m are recovered from solvability conditions obtained by setting $v = 1$ in (15) and proceeding iteratively. The complete algorithm together with

explicit boundary value problems necessary for the determination of the sequences $\{\psi_m\}_{m=0}^\infty, \{\xi_m\}_{m=0}^\infty$ is described in Section 3 and Theorem 3.

The existence theory for the solution of the sequence of boundary value problems is based on the sesquilinear form $B_z(u, v)$ defined for functions u and v belonging to $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. Here $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ is the subspace of functions u belonging to $H^1_{\text{per}}(Y \setminus R)$ with zero mean $\int_{Y \setminus R} u \, dy = 0$. This space is a Hilbert space with inner product (9). Although in this treatment ε_P is given by (1) we are motivated by the general case (2) and proceed in full generality allowing for the possibility that z can lie anywhere on the complex plane \mathbb{C} including the negative real axis. Thus, for each z in \mathbb{C} we are required to characterize the range of the map $u \mapsto B_z(u, \cdot)$ viewed as a linear transformation T_z mapping u into the space of bounded skew linear functionals on $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. This is linked to the following eigenvalue problem characterizing all pairs λ in \mathbb{C} , ψ in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ that solve

$$-\frac{1}{2} \int_P \nabla \psi \cdot \nabla \bar{v} \, dy + \frac{1}{2} \int_H \nabla \psi \cdot \nabla \bar{v} \, dy = (\lambda \psi, v), \tag{16}$$

for every v in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. Inspection shows that for $z = (\lambda + 1/2)/(\lambda - 1/2)$ that $B_z(\psi, v) = 0$, for every v in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. In other words the kernel of the operator T_z is nonempty for $z = (\lambda + 1/2)/(\lambda - 1/2)$. The eigenvalues λ will be referred to as generalized electrostatic resonances. In what follows we show that T_z is a one to one and onto map from $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ into the dual space provided that $z \neq (\lambda + 1/2)/(\lambda - 1/2)$.

The generalized electrostatic resonances are characterized by introducing a suitable orthogonal decomposition of $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$. We introduce the subspace $H^1_0(P)$ given by the closure in the H^1 norm of smooth functions v with compact support on P and the subspace $H^1_{0,\text{per}}(H)$ given by the closure in the H^1 norm of all periodic continuously differentiable functions with support outside P . Extending elements of $H^1_{0,\text{per}}(H)$ by zero to $Y \setminus R$ delivers $W_1 \subset H^1_{\text{per}}(Y \setminus R)$, extending elements of $H^1_0(P)$ by zero to $Y \setminus R$ delivers $W_2 \subset H^1_{\text{per}}(Y \setminus R)$. Define W_3 to be all functions w in $H^1_{\text{per}}(Y \setminus R)$ for which the boundary integral $\int_{\partial P} w \, dS$ vanishes and that belong to the orthogonal complement of $W_1 \cup W_2$ with respect to the inner product (9). It is easily verified that W_1, W_2 , and W_3 are pairwise orthogonal with respect to the inner product (9) and

$$H^1_{\text{per}}(Y \setminus R)/\mathbb{C} = W_1 \oplus W_2 \oplus W_3 \oplus \mathbb{C}, \tag{17}$$

where the constant part of a function u belonging to this space is uniquely determined by the condition $\int_{Y \setminus R} u \, dy = 0$.

The following theorem describing all eigenvalue-eigenfunction pairs is established in Section 5.

Theorem 1. *The eigenvalues for (16) are real and constitute a denumerable set contained inside $[-1/2, 1/2]$ with the only accumulation point being zero. The eigenspaces associated with $\lambda = 1/2$ and $\lambda = -1/2$ are W_1 and W_2 , respectively. Eigenspaces associated with distinct eigenvalues in $(-1/2, 1/2)$ are pairwise orthogonal, and their union spans the subspace W_3 .*

We denote the denumerable set of eigenvalues for (16) by the sequence $\{\lambda_i\}_{i=1}^\infty$. Here we put $\lambda_1 = 0$. The orthogonal projections onto W_1 and W_2 are denoted by \mathcal{P}_1 and \mathcal{P}_2 . The orthogonal projections associated with λ_i in $(-1/2, 1/2)$ are denoted by \mathcal{P}_{λ_i} . For $z \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$ the following existence theorem holds.

Theorem 2. *Suppose $z \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$ for $\lambda_i \in [-\frac{1}{2}, \frac{1}{2}]$ then*

- For any $F \in [H_{\text{per}}^1(Y \setminus R)/\mathbb{C}]^*$ such that $F(v) = 0$ for constant v , there exists a unique solution $u \in H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ of the variational problem $B_z(u, v) = \overline{F(v)}$ for all $v \in H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$.
- The transformation T_z from $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ onto itself has the representation formula given by

$$T_z = \mathcal{P}_1 + z\mathcal{P}_2 + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(1 + (z - 1) \left(\frac{1}{2} - \lambda_n \right) \right) \mathcal{P}_{\lambda_n}, \quad (18)$$

with inverse

$$T_z^{-1} = \mathcal{P}_1 + z^{-1}\mathcal{P}_2 + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(1 + (z - 1) \left(\frac{1}{2} - \lambda_n \right) \right)^{-1} \mathcal{P}_{\lambda_n}, \quad (19)$$

and $B_z(u, v) = (T_z u, v)$ for all u, v in $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$.

This theorem is proved in Section 6.

In the following sections we present the sequence of boundary value problems for determining ψ_m in $Y \setminus R$ and R together with the sequence of solvability conditions characterizing ξ_m . In this section we use the complete orthonormal systems of eigenfunctions associated with electrostatic resonances and Dirichlet eigenvalues to explicitly solve for fields ψ_0 in Y and ψ_1 in $Y \setminus R$ and provide an explicit formula for ξ_0 . In the following section the boundary value problems used to determine ψ_m in $Y \setminus R$, $m \geq 2$ and ψ_m in R for $m \geq 1$ together with solvability conditions for ξ_m for $m \geq 1$ are shown to be a well posed infinite system of equations see Theorem 3.

Applying the convention $\psi_m = 0$ for $m < 0$ in (15) shows that ψ_0 is the solution of

$$B_z(\psi_0, v) = 0, \quad \text{for all } v \text{ in } H^1(Y \setminus R). \quad (20)$$

From Theorem 1 and (17) we have the dichotomy:

1. ξ_0 satisfies $\varepsilon_p^{-1}(\xi_0) = (\lambda_i + 1/2)/(\lambda_i - 1/2)$ and ψ_0 is an eigenfunction for (16).
2. ξ_0 satisfies $\varepsilon_p^{-1}(\xi_0) \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$, $i = 1, 2, \dots$ and $\psi_0 = \text{constant}$.

In this article we assume the second alternative. Subsequent work will investigate the case when the first alternative is applied. For future reference the condition $\varepsilon_p^{-1}(\xi_0) \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$ is equivalent to

$$\xi_0 \neq \zeta_i, \quad \zeta_i \equiv \left(\lambda_i + \frac{1}{2} \right) \frac{\varepsilon_r \omega_p^2}{c^2}, \quad 0 \leq \zeta_i \leq \frac{\varepsilon_r \omega_p^2}{c^2} \quad i = 1, 2, \dots \quad (21)$$

Restricting to test functions v with support in R in (15) we get

$$\int_R (\nabla \psi_0 \cdot \nabla \bar{v} - \xi_0 \psi_0 \bar{v}) \, dy = 0. \tag{22}$$

From continuity we have the boundary condition for ψ_0 on R given by $\psi_0 = \text{const}$. We denote the Dirichlet eigenvalues for R by $v_j, j = 1, 2, \dots$. Here we have the alternative:

1. If ξ_0 is a Dirichlet eigenvalue v_i of $-\Delta$ in R then $\psi_0(\mathbf{y}) = 0$ for \mathbf{y} in $Y \setminus R$.
2. If $\xi_0 \neq v_i, i = 1, 2, \dots$ then ψ_0 is the unique solution of the Helmholtz equation (22) and $\psi_0 = \text{const}$. in $Y \setminus R$.

In this treatment we will choose the second alternative $\xi_0 \neq v_i$. The case when the first alternative is chosen will be taken up in future investigation. Since $u_0 = \underline{u}_0 \psi_0$ where \underline{u}_0 is an arbitrary constant we can, without loss of generality, make the choice $\psi_0 = 1$ for \mathbf{y} in $Y \setminus R$. Since $\xi_0 \neq v_i, i = 1, \dots$ and $\psi_0 = 1$ in $Y \setminus R$ a straight forward calculation gives ψ_0 in R in terms of the complete set of Dirichlet eigenfunctions and eigenvalues:

$$\psi_0 = \sum_{n=1}^{\infty} \frac{\mu_n \langle \phi_n \rangle_R}{\mu_n - \xi_0} \phi_n, \quad \text{in } R. \tag{23}$$

Note here that μ_n denote the Dirichlet eigenvalues of $-\Delta$ in R whose eigenfunctions ϕ_n have nonzero mean, $\langle \phi_n \rangle_R = \int_R \phi_n(y) \, dy \neq 0$. The Dirichlet eigenvalues associated with zero mean eigenfunctions are denoted by μ'_n and $\{v_n\}_{n=1}^{\infty} = \{\mu_n\}_{n=1}^{\infty} \cup \{\mu'_n\}_{n=1}^{\infty}$.

To find ψ_1 in $Y \setminus R$, we appeal to (15) with $\psi_0 = 1$ in $Y \setminus R$ to discover

$$B_z(\psi_1, v) = - \int_H \hat{\kappa} \cdot \nabla \bar{v} - \int_P \varepsilon_P^{-1}(\xi_0) \hat{\kappa} \cdot \nabla \bar{v} \quad \forall v \in H_{\text{per}}^1(Y) \tag{24}$$

It follows from Theorem 2 that the problem has a unique solution subject to the mean-zero condition: $\int_{Y \setminus R} \psi_1 = 0$ provided that $\xi_0 \neq \xi_i, i = 1, 2, \dots$. We apply the decomposition of $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ given by (17) and represent ψ_1 in terms of the complete set of orthonormal eigenfunctions $\{\psi_{\lambda_n}\} \subset W_3$ associated with $-1/2 < \lambda_n < 1/2$ together with the complete orthonormal sets of functions for W_1 and W_2 , denoted by $\{\psi_n^1\}_{n=1}^{\infty}$ and $\{\psi_n^2\}_{n=1}^{\infty}$, respectively. A straight forward calculation gives the representation for ψ_1 in $Y \setminus R$

$$\psi_1 = - \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(\frac{(\alpha_{\lambda_n}^1 + \varepsilon_P^{-1}(\xi_0) \alpha_{\lambda_n}^2)}{1 + (\varepsilon_P^{-1}(\xi_0) - 1)(1 - \lambda_n)} \right) \psi_{\lambda_n} + \sum_{n=1}^{\infty} \alpha_{1,n} \psi_n^1, \quad \text{in } Y \setminus R \tag{25}$$

with

$$\alpha_{\lambda_n}^1 = \hat{\kappa} \cdot \int_H \nabla \psi_{\lambda_n} \, dy, \quad \alpha_{\lambda_n}^2 = \hat{\kappa} \cdot \int_P \nabla \psi_{\lambda_n} \, dy, \quad \text{and} \quad \alpha_{1,n} = \hat{\kappa} \cdot \int_H \nabla \psi_n^1 \, dy. \tag{26}$$

Setting $v = 1$ and $m = 2$ in (15) we recover the solvability condition given by

$$\begin{aligned} &\tau^2 \int_{H \cup P} [-\hat{k} \cdot \xi_0 \nabla \psi_1 + \xi_0] - \tau^2 \varepsilon_r \frac{\omega_p^2}{c^2} \int_H (-\hat{k} \nabla \psi_1 + 1) \\ &= \int_Y \left(\xi_0^2 \psi_0 - \varepsilon_r \frac{\omega_p^2}{c^2} \xi_0 \psi_0 \right) \end{aligned} \tag{27}$$

Substitution of the spectral representations for ψ_1 and ψ_0 given by (23) and (25) into (27) delivers the quasistatic dispersion relation

$$\xi_0 = \tau^2 n_{\text{eff}}^{-2}(\xi_0), \tag{28}$$

where the effective index of diffraction n_{eff}^2 depends upon the direction of propagation \hat{k} and is written

$$n_{\text{eff}}^2(\xi_0) = \mu_{\text{eff}}(\xi_0) / \varepsilon_{\text{eff}}^{-1}(\xi_0) \hat{k} \cdot \hat{k}. \tag{29}$$

The frequency dependent effective magnetic permeability μ_{eff} and effective dielectric permittivity ε_{eff} are given by

$$\mu_{\text{eff}}(\xi_0) = \int_Y \psi_0 = \theta_H + \theta_P + \sum_{n=1}^{\infty} \frac{\mu_n \langle \phi_n \rangle_R^2}{\mu_n - \xi_0} \tag{30}$$

and

$$\begin{aligned} \varepsilon_{\text{eff}}^{-1}(\xi_0) \hat{k} \cdot \hat{k} &= \int_H (I - \mathbb{P}) \hat{k} \cdot \hat{k} \, \mathbf{d}\mathbf{y} + \frac{\xi_0}{\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2}} \theta_P \\ &- \sum_{-\frac{1}{2} < \lambda_h < \frac{1}{2}} \left(\frac{\left(\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2} \right) |\alpha_{\lambda_h}^{(1)}|^2 + 2 \frac{\varepsilon_r \omega_p^2}{c^2} \alpha_{\lambda_h}^{(1)} \alpha_{\lambda_h}^{(2)} + \frac{\left(\frac{\varepsilon_r \omega_p^2}{c^2} \right)^2}{\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2}} |\alpha_{\lambda_h}^{(2)}|^2}{\xi_0 - s_h} \right), \end{aligned} \tag{31}$$

where θ_H and θ_P are the areas occupied by regions H and P , respectively. The first term on the right hand side of (31) is positive and frequency independent. It is written in terms of the spectral projection \mathbb{P} of square integrable vector fields over H onto the subspace of gradients of potentials ψ in $H_{0,\text{per}}^1(H)$. Here $\int_H \mathbb{P} \hat{k} \cdot \hat{k} \, \mathbf{d}\mathbf{y} < \theta_H$ and $\mathbb{P} \sigma = \sum_{n=1}^{\infty} \left(\int_H \nabla \psi_n^1 \cdot \sigma \, \mathbf{d}\mathbf{y} \right) \nabla \psi_n^1$ with $\sum_{n=1}^{\infty} |\alpha_{1,n}|^2 = \int_H \mathbb{P} \hat{k} \cdot \hat{k} \, \mathbf{d}\mathbf{y}$. The poles s_h are the subset of values ζ_h for which at least one of the weights $\alpha_{\lambda_h}^{(1)}$ and $\alpha_{\lambda_h}^{(2)}$ are nonzero. The values of ζ_h for which both weights vanish are denoted by s'_h and $\{\zeta_h\}_{h=1}^{\infty} = \{s_h\}_{h=1}^{\infty} \cup \{s'_h\}_{h=1}^{\infty}$. The graphs of μ_{eff} and $\varepsilon_{\text{eff}}^{-1} \hat{k} \cdot \hat{k}$ as functions of ξ_0 are displayed in Figs. 2 and 3. Here the intervals $a' < \xi_0 < b'$ and $a'' < \xi_0 < b''$ are the same in all graphs.

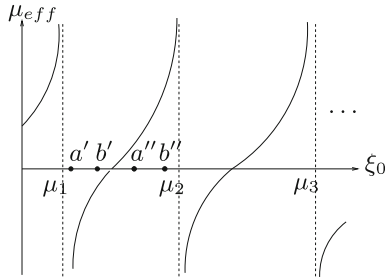


Fig. 2. The relation between μ_{eff} and ξ_0

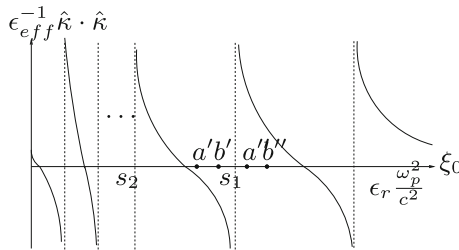


Fig. 3. The relation between $\epsilon_{eff}^{-1} \hat{k} \cdot \hat{k}$ and ξ_0

For future reference is convenient to write the dispersion relation explicitly in terms of μ_{eff} and ϵ_{eff} and

$$\mu_{eff}(\xi_0)\xi_0 = \tau^2 \epsilon_{eff}^{-1}(\xi_0) \hat{k} \cdot \hat{k}. \tag{32}$$

The branches of the quasistatic dispersion relation $(\tau, \hat{k}) \mapsto \xi_0$ are controlled by the poles and zeros of μ_{eff} and ϵ_{eff}^{-1} explicitly determined through the Dirichlet spectra and generalized electrostatic resonances.

3. Solution of Higher Order Problems

Having determined ψ_0 in Y , ψ_1 in $Y \setminus R$, and ξ_0 we now provide the algorithm for determining the rest of the elements in the sequences $\{\psi_m\}_{m=0}^\infty, \{\xi_m\}_{m=0}^\infty$. The algorithm is summarized in Theorem 3.

Step I. Solution of ψ_m in R for $m \geq 1$.

We restrict the trial space to test functions v with support in R and put $m \mapsto m + 2$ in (15). We decompose ψ_m according to

$$\psi_m = \tilde{\psi}_m + (-i)^m \xi_m \psi_* \tag{33}$$

and substitute into (15). This decomposition is chosen such that $\tilde{\psi}_m$ depends on ξ_n and ψ_n inside R for $n \leq m - 1$. The function ψ_* depends only upon ξ_0 and ψ_0 in

R . The function $\tilde{\psi}_m$ solves the Dirichlet boundary value problem with $\tilde{\psi}_m|_{\partial R^-} = \psi_m|_{\partial R^+}$ and

$$\int_R (\nabla \tilde{\psi}_m \cdot \nabla \bar{v} - \xi_0 \tilde{\psi}_m \bar{v}) \, dy = \int_R F \cdot \nabla \bar{v} \, dy + \int_R G \bar{v} \, dy, \tag{34}$$

where

$$F = -\xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \left[\sum_{l=1}^{m-1} (-i)^l \xi_l \nabla \psi_{m-l} + \hat{\kappa} \left(\sum_{l=0}^{m-1} (-i)^l \xi_l \psi_{m-1-l} \right) \right] + \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \varepsilon_r \frac{\omega_p^2}{c^2} \hat{\kappa} \psi_{m-1}, \tag{35}$$

and

$$G = \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \left(\hat{\kappa} \cdot \sum_{l=0}^{m-1} (-i)^l \xi_l \nabla \psi_{m-1-l} + \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \right) - \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) (\hat{\kappa} \cdot \nabla \psi_{m-1} + \psi_{m-2}) + \xi_0^{-1} \varepsilon_p^{-1}(\xi_0) \left[\sum_{l=1}^{m-1} \sum_{n=0}^l \xi_{m-l} \xi_n \psi_{l-n} i^{l-n-m} - \varepsilon_r \frac{\omega_p^2}{c^2} \sum_{l=1}^{m-1} (-i)^l \xi_l \psi_{m-l} + \xi_0 \sum_{n=1}^{m-1} \xi_n \psi_{m-n} i^{-n} \right]. \tag{36}$$

This boundary value problem has a unique solution for $\xi_0 \neq \nu_i, i = 1, \dots$. The function ψ_* is the solution of

$$\begin{cases} \int_R (-\nabla \psi_* \cdot \nabla \bar{v} + \xi_0 \psi_* \bar{v}) + \int_R \psi_0 \bar{v} = 0 \\ \psi_*|_{\partial R^-} = 0 \end{cases} \tag{37}$$

The explicit representation for ψ_* is obtained using the Dirichlet eigenfunctions on R and is given by,

$$\psi_* = \sum_{n=1}^{\infty} \frac{\mu_n \langle \phi_n \rangle_R}{(\mu_n - \xi_0)^2} \phi_n \quad \text{in } R. \tag{38}$$

Step II. Solution of ψ_m in $Y \setminus R$ for $m \geq 2$

We decompose ψ_m

$$\psi_m = \psi'_m + (-i)^{m-1} \xi_{m-1} \hat{\psi} \tag{39}$$

and substitute into (15). This decomposition is chosen such that ψ'_m depends on ξ_n for $0 \leq n \leq m - 2$, the functions $\psi_n, n \leq m - 1$ in $Y \setminus R$ and the functions $\psi_n, n \leq m - 2$ in R . The function $\hat{\psi}$ depends only on ξ_0 and ψ_1 in $Y \setminus R$.

For $m \geq 2$, ψ'_m in $Y \setminus R$ subject to the mean zero condition $\int_{Y \setminus R} \psi'_m \, dy = 0$ is the solution of

$$\begin{aligned} \tau^2 B_z(\psi'_m, v) &= \int_{Y \setminus R} (F_1 \cdot \nabla \bar{v} + G_1 \bar{v}) \, dy \\ &+ \int_R (F_2 \cdot \nabla \bar{v} + G_2 \bar{v}) \, dy, \quad \text{for all } v \text{ in } H^1_{\text{per}}(Y), \end{aligned} \tag{40}$$

where

$$\begin{aligned} F_1 &= \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \tau^2 \left[\sum_{l=1}^{m-2} (-i)^l \xi_l \nabla \psi_{m-l} + \left(\sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-1-l} \right) \hat{\kappa} \right] \\ &- \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \tau^2 \varepsilon_r \frac{\omega_p^2}{c^2} \chi_H \hat{\kappa} \psi_{m-1}, \end{aligned} \tag{41}$$

$$\begin{aligned} G_1 &= \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \tau^2 \left[\hat{\kappa} \cdot \sum_{l=0}^{m-2} (-i)^l \xi_l (-\nabla \psi_{m-1-l}) - \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \right] \\ &- \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \tau^2 \varepsilon_r \frac{\omega_p^2}{c^2} \chi_H [\hat{\kappa} \cdot (-\nabla \psi_{m-1}) - \psi_{m-2}] \\ &+ \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \left[\sum_{l=0}^{m-2} \sum_{n=0}^l \xi_{m-2-l} \xi_n \psi_{l-n} i^{l-n-m} \right. \\ &\left. - \varepsilon_r \frac{\omega_p^2}{c^2} \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \right], \end{aligned} \tag{42}$$

$$\begin{aligned} F_2 &= -\xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \left[\sum_{l=0}^{m-2} (-i)^l \xi_l \nabla \psi_{m-2-l} + \hat{\kappa} \left(\sum_{l=0}^{m-3} (-i)^l \xi_l \psi_{m-3-l} \right) \right] \\ &+ \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \varepsilon_r \frac{\omega_p^2}{c^2} [\nabla \psi_{m-2} + \hat{\kappa} (\psi_{m-3})], \end{aligned} \tag{43}$$

and

$$\begin{aligned} G_2 &= -\xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \left[+\hat{\kappa} \left(\sum_{l=0}^{m-3} (-i)^l \xi_l (-\nabla \psi_{m-3-l}) \right) \right. \\ &- \left. \sum_{l=0}^{m-4} (-i)^l \xi_l \psi_{m-4-l} \right] + \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \varepsilon_r \frac{\omega_p^2}{c^2} [\hat{\kappa} \cdot (-\nabla \psi_{m-3}) + \psi_{m-4}] \\ &+ \xi_0^{-1} \varepsilon_p^{-1} (\xi_0) \left[\sum_{l=0}^{m-2} \sum_{n=0}^l \xi_{m-2-l} \xi_n \psi_{l-n} i^{l-n-m} \bar{v} \right. \\ &\left. - \varepsilon_r \frac{\omega_p^2}{c^2} \sum_{l=0}^{m-2} (-i)^l \xi_l \psi_{m-2-l} \bar{v} \right], \end{aligned} \tag{44}$$

where χ_H is the indicator function of the host domain H taking 1 inside and zero outside. Here we have used the identity $\psi_0(\mathbf{y}) = 1$, for $\mathbf{y} \in Y \setminus R$ in determining the formulas for the right hand side of (40).

Setting $v = 1$ in (40) delivers the solvability condition

$$\int_{Y \setminus R} G_1 \, d\mathbf{y} + \int_R G_2 \, d\mathbf{y} = 0. \tag{45}$$

The condition

$$-\nabla \cdot F_2 + G_2 = 0 \quad \text{for } \mathbf{y} \text{ in } R \tag{46}$$

follows from the solution of Step I see, (33), (34) and (37). Integration by parts on the right hand side of (40) together with (46) transforms (40) into the equivalent Neumann boundary value problem for ψ'_m , $m \geq 2$, given by

$$\begin{aligned} \tau^2 B_z(\psi'_m, v) &= \int_{Y \setminus R} (F_1 \cdot \nabla \bar{v} + G_1 \bar{v}) \, d\mathbf{y} \\ &+ \int_{\partial R} F_2 \cdot n \bar{v} \, ds, \quad \text{for all } v \text{ in } H^1_{\text{per}}(Y \setminus R), \end{aligned} \tag{47}$$

where n is the outward directed unit normal to ∂R . Finally we observe that the solvability condition for the Neumann problem (47) given by

$$\int_{Y \setminus R} G_1 \, d\mathbf{y} + \int_{\partial R} F_2 \cdot n \, ds = 0, \tag{48}$$

follows immediately from (45) and (46). This Neumann problem satisfies the hypotheses of Theorem 2 and we assert the existence of a solution ψ'_m for (47), uniquely determined by the condition $\int_{Y \setminus R} \psi'_m \, d\mathbf{y} = 0$, provided that $\xi_0 \neq \zeta_i$, $0 \leq \zeta_i \leq \frac{\epsilon_r \omega_p^2}{c^2}$, $i = 1, 2, \dots$

The field $\hat{\psi}$ solves

$$\begin{cases} B_z(\hat{\psi}, v) = -\xi_0^{-1} \epsilon_p^{-1}(\xi_0) \int_{Y \setminus R} (\nabla \psi_1 + \hat{\kappa}) \cdot \nabla \bar{v}, \\ \int_{Y \setminus R} \hat{\psi} = 0, \end{cases} \tag{49}$$

for all trials $v \in H^1_{\text{per}}(Y \setminus R)$. The solution $\hat{\psi}$ is represented explicitly in terms of the eigenvectors associated with the generalized electrostatic resonances. A straight forward calculation gives

$$\begin{aligned} \hat{\psi} &= \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \frac{\psi_{\lambda_n}}{\xi_0 - \left(\frac{1}{2} + \lambda_n\right) \frac{\epsilon_r \omega_p^2}{c^2}} \\ &\times \left(-(\alpha_{\lambda_n}^1 + \alpha_{\lambda_n}^2) + \frac{\left(\xi_0 - \frac{\epsilon_r \omega_p^2}{c^2}\right) \left(\alpha_{\lambda_n}^1 + \alpha_{\lambda_n}^2 \left(\frac{\xi_0}{\xi_0 - \frac{\epsilon_r \omega_p^2}{c^2}}\right)\right)}{\xi_0 - s_n} \right). \end{aligned} \tag{50}$$

Step III. Solution of ξ_m for $m \geq 1$.

We apply the decomposition (33) to ψ_{m-2} in R and (39) to ψ_{m-1} in $Y \setminus R$. Substitution of the decompositions into the solvability condition (45) and applying the explicit spectral representations (38), (50) for ψ_* and $\hat{\psi}$ we obtain the recursion relation for determining ξ_1, ξ_2, \dots given by

$$\begin{aligned} \xi_{m-2}\mathcal{G}(\xi_0) = & \tau^2 \int_{Y \setminus R} \left[\hat{\kappa} \cdot \left(\sum_{l=1}^{m-3} (-i)^l \xi_l \nabla \psi_{m-1-l} + \xi_0 \nabla \psi'_{m-1} \right) \right. \\ & \left. + \sum_{l=0}^{m-3} (-i)^l \xi_l \psi_{m-2-l} \right] \\ & - \tau^2 \int_H \varepsilon_r \frac{\omega_p^2}{c^2} (\hat{\kappa} \cdot \nabla \psi'_{m-1} + \psi_{m-2}) \\ & - \int_R \left[\hat{\kappa} \cdot \sum_{l=0}^{m-3} (-i)^l \xi_l \nabla \psi_{m-3-l} + \sum_{l=0}^{m-4} (-i)^l \xi_l \psi_{m-4-l} \right] \\ & - \int_Y \left[\sum_{l=1}^{m-3} \sum_{n=0}^l i^{l-n-m} \xi_{m-2-l} \xi_n \psi_{l-n} - \sum_{l=1}^{m-3} (-i)^l \xi_0 \xi_l \psi_{m-2-l} \right] \\ & - \int_R \xi_0^2 \tilde{\psi}_{m-2} - \int_{Y \setminus R} \xi_0^2 \psi_{m-2} + \int_Y \varepsilon_r \frac{\omega_p^2}{c^2} \sum_{l=1}^{m-3} (-i)^l \xi_l \psi_{m-2-l} \\ & + \int_R \varepsilon_r \frac{\omega_p^2}{c^2} \xi_0 \tilde{\psi}_{m-2} + \int_{Y \setminus R} \varepsilon_r \frac{\omega_p^2}{c^2} \xi_0 \psi_{m-2}. \end{aligned} \tag{51}$$

Noting that (33) and (39) can also be applied to the lower order terms on the right hand side of (51) we see that (51) determines ξ_{m-2} from ξ_n, ψ_n in $Y \setminus R$, and ψ_n in R with $0 \leq n \leq m - 3$. Here the ‘‘solvability matrix,’’ \mathcal{G} depends explicitly on ξ_0 and is given by

$$\begin{aligned} \mathcal{G}(\xi_0) = & \theta_H + \theta_P \\ & + \tau^2 \left[\int_H \mathbb{P} \hat{\kappa} \cdot \hat{\kappa} \, d\mathbf{y} - \sum_{n=1}^{\infty} (g_n^2(\xi_0) |\alpha_{\lambda_n}^1 + \varepsilon_P^{-1}(\xi_0) \alpha_{\lambda_n}^2|^2) \right. \\ & \left. + 2g_n(\xi_0) (\alpha_{\lambda_n}^1 + \alpha_{\lambda_n}^2) (\alpha_{\lambda_n}^1 + \varepsilon_P^{-1}(\xi_0) \alpha_{\lambda_n}^2) \right] \\ & - 2 \left(\xi_0 + \frac{\varepsilon_r \omega_p^2}{c^2} \right) \mu_{\text{eff}}(\xi_0) + \xi_0 \left(\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2} \right) \sum_{n=1}^{\infty} \frac{\mu_n \langle \phi_n \rangle_R^2}{(\mu_n - \xi_0)^2}, \end{aligned} \tag{52}$$

where $g_n(\xi_0) = (\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2}) / (\xi_0 - s_n)$ and $\varepsilon_P^{-1}(\xi_0) = \xi_0 / (\xi_0 - \frac{\varepsilon_r \omega_p^2}{c^2})$. The set of poles and zeros for $\mathcal{G}(\xi_0)$ are explicitly controlled by the generalized electrostatic resonances and Dirichlet spectra. The zeros of \mathcal{G} are denoted by $\gamma_n, n = 1, \dots$. It is now evident that the recursion relation (51) holds provided that $\xi_0 \neq \nu_n, \xi_0 \neq \zeta_n$, and $\xi_0 \neq \gamma_n$.

Denote the union of the Dirichlet spectra, the electrostatic resonances, and zeros of \mathcal{G} by

$$U \equiv \{v_j\}_{j=1}^\infty \cup \{\zeta_j\}_{j=1}^\infty \cup \{\gamma_j\}_{j=1}^\infty \quad (53)$$

where $\{v_j\}_{j=1}^\infty \subset \mathbb{R}^+$, $\{\gamma_j\}_{j=1}^\infty \subset \mathbb{R}^+$, and $\{\zeta_j\}_{j=1}^\infty \subset [0, \frac{\varepsilon_r \omega_p^2}{c^2}]$. We collect results and state the following theorem.

Theorem 3. *If ξ_0 belongs to $\mathbb{R}^+ \setminus U$ then the sequences $\{\psi_m\}_{m=0}^\infty \subset H_{\text{per}}^1(Y)$, $\{\xi_m\}_{m=0}^\infty \subset \mathbb{C}$ exist and are uniquely determined.*

Proof. The functions ψ_0 in $H^1(Y)$ and ψ_1 in $H^1(Y \setminus R)$ with $\int_{Y \setminus R} \psi_1 \, dy = 0$ exist as does ξ_0 . Application of Step I uniquely determines $\tilde{\psi}_1$ in R . Application of Step II uniquely determines ψ'_2 in $Y \setminus R$. Step III uniquely determines ξ_1 and we apply (33) and (39) to recover ψ_1 in $H^1(Y)$ and ψ_2 in $H^1(Y \setminus R)$ with $\int_{Y \setminus R} \psi_2 \, dy = 0$. We now adopt the induction hypotheses $H_n, n \geq 2$:

There exist

1. $\psi_n \in H_{\text{per}}^1(Y \setminus R)$ with $\int_{Y \setminus R} \psi_n = 0$;
2. $\psi_j \in H_{\text{per}}^1(Y)$ for $0 \leq j \leq n - 1$ with $\int_{Y \setminus R} \psi_j = 0, 1 \leq j \leq n - 1$;
3. $\xi_{n-1} \in \mathbb{C}$.

Application of Step I uniquely determines $\tilde{\psi}_n$ in R . Application of Step II uniquely determines ψ'_{n+1} in $Y \setminus R$. Step III uniquely determines ξ_n and we apply (33) and (39) to recover ψ_n in $H^1(Y)$ and ψ_{n+1} in $H^1(Y \setminus R)$ with $\int_{Y \setminus R} \psi_{n+1} \, dy = 0$. \square

4. Band Structure for the Metamaterial Crystal

In this section we present explicit formulas describing the band structure for the metamaterial crystal in the dynamic, sub-wavelength regime $1 > \eta > 0$. The formulas show that the frequency intervals associated with pass bands and stop bands are governed by the poles and zeros of the effective magnetic permittivity and dielectric permittivity tensors. The poles and zeros are explicitly determined by the Dirichlet spectra of R and the generalized electrostatic resonances of $Y \setminus R$ (see (30) and (31)). For the general class of parallel rod configurations treated here the pass bands and stop bands can exhibit anisotropy, that is, dependence on the direction of propagation described by \hat{k} . The anisotropy is governed by the projection of \hat{k} onto the eigenfunctions associated with the generalized electrostatic resonances given by (26). In what follows, pass bands are explicitly linked to frequency intervals and propagation directions for which both the effective magnetic permittivity and dielectric permeability have the same sign. This includes frequency intervals where effective tensors are either simultaneously positive or negative.

We begin by identifying the locations of the branches for the dispersion relation associated with the metamaterial crystal. We introduce the union of open intervals $\bigcup_n O_n$ on the positive real axis \mathbb{R}^+ obtained by removing the points $\{\zeta_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty, \{\gamma_j\}_{j=1}^\infty, \{s_j^*\}_{j=1}^\infty$ and $\{\mu_j^*\}_{j=1}^\infty$. Here s_j^* is the zero of $\varepsilon_{\text{eff}}^{-1}(\xi_0)\hat{k} \cdot \hat{k}$

between s_j and s_{j+1} and μ_j^* is the zero of $\mu_{\text{eff}}(\xi_0)$ between μ_j and μ_{j+1} . The leading order dispersion relation (32) implicitly defines the map $(\tau, \hat{\kappa}) \mapsto \xi_0$. We denote the branch associated with O_n by $\xi_0^n = \xi_0^{(n)}(\tau, \hat{\kappa})$. Let I_n be an open interval strictly contained inside O_n . For this choice $I_n \subset O_n$ does not intersect the union of the Dirichlet spectra, generalized electrostatic spectra and the zeros of the solvability matrix \mathcal{G} (see (53)).

The band structure for the metamaterial in the dynamic sub-wavelength regime $1 > \eta > 0$ is given by the following theorem.

Theorem 4. *The dispersion relation for the metamaterial crystal contains an infinite sequence of branches with leading order behavior given by the functions $\xi_0^n = \xi_0^n(\tau, \hat{\kappa})$. For τ such that ξ_0^n belongs to I_n , there exists a constant R depending only on I_n such that for $0 < \rho < R$, the branch of the dispersion relation for the metamaterial crystal is given by*

$$\xi = \xi_0^n(\tau, \hat{\kappa}) + \sum_{l=1}^{\infty} (\rho\tau)^l \xi_l^n, \tag{54}$$

for

$$\{-2\pi \leq \tau\rho\hat{\kappa}_1 \leq 2\pi, -2\pi \leq \tau\rho\hat{\kappa}_2 \leq 2\pi\}. \tag{55}$$

Here the higher order terms ξ_l^n are real and are uniquely determined by ξ_0^n according to Theorem 3.

The power series representation for the transverse magnetic Bloch wave (3) is given by the following theorem.

Theorem 5. *For I_n and R as in Theorem 4 and for τ such that ξ_0^n belongs to I_n there exist transverse magnetic Bloch waves given by the expansion*

$$H_3 = \underline{u}_0 \left(\psi_0(\mathbf{x}/d) + \sum_{l=1}^{\infty} (\rho\tau)^l i^l \psi_l(\mathbf{x}/d) \right) \exp\{i(k\hat{\kappa} \cdot \mathbf{x} - t\omega)\}, \tag{56}$$

where the series

$$\psi_0(\mathbf{y}) + \sum_{l=1}^{\infty} (\rho\tau)^l i^l \psi_l(\mathbf{y}) \tag{57}$$

is summable in $H^1(Y)$ for $0 \leq \rho < R$ and $\frac{\omega}{c} = \sqrt{\frac{\xi}{\varepsilon_r}}$.

Theorem 4 explicitly shows that the leading order behavior determines the existence of pass bands or stop bands when $\rho = d/\sqrt{\varepsilon_r}$ is sufficiently small. With this in mind we can appeal to (32) and state the following theorem.

Theorem 6. *For $\rho > 0$ sufficiently small:*

- There exist propagating Bloch wave solutions along directions $\hat{\kappa}$ over intervals I_n for which $\mu_{\text{eff}}(\xi_0^n)$ and $\varepsilon_{\text{eff}}^{-1}(\xi_0^n)\hat{\kappa} \cdot \hat{\kappa}$ have the same sign.

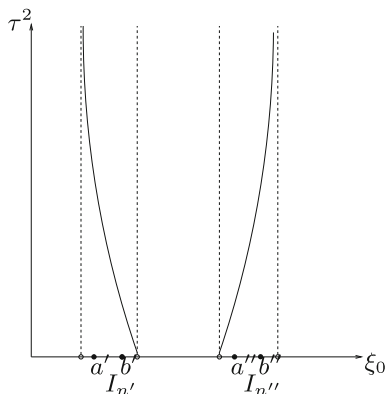


Fig. 4. The leading order dispersion relation over two selected intervals $I_{n'}$ and $I_{n''}$

– Intervals I_n for which $\mu_{\text{eff}}(\xi_0^n)$ and $\varepsilon_{\text{eff}}^{-1}(\xi_0^n)\hat{k} \cdot \hat{k}$ have the opposite sign lie within stop bands.

Leading order behavior for pass bands associated with double negative and double positive behavior are illustrated in Fig. 4 for two intervals $I_{n'}$ and $I_{n''}$. Here the effective properties are both negative over the interval $I_{n'}$ while they are both positive over $I_{n''}$.

5. Generalized Electrostatic Spectra

In this section we establish Theorem 1 and characterize all pairs (λ, ψ) in $\mathbb{C} \times H_{\text{per}}^1(Y \setminus R)$ satisfying (16). We start by recalling the bilinear form in (16) and forming the quotient

$$Q(u) = \frac{-\frac{1}{2} \int_P |\nabla u|^2 dx + \frac{1}{2} \int_H |\nabla u|^2 dx}{(u, u)}. \tag{58}$$

From (58) it is evident that λ lies inside $[-1/2, 1/2]$. It is easily seen that the solutions (λ, ψ) of (16) for the choices $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$ correspond to ψ in W_1 and W_2 , respectively. Only the 0 element of W_3 satisfies (16) for the choices $\lambda = \frac{1}{2}$ and $\lambda = -\frac{1}{2}$.

We now investigate solutions (λ, ψ) for ψ belonging to W_3 . The bilinear form (16) defines a map T from W_3 into the space of skew linear functionals on W_3 . Here W_3 is a Hilbert space and

$$(Tu, v) = -\frac{1}{2} \int_P \nabla u \cdot \nabla \bar{v} dy + \frac{1}{2} \int_H \nabla u \cdot \nabla \bar{v} dy, \tag{59}$$

for u and v in W_3 . In what follows we show that T is a compact map from W_3 onto W_3 with eigenvalues contained inside the open interval $(-1/2, 1/2)$. We do this by providing an explicit representation for T given in terms of a composition of single and double layer potentials.

Set $\mathbb{D} = \cup_{n \in \mathbb{Z}^2} ((Y \setminus R) + n)$ and introduce the Green's function $G(\mathbf{x}, \mathbf{y})$ on \mathbb{D} such that: G is separately periodic in \mathbf{x} and \mathbf{y} with period Y , G is C^2 in each of the variables \mathbf{x} and \mathbf{y} for $\mathbf{x} \neq \mathbf{y}$, and

$$\begin{aligned} \Delta_{\mathbf{x}} G(\mathbf{x}, \mathbf{y}) &= \sum_{n \in \mathbb{Z}^2} \delta(\mathbf{x} - \mathbf{y} + n) - 1 \quad \text{in } \mathbb{D}, \\ \partial_{n_{\mathbf{x}}} G(\mathbf{x}, \mathbf{y}) &= \frac{|R|}{|\partial R|}, \quad \text{for } \mathbf{x} \in \cup_{n \in \mathbb{Z}^2} (\partial R + n), \end{aligned} \tag{60}$$

where $\delta_{\mathbf{x}}$ is the Dirac delta function located at \mathbf{x} and $|R|$ and $|\partial R|$ are the area and arc-length of R and ∂R , respectively. We can find $G(\mathbf{x}, \mathbf{y})$ as a sum of the periodic free space Green's function F for the Laplacian and a corrector ϕ^* : $G(\mathbf{x}, \mathbf{y}) = F(\mathbf{x}, \mathbf{y}) + \phi^*(\mathbf{x}, \mathbf{y})$, where

$$F(x, y) = - \sum_{n \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{i2\pi n \cdot (x-y)}}{4\pi^2 |n|^2}, \tag{61}$$

$\phi^*(\mathbf{x}, \mathbf{y})$ is periodic in \mathbf{x} and \mathbf{y} with period Y , C^2 in \mathbf{x}, \mathbf{y} , and solves

$$\begin{aligned} \Delta_{\mathbf{x}} \phi^*(\mathbf{x}, \mathbf{y}) &= 0 \quad \text{for } \mathbf{x}, \mathbf{y} \text{ in } \mathbb{D}, \\ \partial_{n_{\mathbf{x}}} \phi^*(\mathbf{x}, \mathbf{y})|_{\partial R}^+ &= -\partial_{n_{\mathbf{x}}} F(\mathbf{x}, \mathbf{y})|_{\partial R}^+ + \frac{|R|}{|\partial R|}, \quad \text{for } \mathbf{x} \text{ on } \partial R, \\ \text{and } \int_{\partial R} \phi^*(\mathbf{x}, \mathbf{y}) \, dS_{\mathbf{x}} &= 0. \end{aligned} \tag{62}$$

Here the subscript ∂R^+ , indicates traces of functions on ∂R taken from the outside of R .

The space of mean zero square integrable functions defined on ∂P is denoted by $L_0^2(\partial P)$ and for ϕ in $L_0^2(\partial P)$ we introduce the single layer potential \tilde{S}_P given by

$$\tilde{S}_P(\phi) = \int_{\partial P} G(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \, dS_{\mathbf{y}}, \tag{63}$$

and the modified single layer potential S_P mapping $L_0^2(\partial P)$ into W_3 given by

$$S_P(\phi) = \tilde{S}_P(\phi) - |\partial P|^{-1} \int_{\partial P} \tilde{S}_P(\phi)(\mathbf{y}) \, dS_{\mathbf{y}}. \tag{64}$$

On restricting $S_P(\phi)(\mathbf{x})$ to \mathbf{x} on ∂P , one readily verifies that S_P is self-adjoint with respect to the $L^2(\partial P)$ inner product and is a bounded one to one linear map from $L_0^2(\partial P)$ onto $H^{1/2}(\partial P) \cap L_0^2(\partial P)$. The inverse denoted by $S_{\partial P}^{-1} : H^{1/2}(\partial P) \cap L_0^2(\partial P) \rightarrow L_0^2(\partial P)$ is linear and continuous. We introduce the trace operator γ mapping W_3 into $H^{1/2}(\partial P) \cap L_0^2(\partial P)$, this map is bounded and onto [2]. Collecting results we have the following

Lemma 1. S_P is a one to one, bounded linear transformation from $L_0^2(\partial P)$ onto W_3 with $S_P^{-1} : W_3 \rightarrow L_0^2(\partial P)$ linear and continuous given by $S_P^{-1} = S_{\partial P}^{-1} \gamma$.

Proof. Given u in W_3 consider its trace on ∂P given by $\gamma u = u|_{\partial P}$ and $u|_{\partial P}$ belongs to $H^{1/2}(\partial P) \cap L_0^2(\partial P)$. For \mathbf{x} in $Y \setminus R$ set $w(\mathbf{x}) = S_P(S_{\partial P}^{-1}(u|_{\partial P}))(\mathbf{x})$. Since w belongs to W_3 the difference $w - u$ also belongs to W_3 . Noting further that the traces of w and u agree on ∂P we conclude that $w - u$ also belongs to $W_1 \oplus W_2$. However, since $W_3 = (W_1 \oplus W_2)^\perp$ it is evident that $w - u = 0$ and we conclude that $S_P^{-1} = S_{\partial P}^{-1}\gamma$. \square

The outward pointing normal derivative of a quantity q on ∂P is denoted by $\frac{\partial q}{\partial n_x}$ and the jump relations satisfied by the single layer potential are given by

$$S_P(\phi)|_{\partial P}^+ = S_P(\phi)|_{\partial P}^-, \quad \text{on } \partial P, \quad \frac{\partial S_P(\phi)}{\partial n_x}\Big|_{\partial P}^\pm = \pm \frac{1}{2}\phi + K_P^*(\phi), \quad \text{on } \partial P \quad (65)$$

where the double layer K_P^* is a bounded linear operator mapping $L_0^2(\partial P)$ into $L_0^2(\partial P)$ defined by

$$K_P^*(\phi) = \int_{\partial P} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial n_x} \phi(\mathbf{y}) \, dS_y, \quad (66)$$

and the subscripts $+$ and $-$ indicate traces from the outside and inside of P , respectively. Here $\frac{\partial G}{\partial n_x}$ is a continuous kernel of order zero and it follows [22, 14] that K_P^* is a compact operator mapping $L_0^2(\partial P)$ into $L_0^2(\partial P)$. Now we identify the transform T defined by (59).

Theorem 7. *The linear map $T : W_3 \rightarrow W_3$ defined by the sesquilinear form (59) is given by*

$$T = S_P K_P^* S_P^{-1}, \quad (67)$$

and is a compact bounded self-adjoint operator on W_3 with eigenvalues lying inside $(-1/2, 1/2)$.

Since T is self-adjoint, bounded, and compact: the spectrum is discrete with only zero as an accumulation point, the eigenspaces associated with distinct eigenvalues are pairwise orthogonal, finite dimensional, and their union together with the null space of T is W_3 . Theorem 1 follows immediately from Theorem 7 noting that the choices $\lambda = 1/2, \lambda = -1/2$ in (16) correspond to ϕ belonging to W_1 and W_2 , respectively.

We now prove Theorem 7.

Proof. *Step I.*

We establish (67). For u, v , consider

$$(S_P K_P^* S_P^{-1}(u), v) = \int_{Y \setminus R} \nabla S_P K_P^* S_P^{-1}(u) \cdot \nabla \bar{v} \, dx. \quad (68)$$

Integration by parts gives

$$(S_P K_P^* S_P^{-1}(u), v) = \int_{\partial P} \left[\frac{\partial (S_P K_P^* S_P^{-1}(u))}{\partial n_x} \right]_+^- \bar{v} \, dS, \quad (69)$$

where $[q]_+^- = q|_{\partial P^-} - q|_{\partial P^+}$. Applying the jump condition (65) we get

$$(S_P K_P^* S_P^{-1}(u), v) = - \int_{\partial P} K_P^* S_P^{-1}(u) \bar{v} \, dS, \tag{70}$$

applying (65) again gives

$$\begin{aligned} K_P^* S_P^{-1}(u) &= \frac{1}{2} \frac{\partial S_P S_P^{-1}(u)}{\partial n_{\mathbf{x}}} \Big|_{\partial P^-} + \frac{1}{2} \frac{\partial S_P S_P^{-1}(u)}{\partial n_{\mathbf{x}}} \Big|_{\partial P^+} \\ &= \frac{1}{2} \frac{\partial u}{\partial n_{\mathbf{x}}} \Big|_{\partial P^-} + \frac{1}{2} \frac{\partial u}{\partial n_{\mathbf{x}}} \Big|_{\partial P^+}. \end{aligned} \tag{71}$$

Step I now follows on substitution of (71) into (70) and integration by parts.

Step II. The remaining properties of the operator T follow directly from the representation $T = S_P K_P^* S_P^{-1}$ and the properties of S_P and K_P^* . \square

We conclude noting that the spectrum of T is the same as the spectrum of K_P^* . This is stated in the following lemma.

Lemma 2. $Tu = \lambda u$ if and only if λ corresponds to an eigenvalue of K_P^* .

Proof. If a pair (λ, u) belonging to $(-1/2, 1/2) \times W_3$ satisfies $Tu = \lambda u$ then $S_P K_P^* S_P^{-1}u = \lambda u$. Multiplication of both sides by S_P^{-1} shows that $S_P^{-1}u$ is an eigenfunction for K_P^* associated with λ . Suppose the pair (λ, w) belongs to $(-1/2, 1/2) \times L_0^2(\partial P)$ and satisfies $K_P^*w = \lambda w$. Since the trace map from W_3 to $H^{1/2}(\partial P) \cap L_0^2(\partial P)$ is onto then there is a u in W_3 for which $w = S_P^{-1}u$ and $K_P^*S_P^{-1}u = \lambda S_P^{-1}u$. Multiplication of this identity by S_P shows that u is an eigenfunction for T associated with λ . \square

6. Existence for Exterior Problems with a Dielectric Permittivity in \mathbb{C}

In this section we establish Theorem 2. Recall that any element u of $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ can be written as

$$u = \mathcal{P}_1 u + \mathcal{P}_2 u + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \mathcal{P}_{\lambda_n} u + d. \tag{72}$$

where d is chosen such that $\int_{Y \setminus R} u \, dy = 0$. Here \mathcal{P}_i are the orthogonal projections onto $W_i, i = 1, 2$ and \mathcal{P}_{λ_n} are the orthogonal projections associated with λ_n in $(-1/2, 1/2)$. The orthogonal decomposition (72) is used in the proof of Theorem 2 given below.

Proof. For u, v in $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ we apply (72) to see that

$$\begin{aligned} B_z(u, v) &= \sum_{i=1}^2 B_z(\mathcal{P}_i u, \mathcal{P}_i v) + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} B_z(\mathcal{P}_{\lambda_n} u, \mathcal{P}_{\lambda_n} v) \\ &= (\mathcal{P}_1 u, v) + z(\mathcal{P}_2 u, v) + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(1 + (z - 1) \left(\frac{1}{2} - \lambda_n \right) \right) (\mathcal{P}_{\lambda_n} u, v) \\ &= (T_z u, v). \end{aligned} \tag{73}$$

From (73) we conclude that

$$T_z = \mathcal{P}_1 + z\mathcal{P}_2 + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(1 + (z - 1) \left(\frac{1}{2} - \lambda_n \right) \right) \mathcal{P}_{\lambda_n}. \tag{74}$$

It is evident from (74) that for $z \neq (\lambda_i + 1/2)/(\lambda_i - 1/2)$ that T_z is a bounded one to one and onto map in $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$. The formula for T_z^{-1} is given by

$$T_z^{-1} = \mathcal{P}_1 + z^{-1}\mathcal{P}_2 + \sum_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \left(1 + (z - 1) \left(\frac{1}{2} - \lambda_n \right) \right)^{-1} \mathcal{P}_{\lambda_n}. \tag{75}$$

Taking conjugates on both sides of (73) gives $\overline{B_z(u, v)} = (v, T_z u)$ and choosing $u = T_z^{-1}q$ for q in $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ delivers the identity

$$\overline{B_z(T_z^{-1}q, v)} = (v, q). \tag{76}$$

To complete the proof consider any linear functional F in $[H_{\text{per}}^1(Y \setminus R)/\mathbb{C}]^*$ with $F(v) = 0$ for $v = \text{const}$. Applying the Reisz representation theorem shows that there exists a unique solution u in $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ of

$$B_z(u, v) = \overline{F(v)}, \quad \text{for all } v \text{ in } H_{\text{per}}^1(Y \setminus R)/\mathbb{C}, \tag{77}$$

and Theorem 2 is proved. \square

7. Convergence of the Power Series

In this section we show that the series (54), (57) identified in Theorems 4 and 5 are convergent. Here the convergence radius R depends upon the branch of the quasistatic dispersion relation $\xi_0^n = \xi_0^n(\tau, \hat{\kappa})$. The methodology applied here uses generating functions and follows the approach presented in [24].

Step I. A priori estimates.

We establish a priori bounds for the solutions of (34) and (47). Before proceeding recall the Friderich’s inequality and trace estimate satisfied by elements ψ of $H_{\text{per}}^1(Y \setminus R)/\mathbb{C}$ given by

$$\|\psi\|_{Y \setminus R} = \left(\int_{Y \setminus R} |\nabla \psi|^2 + |\psi|^2 \, dy \right)^{1/2} \leq \Omega \|\psi\| \tag{78}$$

and

$$\|\psi\|_{H^{1/2}(\partial R)} \leq A \|\psi\| \tag{79}$$

where Ω and A are positive constants depending on R and $\|\psi\|$ is the norm on $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ defined by (9).

The solutions $\tilde{\psi}$ of (34) solve boundary value problems of the following generic form. Given $\psi \in H^1_{\text{per}}(Y \setminus R)$, $G \in L^2(R)$ and $F \in [L^2(R)]^2$ the function $\tilde{\psi}$ is the $H^1(R)$ solution of

$$\begin{cases} \int_R (\nabla \tilde{\psi} \cdot \nabla \bar{v} - \xi_0 \tilde{\psi} \bar{v}) = \int_R G \bar{v} + \int_R F \cdot \nabla \bar{v} \quad \forall v \in H^1_0(R) \\ \tilde{\psi}|_{\partial R^-} = \psi|_{\partial R^+}. \end{cases} \tag{80}$$

Decompose $\tilde{\psi}$ according to $\tilde{\psi} = \psi_0 + \psi_1 + \psi_2$ such that

$$\begin{cases} \int_R (\nabla \psi_0 \cdot \nabla \bar{v} - \xi_0 \psi_0 \bar{v}) = \int_R [G + (1 + \xi_0)\psi_1 + (1 + \xi_0)\psi_2] \bar{v} \quad \forall v \in H^1_0(R) \\ \psi_0|_{\partial R^-} = 0, \end{cases} \tag{81}$$

$$\begin{cases} \int_R (\nabla \psi_1 \cdot \nabla \bar{v} + \psi_1 \bar{v}) = 0 \quad \forall v \in H^1_0(R) \\ \psi_1|_{\partial R^-} = \psi|_{\partial R^+}, \end{cases} \tag{82}$$

and

$$\begin{cases} \int_R (\nabla \psi_2 \cdot \nabla \bar{v} + \psi_2 \bar{v}) = \int_R F \cdot \nabla \bar{v} \quad \forall v \in H^1_0(R) \\ \psi_2|_{\partial R^-} = 0. \end{cases} \tag{83}$$

Let $J = G + (1 + \xi_0)\psi_1 + (1 + \xi_0)\psi_2$ denote the right hand side for (81). Expanding ψ_0 and J with respect to the complete orthonormal set of Dirichlet eigenfunctions $\{\phi_j\}_{j=1}^\infty, \{v_j\}_{j=1}^\infty$ gives

$$\|\psi_0\|_{H^1(R)}^2 = \sum_{j=1}^\infty (1 + v_j) \left| \frac{\langle J \psi_j \rangle_R}{(\xi_0 - v_j)} \right|^2 \leq C_{1,n}^2 \|J\|_{L^2(R)}^2, \tag{84}$$

where

$$C_{1,n} = \max_{\xi_0 \in I_n} \left\{ \max_j \left\{ \frac{(1 + v_j)^{1/2}}{|\xi_0 - v_j|} \right\} \right\}. \tag{85}$$

Collecting results gives

$$\begin{aligned} \|\psi_1\|_{H^1(R)} &= \|\psi_2\|_{H^{1/2}(\partial R)} \leq A \|\psi\| \\ \|\psi_2\|_{H^1(R)} &\leq \|F\|_{L^2(R)} \\ \|\psi_0\|_{H^1(R)} &\leq C_{1,n} \|G + (1 + \xi)\psi_1 + (1 + \xi)\psi_2\|_{L^2(R)} \end{aligned} \tag{86}$$

and

$$\|\tilde{\psi}\|_{H^1(R)} \leq B_n (\|G\|_{L^2(R)} + \|F\|_{L^2(R)} + \|\psi\|), \tag{87}$$

where

$$B_n = \max_{\xi_0 \in I_n} \{C_{1,n}, C_{1,n}(1 + \xi_0), A(C_{1,n}(1 + \xi_0) + 1)\}. \tag{88}$$

Now we estimate the field ψ_* . The explicit expression (38) for ψ_* gives

$$\|\psi_*\|_{H^1(R)}^2 = \sum_{n=1}^{\infty} \frac{(1 + \mu_n)\mu_n^2(\phi_n)_R^2}{(\mu_n - \xi_0)^4}, \tag{89}$$

and

$$\|\psi_*\|_{H^1(R)} \leq L_n. \tag{90}$$

where L_n is the maximum of the right hand side of (89) for $\xi_0(\tau, \hat{\kappa}) \in I_n$.

Next we provide an upper bound for the solutions of ψ'_m of (47). We may continuously extend elements v in $H^1_{\text{per}}(Y \setminus R)/\mathbb{C}$ onto R as $H^1_{\text{per}}(Y)$ functions such that the extension v^{ext} satisfies

$$\|v^{\text{ext}}\|_Y \leq C\|v\|. \tag{91}$$

In what follows we continue to denote these extensions by v and (47) takes the equivalent form (40). Application of Hölders inequality to the right hand side of (40) gives

$$\begin{aligned} \tau^2|B_z(\psi'_m, v)| \\ \leq C(\|F_1\|_{L^2(Y \setminus R)} + \|G_1\|_{L^2(Y \setminus R)} + \|F_2\|_{L^2(R)} + \|G_2\|_{L^2(R)})\|v\|. \end{aligned} \tag{92}$$

It is evident that $B_z(\psi'_m, v) = (T_z\psi'_m, v)$ and choosing $v = T_z^{-1}\psi'_m$ gives $B_z(\psi'_m, v) = \|\psi'_m\|^2$, $\|v\| \leq G_z\|\psi'_m\|$ and delivers the estimate

$$\begin{aligned} \tau^2\|\psi'_m\|^2 &= \tau^2|B_z(\psi'_m, v)| \\ &\leq C \times G_z(\|F_1\|_{L^2(Y \setminus R)} + \|G_1\|_{L^2(Y \setminus R)} \\ &\quad + \|F_2\|_{L^2(R)} + \|G_2\|_{L^2(R)})\|\psi'_m\|, \end{aligned} \tag{93}$$

with

$$G_z = \left(\max\{1, |z|^{-1}, \sup_{-\frac{1}{2} < \lambda_n < \frac{1}{2}} \{|1 + (z - 1)(1/2 - \lambda_n)|^{-1}\}\} \right). \tag{94}$$

For $z = \varepsilon_p(\xi_0)$ we maximize (94) for $\xi_0 \in I_n$ to obtain

$$\tau^2\|\psi'_m\| \leq G_n(\|F_1\|_{L^2(Y \setminus R)} + \|G_1\|_{L^2(Y \setminus R)} + \|F_2\|_{L^2(R)} + \|G_2\|_{L^2(R)}). \tag{95}$$

Applying Freidrich's inequality we arrive at the desired upper bound given by

$$\tau^2\|\psi'_m\|_{Y \setminus R} \leq E_n(\|F_1\|_{L^2(Y \setminus R)} + \|G_1\|_{L^2(Y \setminus R)} + \|F_2\|_{L^2(R)} + \|G_2\|_{L^2(R)}), \tag{96}$$

where E_n denotes a generic constant depending only on I_n .

The a priori estimate for $\hat{\psi}$ follows from the representation formula (50) and

$$\|\hat{\psi}\|_{Y \setminus R} \leq H_n, \tag{97}$$

where the constant H_n depends only on I_n .

Step II. System of inequalities.

We apply the a priori estimates developed in Step I to the system of Equations (34), (47), and solvability conditions (51) derived in Section 3. From (40) and (96), it follows that

$$\begin{aligned}
 \tau^2 \|\psi'_m\|_{Y \setminus R} \leq E_n \left[\tau^2 \left(\sum_{l=1}^{m-2} |\xi_l| \|\psi_{m-l}\|_{Y \setminus R} \right. \right. \\
 + \sum_{l=1}^{m-2} |\xi_l| \|\psi_{m-1-l}\|_{Y \setminus R} + \sum_{l=1}^{m-2} |\xi_l| \|\psi_{m-2-l}\|_{Y \setminus R} \\
 + \|\psi_{m-1}\|_{Y \setminus R} + \|\psi_{m-2}\|_{Y \setminus R} \left. \right) + \sum_{l=0}^{m-2} |\xi_l| \|\psi_{m-2-l}\|_{H^1(R)} \\
 + \sum_{l=0}^{m-3} |\xi_l| \|\psi_{m-3-l}\|_{H^1(R)} \\
 + \sum_{l=1}^{m-4} |\xi_l| \|\psi_{m-4-l}\|_{H^1(R)} + \|\psi_{m-2}\|_{H^1(R)} \\
 + \|\psi_{m-3}\|_{H^1(R)} + \|\psi_{m-4}\|_{H^1(R)} \\
 + \sum_{l=0}^{m-2} \sum_{n=0}^l |\xi_{m-2-l}| |\xi_n| (\|\psi_{l-n}\|_{Y \setminus R} + \|\psi_{l-n}\|_{H^1(R)}) \\
 + \sum_{l=0}^{m-2} |\xi_l| (\|\psi_{m-2-1}\|_{Y \setminus R} + \|\psi_{m-2-1}\|_{H^1(R)}) \left. \right]. \tag{98}
 \end{aligned}$$

From (34) and (87), it follows that

$$\begin{aligned}
 \|\tilde{\psi}_m\|_{H^1(R)} \leq B_n \left[\sum_{l=1}^{m-1} \xi_l \|\psi_{m-l}\|_{H^1(R)} \right. \\
 + \sum_{l=0}^{m-1} \xi_l \|\psi_{m-1-l}\|_{H^1(R)} \\
 + \sum_{l=0}^{m-2} \xi_l \|\psi_{m-2-l}\|_{H^1(R)} + \|\psi_{m-1}\|_{H^1(R)} + \|\psi_{m-2}\|_{H^1(R)} \\
 + \sum_{l=1}^{m-1} \sum_{n=0}^l \xi_{m-l} \xi_n \|\psi_{l-n}\|_{H^1(R)} + \sum_{l=1}^{m-1} \xi_l \|\psi_{m-l}\|_{H^1(R)} + \|\psi_m\|_{Y \setminus R} \left. \right] \tag{99}
 \end{aligned}$$

The solvability constant \mathcal{G} in (51) is bounded away from zero and ∞ for $\xi_0 \in I_n$. Put $m - 2 \mapsto m$ in (51) to obtain the inequality for $|\xi_m|$. The inequality is in terms

of a constant C_n depending only on I_n and is given by

$$\begin{aligned}
 |\xi_m| \leq C_n \left[\tau^2 \left(\sum_{l=0}^{m-1} |\xi_l| \|\psi_{m+1-l}\|_{Y \setminus R} \right. \right. \\
 + \sum_{l=0}^{m-1} |\xi_l| \|\psi_{m-l}\|_{Y \setminus R} + 2\|\psi'_{m+1}\|_{Y \setminus R} + \|\psi_m\|_{Y \setminus R} \Big) \\
 + \sum_{l=0}^{m-1} |\xi_l| \|\psi_{m-1-l}\|_{H^1(R)} + \sum_{l=0}^{m-2} |\xi_l| \|\psi_{m-2-l}\|_{H^1(R)} \\
 + \sum_{l=1}^{m-1} \sum_{n=0}^l |\xi_{m-l}| |\xi_n| (\|\psi_{l-n}\|_{H^1(R)} + \|\psi_{l-n}\|_{Y \setminus R}) \\
 + 2 \sum_{l=1}^{m-1} |\xi_l| (\|\psi_{m-l}\|_{H^1(R)} + \|\psi_{m-l}\|_{Y \setminus R}) \\
 \left. + (\|\tilde{\psi}_m\|_{H^1(R)} + \|\psi_m\|_{Y \setminus R}) + (\|\tilde{\psi}_m\|_{H^1(R)} + \|\psi_m\|_{Y \setminus R}) \right] \quad (100)
 \end{aligned}$$

The decompositions (33), (39) and bounds (90), (97) give

$$\|\psi_m\|_{H^1(R)} \leq \|\tilde{\psi}_m\|_{H^1(R)} + L_n |\xi_m| \quad \text{and} \quad (101)$$

$$\|\psi_m\|_{Y \setminus R} \leq \|\psi'_m\|_{Y \setminus R} + H_n |\xi_{m-1}|. \quad (102)$$

Here the upper bounds (98)–(102) hold for all values of τ for which the branch of the quasistatic dispersion relation ξ_n^0 lies in interval I_n .

Step III. Majorizing sequence and analyticity of generating functions.

We simplify the exposition by introducing

$$\bar{p}_m = \|\tau^m \psi_m\|_{Y \setminus R} \quad (103)$$

$$p_m = \|\tau^m \psi_m\|_{H^1(R)} \quad (104)$$

$$\tilde{p}_m = \|\tau^m \tilde{\psi}_m\|_{H^1(R)} \quad (105)$$

$$p'_m = \|\tau^m \psi'_m\|_{Y \setminus R} \quad (106)$$

$$s_m = |\tau^m \xi_m|, \quad (107)$$

and show that the corresponding series $\sum \rho^m \bar{p}_m$, $\sum \rho^m p_m$, $\sum \rho^m \tilde{p}_m$, $\sum \rho^m p'_m$, and $\sum \rho^m s_m$ converge. This is sufficient to establish summability and convergence for the series (57) and (54). Writing (98)–(102) in terms of the new notation gives

the following system of inequalities. For $m \geq 2$ we have:

$$\begin{aligned}
 p'_m \leq E_n & \left[\sum_{l=1}^{m-2} s_l \bar{p}_{m-l} \right. \\
 & + \tau \sum_{l=0}^{m-2} s_l \bar{p}_{m-1-l} + \tau^2 \sum_{l=0}^{m-2} s_l \bar{p}_{m-2-l} \\
 & + \tau \bar{p}_{m-1} + \tau^2 \bar{p}_{m-2} + \sum_{l=0}^{m-2} s_l p_{m-2-l} + \tau \sum_{l=0}^{m-3} s_l p_{m-3-l} + \tau^2 \sum_{l=0}^{m-4} s_l p_{m-4-l} \\
 & + p_{m-2} + \tau p_{m-3} + \tau^2 p_{m-4} + \sum_{l=0}^{m-2} \sum_{n=0}^l s_{m-2-l} s_n (\bar{p}_{l-n} + p_{l-n}) \\
 & \left. + \sum_{l=0}^{m-2} s_l (\bar{p}_{m-2-l} + p_{m-2-l}) \right], \tag{108}
 \end{aligned}$$

and for $m \geq 1$,

$$\begin{aligned}
 \tilde{p}_m \leq B_n & \left[\sum_{l=1}^{m-1} s_l p_{m-l} \right. \\
 & + \tau \sum_{l=0}^{m-1} s_l p_{m-1-l} + \tau^2 \sum_{l=0}^{m-2} s_l p_{m-2-l} + \tau p_{m-1} \\
 & \left. + \tau^2 p_{m-2} + \sum_{l=1}^{m-1} \sum_{n=0}^l s_{m-l} s_n p_{l-n} + \sum_{l=1}^{m-1} s_l p_{m-l} + \bar{p}_m \right]. \tag{109}
 \end{aligned}$$

The bounds (100)–(102) yield the following bounds for $m \geq 1$:

$$\begin{aligned}
 s_m \leq C_n & \left[2\tau p'_{m+1} \right. \\
 & + \tau \sum_{l=1}^{m-1} s_l \bar{p}_{m+1-l} + \tau^2 \sum_{l=0}^{m-1} s_l \bar{p}_{m-l} \\
 & + \tau \sum_{l=0}^{m-1} s_l p_{m-1-l} + \tau^2 \sum_{l=0}^{m-2} s_l p_{m-2-l} + \sum_{l=0}^{m-3} \sum_{n=0}^l s_{m-l} s_n (\bar{p}_{l-n} + p_{l-n}) \\
 & \left. + 2 \sum_{l=1}^m s_l (\bar{p}_{m-l} + p_{m-l}) + 2\tilde{p}_m + (2 + \tau^2) \bar{p}_m \right], \tag{110}
 \end{aligned}$$

$$p_m \leq \tilde{p}_m + L_n s_m, \tag{111}$$

$$\bar{p}_m \leq p'_m + \tau H_n s_{m-1}. \tag{112}$$

The inequalities presented above hold for all τ for which ξ_0^n belongs to I_n . In what follows it is convenient to increase the upper bounds by choosing the maximum

value of τ for ξ_0^n in I_n and absorbing this value into the constants B_n, C_n, E_n . Additionally any constant factors other than unity that multiply terms in these upper bounds are absorbed into these constants. With these choices the system is written:

$$\begin{aligned}
 p'_m \leq & E_n \left[\sum_{l=1}^{m-2} s_l \bar{p}_{m-l} \right. \\
 & + \sum_{l=0}^{m-2} s_l \bar{p}_{m-1-l} + \sum_{l=0}^{m-2} s_l \bar{p}_{m-2-l} \\
 & + \bar{p}_{m-1} + \bar{p}_{m-2} + \sum_{l=0}^{m-2} s_l p_{m-2-l} + \sum_{l=0}^{m-3} s_l p_{m-3-l} + \sum_{l=0}^{m-4} s_l p_{m-4-l} \\
 & + p_{m-2} + p_{m-3} + p_{m-4} + \sum_{l=0}^{m-2} \sum_{n=0}^l s_{m-2-l} s_n (\bar{p}_{l-n} + p_{l-n}) \\
 & \left. + \sum_{l=0}^{m-2} s_l (\bar{p}_{m-2-l} + p_{m-2-l}) \right], \tag{113}
 \end{aligned}$$

$$\begin{aligned}
 \tilde{p}_m \leq & B_n \left[\sum_{l=1}^{m-1} s_l p_{m-l} \right. \\
 & + \sum_{l=0}^{m-1} s_l p_{m-1-l} + \sum_{l=0}^{m-2} s_l p_{m-2-l} + p_{m-1} + p_{m-2} \\
 & \left. + \sum_{l=1}^{m-1} \sum_{n=0}^l s_{m-l} s_n p_{l-n} + \sum_{l=1}^{m-1} s_l p_{m-l} + \bar{p}_m \right], \tag{114}
 \end{aligned}$$

$$\begin{aligned}
 s_m \leq & C_n \left[p'_{m+1} + \sum_{l=1}^{m-1} s_l \bar{p}_{m+1-l} \right. \\
 & + \sum_{l=0}^{m-1} s_l \bar{p}_{m-l} \\
 & + \sum_{l=0}^{m-1} s_l p_{m-1-l} + \sum_{l=0}^{m-2} s_l p_{m-2-l} + \sum_{l=0}^{m-3} \sum_{n=0}^l s_{m-l} s_n (\bar{p}_{l-n} + p_{l-n}) \\
 & \left. + \sum_{l=1}^m s_l (\bar{p}_{m-l} + p_{m-l}) + \tilde{p}_m + \bar{p}_m \right], \tag{115}
 \end{aligned}$$

$$p_m \leq C_4 (\tilde{p}_m + s_m), \tag{116}$$

$$\bar{p}_m \leq C_5 (p'_m + s_{m-1}), \tag{117}$$

where C_4, C_5 are positive constants depending only on I_n and $m \geq 1$.

We now introduce a majorizing sequence $\{(a_m, b_m, c_m, d_m, e_m)\}_{m=0}^\infty$ for which

$$\begin{aligned}
 \sum_{n=0}^\infty a_n \rho^n &= \rho a(\rho) + \bar{p}_0 \geq \sum_{n=0}^\infty \bar{p}_n \rho^n \\
 \sum_{n=1}^\infty b_n \rho^n &= b(\rho) \geq \sum_{n=1}^\infty \tilde{p}_n \rho^n \\
 \sum_{n=0}^\infty c_n \rho^n &= c(\rho) \geq \sum_{n=0}^\infty s_n \rho^n \\
 \sum_{n=0}^\infty d_n \rho^n &= d(\rho) \geq \sum_{n=0}^\infty p_n \rho^n \\
 \sum_{n=2}^\infty e_n \rho^n &= \rho e(\rho) \geq \sum_{n=2}^\infty p'_n \rho^n,
 \end{aligned} \tag{118}$$

to show that the generating functions $a(\rho), \dots, e(\rho)$ are analytic in a neighborhood of the origin $\rho = 0$.

The majorizing sequence is chosen so that the system of inequalities (113)–(117) holds with equality for $a_m = \bar{p}_m, b_m = \tilde{p}_m, c_m = s_m, p_m = d_m,$ and $e_m = p'_m$. Indeed, for this choice one observes that $\bar{p}_m \leq a_m, \tilde{p}_m \leq b_m, s_m \leq c_m, p_m \leq d_m,$ and $p'_m \leq e_m$. Enforcing equality in (113)–(117), multiplying each by the appropriate power of ρ and summation delivers the equivalent system for the generating functions $a(\rho), b(\rho), c(\rho), d(\rho), e(\rho)$ given by:

$$A_i(a, b, c, d, e, \rho) = 0, \quad i = 1, \dots, 5, \tag{119}$$

where:

$$A_1(a, b, c, d, e, \rho) = \bar{p}_1 - a + C_5(e + (c - s_0)), \tag{120}$$

$$\begin{aligned}
 A_2(a, b, c, d, e, \rho) &= -b + B_n[(c - s_0)(d - p_0) \\
 &\quad + (cd + d)(\rho + \rho^2) + c^2d - s_0^2p_0 - \rho s_0s_1(p_0 + p_1) \\
 &\quad - s_0p_0(c - s_0) - s_0cd + s_0(\rho(s_0p_1 + s_1p_0) \\
 &\quad + s_0p_0) + (c - s_0)(d - p_0) + \rho a]
 \end{aligned} \tag{121}$$

$$\begin{aligned}
 A_3(a, b, c, d, e, \rho) &= -(c - s_0) + C_n[(a - \bar{p}_1)(c - s_0) + e + \rho ac + \rho(a + cd) \\
 &\quad + \rho^2cd + c^2(\rho a + \bar{p}_0) - s_0\bar{p}_0c - s_0((\rho a \\
 &\quad + \bar{p}_0)c - s_0\bar{p}_0) + (c - s_0)(d - p_0) + b \\
 &\quad - \rho(\rho a + \bar{p}_0)cs_1 - \rho^2s_2(\rho a \\
 &\quad + \bar{p}_0)c + c^2d - s_0p_0c - s_0(cd - s_0p_0) \\
 &\quad - \rho cd(s_1 + \rho s_2)]
 \end{aligned} \tag{122}$$

$$A_4(a, b, c, d, e, \rho) = (d - p_0) + C_4(b + (c - s_0)). \tag{123}$$

$$\begin{aligned}
 A_5(a, b, c, d, e, \rho) &= -e + E_n[(a - \bar{p}_1)(c - s_0) \\
 &\quad + \rho ac + \rho(\rho a + \bar{p}_0)c + \rho a + \rho(\rho a + \bar{p}_0) \\
 &\quad + (c + 1)d(\rho + \rho^2 + \rho^3) + \rho c(c + 1)(\rho a + \bar{p}_0 + d)]
 \end{aligned} \tag{124}$$

One can check A_1, A_2, A_3, A_4 and A_5 vanish for $a = \bar{p}_1 = \|\psi_1\|_{Y \setminus R}, b = 0, c = s_0 = |\xi_0^n|, d = p_0 = \|\psi_0\|_{H^1(R)}, e = 0,$ and $\rho = 0$. Moreover, the determinant of the Jacobian matrix with respect to the first five variables at this point, that is, $(a, b, c, d, e, \rho) = (\bar{p}_1, 0, s_0, p_0, 0, 0)$, is

$$\det \frac{\partial(A_1, A_2, A_3, A_4, A_5)}{\partial(a, b, c, d, e)}(\bar{p}_1, 0, s_0, p_0, 0, 0) = 1. \tag{125}$$

Thus, the implicit function theorem for functions of several complex variables guarantees that the generating functions a, b, c, d, e are analytic within a neighborhood of $\rho = 0$ and we conclude that the series on the right hand side of the inequalities (118) are convergent. Here $\bar{p}_0 = \|\psi_0\|_{Y \setminus R} = |Y \setminus R|^{1/2}$ and the initial data p_0, \bar{p}_1 are bounded functions of ξ_0^n on I_n . Choosing the maximum values of p_0, \bar{p}_1, ξ_0^n , as ξ_0^n varies over I_n delivers a majorizing sequence with radius of convergence R depending only on I_n . It now follows that the series (54) converges and that (57) is summable for $0 \leq \rho < R$.

8. Power Series Solution of Cell Problem

In this section we show that the functions defined by the power series solve (11). For ξ_0^n in I_n recall the series (12), (13)

$$u = \sum_{m=0}^{\infty} \eta^m u_m \tag{126}$$

$$\xi = \sum_{m=0}^{\infty} \eta^m \xi_m \tag{127}$$

with $u_m = i^m \underline{u}_0 \psi_m$ and $\eta = \rho \tau$ converges for $0 < \rho < R$. For $v \in H^1_{\text{per}}(Y)$ and $0 \leq \rho < R$. We define

$$\begin{aligned} a^\eta(v) := & \int_H \tau^2 \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) (\nabla_y + i\eta \hat{k}) u \cdot \overline{(\nabla_y + i\eta \hat{k}) v} \\ & + \int_P \tau^2 \xi (\nabla_y + i\eta \hat{k}) u \cdot \overline{(\nabla_y + i\eta \hat{k}) v} \\ & + \int_R \eta^2 \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) (\nabla_y + i\eta \hat{k}) u \cdot \overline{(\nabla_y + i\eta \hat{k}) v} - \int_Y \eta^2 \xi \left(\xi - \varepsilon_r \frac{\omega_p^2}{c^2} \right) u \bar{v}. \end{aligned} \tag{128}$$

First, observe that $a^\eta(v)$ has a convergent power series in η that is obtained by inserting u and ξ into the above expression and expanding in powers of η . The coefficients of this expansion are exactly the left hand side of Equation (15). Since ψ_m satisfies (15), all coefficients of the power series expansion of $a^\eta(v)$ are zero and we conclude that $a^\eta(v) = 0$ for all $v \in H^1_{\text{per}}(Y), 0 \leq \rho < R$. Thus, we conclude that the pair (u, ξ) satisfies (11).

Next we record the following property of the sequence $\{\xi_m\}$ that follows directly from the variational formulation of the problem and the power series solution.

Theorem 8. *The functions ξ_m are real for all m .*

Proof. Setting $v = u$ in (11) delivers a quadratic equation for ξ . Since the discriminant greater than zero, we conclude that ξ is real and it follows that ξ_m , $m = 1, \dots$, are real. \square

With these results in hand, Theorems 4 and 5 now follow from the convergence established in Section 7 together with Theorem 3.

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References

1. ALU, A., ENGHETA, N.: Dynamical theory of artificial optical magnetism produced by rings of plasmonic nanoparticles. *Phys. Rev. B* **78**, 085112 (2008)
2. ADAMS, D.: *Sobolev Spaces*. Elsevier, Waltham, 2003
3. AMMARI, H., KANG, H., LEE, H.: Asymptotic analysis of high-contrast phononic crystals and a criterion for the band-gap opening. *Arch. Ration. Mech. Anal.* **193**, 679–714 (2009)
4. AMMARI, H., KANG, H., SOUSSI, S., ZRIBI, H.: Layer potential techniques in spectral analysis. Part II: sensitivity analysis of spectral properties of high contrast band-gap materials. *SIAM Multiscale Model. Simul.* **5**, 646–663 (2006)
5. BERGMAN, D.J.: The dielectric constant of a simple cubic array of identical spheres. *J. Phys. C* **12**, 4947–4960 (1979)
6. BOHREN, C.F., HUFFMAN, D.H.: *Absorption and Scattering of Light by Small Particles*. Wiley, Weinheim, 2004
7. BOUCHITTÉ, G., BOUREL, C.: Homogenization of finite metallic fibers and 3D-effective permittivity tensor. *Proc. SPIE* **7029**, 702914 (2008). doi:10.1117/12.794935
8. BOUCHITTÉ, G., SCHWEIZER, B.: Homogenization of Maxwell's equations in a split ring geometry. *SIAM Multiscale Model. Simul.* **8**(3), 717–750 (2010)
9. BOUCHITTÉ, G., FELBACQ, D.: Negative refraction in periodic and random photonic crystals. *New J. Phys.* **7**, 159 (2005)
10. BOUCHITTÉ, G., FELBACQ, D.: Homogenization near resonances and artificial magnetism from dielectrics. *C. R. Acad. Sci. Paris I* **399**(5), 377–382 (2004)
11. CHEN, Y., LIPTON, R.: Tunable double negative band structure from non-magnetic coated rods. *New J. Phys.* **12**, 083010 (2010)
12. CHERDANTSEV, M.: Spectral convergence for high-contrast elliptic periodic problems with a defect via homogenization. *Mathematika* **55**, 29–57 (2009)
13. CHERN, R.L., FELBACQ, D.: Artificial magnetism and anticrossing interaction in photonic crystals and split-ring structures. *Phys. Rev. B* **79**(7), 075118 (2009)
14. COLTON, D., KRESS, R.: *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer, New York, 1998
15. DOLLING, G., ENRICH, C., WEGENER, M., SOUKOULIS, C.M., LINDEN, S.: Low-loss negative-index metamaterial at telecommunication wavelengths. *Opt. Lett.* **31**, 1800–1802 (2006)
16. FELBACQ, D., BOUCHITTE, G.: Homogenization of wire mesh photonic crystals embedded in a medium with a negative permeability. *Phys. Rev. Lett.* **94**(18), 183902 (2005)
17. FELBACQ, D., GUIZAL, B., BOUCHITTE, G., BOUREL, G.: Resonant homogenization of a dielectric metamaterial. *Microw. Opt. Technol. Lett.* **51**(11), 2695–2701 (2009)

18. FARHAT, M., GUENNEAU, ENOCH, S., MOVCHAN, A.B.: Negative refraction, surface modes, and superlensing effect via homogenization near resonances for a finite array of split-ring resonators. *Phys. Rev. E* **80**(4), 046309 (2009)
19. FIGOTIN, A., KUCHMENT, P.: Band-gap structure of the spectrum of periodic dielectric and acoustic media. I. Scalar model. *SIAM J. Appl. Math.* **56**(1), 68–88 (1996)
20. FIGOTIN, A., KUCHMENT, P.: Band-gap structure of the spectrum of periodic dielectric and acoustic media. II. 2D Photonic crystals. *SIAM J. Appl. Math.* **56**(6), 1561–1620 (1996)
21. FIGOTIN, A., KUCHMENT, P.: Spectral properties of classical waves in high contrast periodic media. *SIAM J. Appl. Math.* **58**(2), 683–702 (1998)
22. FOLLAND, G.: *Introduction to Partial Differential Equations*. Princeton University Press, Princeton, 1995
23. FORTES, S.P., LIPTON, R.P., SHIPMAN, S.P.: Sub-wavelength plasmonic crystals: dispersion relations and effective properties. *Proc. R. Soc. Lond. Ser. A Mat.* (2009). doi:[10.1098/rspa.2009.0542](https://doi.org/10.1098/rspa.2009.0542)
24. FORTES, S.P., LIPTON, R.P., SHIPMAN, S.P.: Convergent power series for fields in positive or negative high-contrast periodic media. *Commun. Partial Differ. Equ.* **36**(6), 1016–1043 (2011)
25. HEMPEL, R., LIENAU, K.: Spectral properties of periodic media in the large coupling limit. *Commun. Partial Differ. Equ.* **25**, 1445–1470 (2000)
26. HUANGFU, J., RAN, L., CHEN, H., ZHANG, X., CHEN, K., GRZEGORCZYK, T.M., KONG, J.A.: Experimental confirmation of negative refractive index of a metamaterial composed of Ω -like metallic patterns. *Appl. Phys. Lett.* **84**, 1537 (2004)
27. HUANG, K.C., POVINELLI, M.L., JOANNOPOULOS, J.D.: Negative effective permeability in polaritonic photonic crystals. *Appl. Phys. Lett.* **85**(4), 543 (2004)
28. JELINEK, L., MARQUES, R.: Artificial magnetism and left-handed media from dielectric rings and rods. *J. Phys. Condens. Matter* **22**(2), 025902 (2010)
29. KAMOTSKI, V., MATTHIES, M., SMYSHLYAEV, V.: Exponential homogenization of linear second order elliptic PDEs with periodic coefficients. *SIAM J. Math. Anal.* **38**(5), 1565–1587 (2007)
30. KOHN, R., SHIPMAN, S.: Magnetism and homogenization of micro-resonators. *SIAM Multiscale Model. Simul.* **7**(1), 62–92 (2008)
31. LEPETIT, T., AKMANSOY, E.: Magnetism in high-contrast dielectric photonic crystals. *Microw. Opt. Technol. Lett.* **50**(4), 909–911 (2008)
32. MCPHEDRAN, R.C., MILTON, G.W.: Bounds and exact theories for transport properties of inhomogeneous media. *Physics A* **26**, 207–220 (1981)
33. MILTON, G.W.: Realizability of metamaterials with prescribed electric permittivity and magnetic permeability tensors. *New J. Phys.* **12**(3), 033035 (2010)
34. MILTON, G.W.: *The Theory of Composites*. Cambridge University Press, Cambridge, 2002
35. O'BRIEN, S., PENDRY, J.B.: Photonic band-gap effects and magnetic activity in dielectric composites. *J. Phys. Condens. Matter* **14**, 4035–4044 (2002)
36. PENDRY, J., HOLDEN, A., ROBBINS, D., STEWART, W.: Magnetism from conductors and enhanced nonlinear phenomena. *IEEE Trans. Microw. Theory Tech.* **47**(11), 2075–2084 (1999)
37. PENDRY, J., HOLDEN, A., ROBBINS, D., STEWART, W.: Low frequency plasmons in thin-wire structures. *J. Phys. Condens. Matter* **10**, 4785–4809 (1998)
38. PENG, L., RAN, L., CHEN, H., ZHANG, H., KONG, L.A., GRZEGORCZYK, T.M.: Experimental observation of left-handed behavior in an array of standard dielectric resonators. *Phys. Rev. Lett.* **98**, 157403 (2007)
39. SHVETS, G., URZHUMOV, Y.: Engineering the electromagnetic properties of periodic nanostructures using electrostatic resonances. *Phys. Rev. Lett.* **93**(24), 243902-1-4 (2004)

40. SHIPMAN, S.: Power series for waves in micro-resonator arrays. *Proceedings of the 13th International Conference on Mathematical Methods in Electrodynamics Theory, MMET 10*, Kyiv, Ukraine, September 6–8, IEEE, 2010
41. VESELAGO, V.G.: The electrodynamics of substances with simultaneously negative values of ϵ and μ . *Sov. Phys. Usp.* **10**, 509 (1968)
42. VYNCK, K., FELBACQ, D., CENTENO, E., CABUZ, A.I., CASSAGNE, D., GUIZAL, B.: All-dielectric rod-type metamaterials at optical frequencies. *Phys. Rev. Lett.* **102**, 133901 (2009)
43. SERVICE, R.F.: Next Wave of metamaterials hopes to fuel the revolution. *Science* **327**(5962), 138–139 (2010)
44. SHALAEV, V.: Optical negative-index metamaterials. *Nat. Photonics* **1**, 41–48 (2007)
45. SHALAEV, V.M., CAI, W., CHETTIAR, U.K., YUAN, H.K., SARYCHEV, A.K., DRACHEV, V.P., KILDISHEV, A.V.: Negative index of refraction in optical metamaterials. *Opt. Lett.* **30**(24), 3356–3358 (2005)
46. SMITH, D., PADILLA, W., VIER, D., NEMAT-NASSER, S., SCHULTZ, S.: Composite medium with simultaneously negative permeability and permittivity. *Phys. Rev. Lett.* **84**(18) 4184–4187 (2000)
47. WHEELER, M.S., AITCHISON, J.S., MOJAHEDI, M.: Coated non-magnetic spheres with a negative index of refraction at infrared frequencies. *Phys. Rev. B* **73**, 045105 (2006)
48. YANNOPAPAS, V.: Negative refractive index in the near-UV from Au-coated CuCl nanoparticle superlattices. *Phys. Stat. Sol. (RRL)* **1**(5), 208–210 (2007)
49. YANNOPAPAS, V.: Artificial magnetism and negative refractive index in three-dimensional metamaterials of spherical particles at near-infrared and visible frequencies. *Appl. Phys. A* **87**(2), 259–264 (2007)
50. ZHANG, F., POTET, S., CARBONELL, J., LHEURETTE, E., VANBESIEEN, O., ZHAO, X., LIPPENS, D.: Negative-zero-positive refractive index in a prism-like omega-type metamaterial. *IEEE Trans. Microw. Theory Tech.* **56**, 2566 (2008)
51. ZHANG, S., FAN, W., MINHAS, B.K., FRAUENGLASS, A., MALLOY, K.J., BRUECK, S.R.J.: Midinfrared resonant magnetic nanostructures exhibiting a negative permeability. *Phys. Rev. Lett.* **94**, 037402 (2005)
52. ZHIKOV, V.V.: Gaps in the spectrum of some elliptic operators in divergent form with periodic coefficients. *Algebra i Analiz* **16**, 34–58 (2004) [English translation *St. Petersburg Math. J.* **16**(5), 773–790 (2005)]
53. ZHOU, X., ZHAO, X.P.: Resonant condition of unitary dendritic structure with overlapping negative permittivity and permeability. *Appl. Phys. Lett.* **91**, 181908 (2007)

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