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An Isoperimetric Inequality for the Torsional Rigidity of Imperfectly Bonded Fiber Reinforced Cylinders

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Abstract. An isoperimetric inequality for the torsional rigidity of imperfectly bonded, fiber reinforced cylinders is found. The fiber cross sections can be simply or multiply connected. The imperfect bonding conditions are given in terms of a flexible interface model.

Mathematics Subject Classifications (1991): 35J20, 73C02, 73K20.

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1. Introduction

We consider cylinders reinforced with cylindrical fibers of greater shear stiffness. We suppose that the cross sections of the cylinder and fibers are uniform. The part of the cylinder cross section occupied by the union of reinforcement fibers is denoted by A_r. The remaining part of the cylinder cross section containing more compliant material is denoted by A_m. The cylinder cross section is denoted by Ω and $\Omega = A_r \cup A_m \cup J$, where J is the interface between the reinforcement fibers and the more compliant material. It is assumed that the cylinder cross section is simply connected while each fiber cross section can be simply or multiply connected. The fibers are taken to be imperfectly bonded to the surrounding material. The area occupied by the union of the fiber cross sections is denoted by θ_r. The area occupied by the more compliant material is denoted by θ_m. We investigate the problem of simultaneously finding the cylinder cross section, fiber configuration, and cross-section of each fiber that yields the maximum torsional rigidity. The only constraint is that we search for the best design over all cylinder cross sections and fiber configurations with θ_r and θ_m prescribed.

Problems of this type are well known and were investigated by B. de Saint-Venant [9], who proposed in 1856 that among all cylinders with given cross-sectional

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area that the greatest torsional rigidity is obtained by a cylinder with circular cross-section. This proposition was proven by G. Polya in (1948), (see [7]), who with A. Weinstein, extended this result to cylinders with multiply connected cross-sections [8]. For perfectly bonded fiber reinforced cylinders, the results of A. Alvino and G. Trombetti (1985) and C. Voas and D. Yaniro (1987) show that a cylinder with circular cross section composed of a centered fiber of circular cross section made from the compliant material reinforced by an outer annular fiber of stiff material gives the maximum torsional rigidity: see, [2, 10]. This result is naturally consistent with the result of Polya and Weinstein (1950) who showed that among all multiply connected cylindrical cross-sections that the annulus yields the maximum torsional rigidity. In this note we show that a similar assertion persists in the case of reinforcement using imperfectly bonded fibers when the fibers can be multiply connected as well as simply connected. It should be pointed out that if we do not allow for the presence of multiply connected fibers, i.e., we reinforce with simply connected fibers only, then the problem of finding the optimal fiber and cylinder configuration becomes dependent on the degree of imperfect bonding and the fiber size. This is shown by the author in [5].

Imperfect bonding or partial adhesion on the fiber surface is often caused by interfacial damage due to service. Imperfect bonds are characterized by the loss of continuity in the displacement across the interface between different elastic materials. In this treatment partial adhesion is modeled by an interfacial surface across which the tangential components of the displacement are discontinuous. The traction is assumed continuous across the interface and the relative tangential displacement is proportional to the tangential traction. No inter-penetration between the fiber and surrounding material is allowed and the normal component of the displacement is continuous across the interface. The stiffness of the interface is characterized by the parameter 'α', relating the tangential traction to the relative tangential displacement. This parameter has dimensions of shear stiffness per unit length and ranges between zero and infinity. The limiting case '$\alpha = \infty$', corresponds to perfect bonding for which the displacement is continuous across the interface. The case of noadhesion along directions parallel to the interface is captured in the limit '$\alpha = 0$'. This interface model was used by Lene and Leguillon [4] in their treatment of the softening of effective moduli arising from damage. Flexible interface models similar to the type treated here can be found in the work of Jones and Whittier (1967). A comprehensive treatment of interface models as they relate to imperfect bonding is provided in the recent book of Aboudi [1].

We suppose that cylinder is of length h with generators parallel to the x_3 axis. The cylinder cross section, Ω, is a simply connected domain with Lipschitz continuous boundary in the x_1-x_2 plane. We suppose that all fibers run the length of the cylinder and that each fiber is a cylinder of constant cross section with generators parallel to the x_3 axis. The boundary of each fiber is assumed Lipschitz continuous. Both the fibers and nonreinforced parts of the cylinder are assumed to be made from linearly elastic isotropic material. The shear moduli of the fibers and surrounding material are denoted by G_r and G_m respectively. The fibers are assumed to provide reinforcement and so we set $G_r > G_m$. The displacement inside the cylinder is given by, $\mathbf{u} = (u_1, u_2, u_3)$ and the associated 3×3 stress tensor is denoted by σ_{ij}. We fix coordinates so that the base of the cylinder lies on the $x_3 = 0$ plane and the x_3 axis lies within the shaft. The sides of the cylinder are kept traction free and we fix $u_1 = u_2 = 0$ and $\sigma_{33} = 0$ on the base of the cylinder. The cylinder is subjected to a twist of angle θ per unit length. At $x_3 = h$ we have $u_1 = -\theta h x_2, u_2 = \theta h x_1$, with $\sigma_{33} = 0$. The torsional rigidity is the ratio between the resultant torsional moment over the cross section Ω and the twist per unit length θ. Denoting the torsional rigidity by $T_\alpha(A_r, \Omega)$ we have

$$\theta T_\alpha(A_r, \Omega) = \int_\Omega (x_1\sigma_{32} - x_2\sigma_{31})\,\mathrm{d}x. \tag{1.1}$$

Here $\mathrm{d}x = \mathrm{d}x_1 \mathrm{d}x_2$.

One naturally expects that the torsional rigidity of an imperfectly bonded fiber reinforced cylinder is less than the torsional rigidity when the fibers are perfectly bonded. However, equality holds for a cylinder of circular cross section, made up of a centered circular cross section of compliant material surrounded by stiff material: see Section 3. Indeed, when the cylinder cross section has a radius of R and the radius of the cross section of compliant material is a, the torsional rigidity is given by:

$$\frac{\pi G_r}{2}(R^4 - a^4) + \frac{\pi G_m}{2} a^4, \tag{1.2}$$

this is computed in Section 3.

Formula (1.2) shows that the torsional rigidity for this configuration is independent of the tangential interfacial stiffness 'α'. This is due to the fact that the traction vanishes at the matrix-fiber interface: see Section 3. Formula (1.2) is naturally found to give the torsional rigidity for this configuration when the fibers are perfectly bonded to the surrounding material.

We consider the problem of finding the cylinder cross section, the cross section of each reinforcement fiber, and fiber configuration that yields the maximum torsional rigidity. Each admissible design is specified by a simply connected cylinder cross section reinforced with a finite number of fiber cross sections. Here the fiber cross sections can be simply or multiply connected. The only constraint is that θ_r and θ_m are prescribed. Here we place no lower bound on the size of the fibers nor do we place any constraint on the number of fibers appearing in any design. The main result of this paper is the following.

THEOREM 1.1. Isoperimetric inequality. *Of all fiber reinforced cylinders, for which θ_r and θ_m are prescribed, the cylinder with circular cross section composed of a centered circular cross section of compliant material reinforced with an outer annular fiber of stiff material has the maximum torsional rigidity.*

Let the πR^2 be the area of the cylinder cross-section and $\pi(R^2 - a^2) = \theta_r$, then the theorem asserts that

$$T_\alpha(A_r, \Omega) \leqslant \frac{\pi G_r}{2}(R^4 - a^4) + \frac{\pi G_m}{2} a^4. \tag{1.3}$$

It should be noted that if we restrict the fiber cross sections to be simply connected then the problem of finding the optimal fiber and cylinder configuration becomes dependent on the degree of imperfect bonding and the fiber size [5]. In Lipton (1998) it is shown that the optimal configuration depends upon the degree of imperfect bonding through the parameter 'R_{cr}', where $R_{\text{cr}} = \alpha^{-1}/(G_m^{-1} - G_r^{-1})$. The dependence of the optimal design on the fiber size is given by the *surface traction to bulk stress quotient* introduced in Lipton (1998). The *surface traction to bulk stress quotient* ρ is a geometric parameter intrinsic to the fiber shape and is a measure of the fiber's response to anti-plane shear. When the total cross-sectional area of fibers is less than πR_{cr}^2 and $R_{\text{cr}} \times \rho > 1$ for each fiber, then the optimal cylinder and shaft configuration is given by a circular cylinder reinforced by a centered circular fiber composed of the stiffer material surrounded by the more compliant material: see, ([5], Theorem 1.6).

The variational formulation of the torsional rigidity given in terms of stress potentials is described in Section 2. The isoperimetric inequality is proved in Section 3.

2. The Torsion Boundary Value Problem and a Variational Formulation for the Torsional Rigidity

We consider a cylinder reinforced with a finite number of fibers. The union of all interfaces between fibers and surrounding material is written as Γ. The jump in a quantity 'q' across Γ is denoted by $[q] = q_r - q_m$, where q_r is the trace of the quantity on the reinforcement fiber side and q_m is the trace on the compliant material side. On the interface, the elastic displacement is decomposed into normal and tangential components given by $u_n = \mathbf{u} \cdot \mathbf{n}$ and $\mathbf{u}_\tau = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$, where $\mathbf{n}$ is the unit normal pointing out of the fiber domain. The stress tensor inside the composite shaft is denoted by σ_{ij} and on the fiber surface the traction is decomposed into normal and tangential components given by $\sigma_n = \sigma_{ij} n_i n_j$ and $(\sigma_\tau)_i = \sigma_{ij} n_j - (\sigma_{kl} n_k n_l) n_i$. Inside each phase we have the equilibrium condition

$$\partial_j \sigma_{ij} = 0, \tag{2.1}$$

and on the interface we have the imperfect bonding conditions described by

$$[u_n] = 0 \quad \text{on} \quad \Gamma, \tag{2.2}$$

$$[\sigma_{ij} n_j] = 0 \quad \text{on} \quad \Gamma, \tag{2.3}$$

and

$$\sigma_\tau = -\alpha[\mathbf{u}_\tau] \quad \text{on} \quad \Gamma. \tag{2.4}$$

The constitutive law is given by $\sigma = \mathcal{C}e(\mathbf{u})$, where $e(\mathbf{u})$ is the strain tensor given by $e(\mathbf{u}) = (\nabla\mathbf{u} + \nabla\mathbf{u}^t)/2$ and $\mathcal{C}$ is the isotropic elastic tensor taking different values in each phase. The elasticity tensor is specified by bulk and shear moduli κ_r and G_r inside the fibers and outside the fibers by κ_m and G_m. The equilibrium condition (2.1) together with the constitutive law, interface conditions (2.2–2.4), and boundary conditions given in Section 1 constitute a well posed boundary value problem for the elastic displacement, see Lipton (1998). The solution is easily seen to be unique up to a constant translation parallel to the axis of the shaft.

Proceeding as in the perfectly bonded case we find that the solution is of Saint-Venant type. That is the displacement in the shaft is given by

$$u_1 = -\theta x_3 x_2, u_2 = \theta x_3 x_1 \tag{2.5}$$

and

$$u_3 = \theta w(x_1, x_2). \tag{2.6}$$

The function $w(x_1, x_2)$ is analogous to the warping function appearing in the torsion problem with perfectly bonded interfaces. However, unlike the perfectly bonded case the warping function introduced here can be discontinuous across the bi-material interface. To make this precise we consider the intersection of the interface Γ with the cylinder cross section Ω. This set is precisely J. We allow the warping function to have jump discontinuities across J. The set of all points in Ω not on J is denoted by Ω/J. The warping function is assumed to belong to the space of square integrable functions with square integrable first derivatives on the region Ω/J. This space is denoted by $H^1(\Omega/J)$.

Equations (2.5) and (2.6) imply that the only nonzero components of the strain tensor are given by

$$e_{13} = \frac{\theta}{2}(\partial_{x_1} w - x_2) \quad \text{and} \quad e_{23} = \frac{\theta}{2}(\partial_{x_2} w + x_1) \quad \text{in } \Omega/J. \tag{2.7}$$

The nonzero components of the stress tensor are

$$\sigma_{13} = \theta G(x)(\partial_{x_1} w - x_2) \quad \text{and} \quad \sigma_{23} = \theta G(x)(\partial_{x_2} w + x_1) \quad \text{in } \Omega/J. \tag{2.8}$$

Here $G(x)$ is the piece-wise constant shear modulus taking the values: G_m and G_r in the matrix and fiber respectively. Substitution of (2.5 and 2.6) into the interface conditions (2.2) and (2.3) gives

$$\sigma_n = 0 \quad \text{on } J, \tag{2.9}$$

$$[G(x)(\nabla w + \tilde{v}) \cdot n] = 0 \quad \text{on } J, \tag{2.10}$$

and

$$G_r(\nabla w + \tilde{v})_r \cdot n = -\alpha[w] \quad \text{on } J. \tag{2.11}$$

Here $\nabla w = (\partial_{x_1} w, \partial_{x_2} w)^t$, $\tilde{v} = (-x_2, x_1)^t$, and $n = (n_1, n_2)^t$ is the unit normal pointing out of the fiber. The traction free condition on the sides of the shaft gives

$$\partial_n w = -n \cdot \tilde{v} \quad \text{on } \partial\Omega \tag{2.12}$$

and the equilibrium condition $\partial_j \sigma_{ij} = 0$ gives

$$\Delta w = 0 \quad \text{in } \Omega/J. \tag{2.13}$$

Equations (2.10)–(2.13) determine the warping function up to an additive constant.

We introduce the harmonic function ϕ conjugate to the warping function w on the region Ω/J. This function is defined uniquely up to an additive constant inside each fiber and in the matrix. The stress potential Φ is defined as

$$\Phi = G(x)(\phi - (x_1^2 + x_2^2)/2). \tag{2.14}$$

One easily calculates that for all points in Ω/J that

$$\nabla\Phi = -RG(x)(\nabla w + \tilde{v}), \tag{2.15}$$

where R is the rotation matrix associated with a clockwise rotation of $\pi/2$ radians. Relations (2.14) and (2.15) allow us to recover the boundary value problem for the stress potential from that of the warping function. From (2.14) we obtain

$$\mathrm{G}^{-1}(x)\Delta\Phi = -2 \quad \text{in } \Omega/J. \tag{2.16}$$

Application of (2.10) and (2.15) gives

$$0 = [G(x)(\nabla w + \tilde{v}) \cdot n] = [R\nabla\Phi \cdot n] \quad \text{on } J \tag{2.17}$$

and we find

$$[\partial_\tau \Phi] = 0 \quad \text{on } J. \tag{2.18}$$

Here ∂_τ indicates tangential differentiation along the interface. It follows from (2.18) that adjustment by a constant in each fiber (if necessary) gives $[\Phi] = 0$ across the interface. Thus the gradient of Φ is square integrable over the whole domain Ω and Φ lies in the Sobolev space $H^1(\Omega)$. From (2.12) we find that $\partial_\tau \Phi = 0$ on $\partial\Omega$ and so Φ is a constant on $\partial\Omega$. We fix the last constant at our disposal to set $\Phi = 0$ on $\partial\Omega$. Lastly we recover the transmission conditions satisfied by the derivatives of the stress potential across the interface. We return to equation (2.15) and apply standard trace theorems to find:

$$[G^{-1}\partial_n \Phi] = -[R(\nabla w + \tilde{v}) \cdot n] \quad \text{on } J. \tag{2.19}$$

Noting that $R^t = -R$ and $Rn = \tau$ where τ is the unit tangent to J, we have

$$[G^{-1}\partial_n \Phi] = [\partial_\tau w] \quad \text{on } J. \tag{2.20}$$

On the other hand from (2.11) we have

$$G_r(\nabla w + \tilde{v})_r \cdot n = -\alpha[w] \quad \text{on } J \tag{2.21}$$

and since $R\nabla\Phi_r \cdot n = G_r(\nabla w + \tilde{v})_r \cdot n$ on J, we obtain

$$-\partial_\tau \Phi = -\alpha(w_r - w_m) \quad \text{on } J. \tag{2.22}$$

(Here we recall from (2.18) that the tangential derivative of Φ is continuous across J). When J is sufficiently regular we may differentiate (2.22) and apply (2.20) to find the desired transmission condition:

$$\alpha^{-1}\partial_\tau^2 \Phi = [G^{-1}\partial_n \Phi]. \tag{2.23}$$

Collecting our results we find that the transmission condition $[\Phi] = 0$ on J, (2.16) and (2.23) together with the boundary condition $\Phi = 0$ on $\partial\Omega$ provides a boundary value problem for the stress potential. Existence and uniqueness follows from an application of the Lax-Milgram Lemma; this is established in the work of Pham Huy and Sanchez Palencia [6].

The torsional rigidity can be expressed in terms of the stress potential. Substitution of the stress potential into equation (1.1) gives

$$T_\alpha(A_r, \Omega) = \int_\Omega G^{-1}(x)|\nabla\Phi|^2 \, \mathrm{d}x + \alpha^{-1} \int_J |\partial_\tau \Phi|^2 \, \mathrm{d}l. \tag{2.24}$$

To proceed with the analysis we formulate the torsional rigidity in terms of the following variational principle. We write

$$T_\alpha(A_r, \Omega) = -2E_\alpha(\mathrm{A_r}, \Omega), \tag{2.25}$$

where

$$E_\alpha(A_r, \Omega) = \min_{\varphi \in H_0^1(\Omega)} \left\{ \frac{1}{2} \left(\int_\Omega G^{-1}(x)|\nabla\varphi|^2 \, \mathrm{d}x + \alpha^{-1} \int_J |\partial_\tau \varphi|^2 \, \mathrm{d}l \right) - 2\int_\Omega \varphi \, \mathrm{d}x \right\}. \tag{2.26}$$

Here $H_0^1(\Omega)$ denotes all functions in $H^1(\Omega)$ that are zero on the boundary of Ω. It is easily checked that the minimizer is precisely the stress potential.

3. Proof of the Isoperimetric Inequality

We now prove the isoperimetric inequality. To do this we start by comparing the torsional rigidity $T_\alpha(A_r, \Omega)$ for finite values of the tangential compliance to the torsional rigidity for perfectly bonded fibers. The torsional rigidity for a perfectly bonded fiber reinforced shaft is written $T_\infty(A_r, \Omega)$ and is given by the well known variational formulation

$$T_\infty(A_r, \Omega) = -2\mathrm{E}_\infty(\mathrm{A_r}, \Omega), \tag{3.1}$$

where

$$E_\infty(\mathrm{A_r}, \Omega) = \min_{\psi \in H_0^1(\Omega)} \frac{1}{2}\left(\int_\Omega G^{-1}|\nabla\psi|^2\,\mathrm{d}x\right) - 2\int_\Omega \psi\,\mathrm{d}x. \tag{3.2}$$

The minimizer is the stress potential in the shaft. It is evident from the definition of the energies $E_\alpha(A_r, \Omega)$ and $E_\infty(A_r, \Omega)$ that for any reinforcement fiber and shaft cross section

$$T_\infty(A_r, \Omega) \geqslant T_\alpha(A_r, \Omega), \tag{3.3}$$

for $0 < \alpha < \infty$. This expresses the intuitive notion that shafts with perfectly bonded fibers are more rigid than ones with imperfectly bonded fibers.

Next we consider the special case of a cylinder of circular cross-section of radius R, containing a centered circular region of radius a, filled with compliant material, surrounded by an annular jacket of stiff material. The region occupied by the more compliant material is a disk of radius a and is denoted by D_a, the cylinder cross section Ω is denoted by D_R and the region A_r occupied by the reinforcement phase is D_R/D_a. For this configuration the stress potential is independent of the interfacial shear stiffness and calculation shows that it is given by

$$\begin{aligned} \psi \;&=\; f(x), \quad a < x < R, \\ &\quad f(x) + v(x), \quad 0 < x < a, \end{aligned} \tag{3.4}$$

where

$$f(x) = -\frac{G_r}{2}|x|^2 + \frac{G_r}{2}R^2 \tag{3.5}$$

and

$$v(x) = -\frac{(G_m - G_r)}{2}|x|^2 + \frac{(G_m - G_r)}{2}a^2, \tag{3.6}$$

where $|x|^2 = x_1^2 + x_2^2$. The associated warping function for this configuration is:

$$w(x_1, x_2) = \text{constant}. \tag{3.7}$$

It is evident that the traction vanishes at the fiber-matrix interface and that the displacement is continuous everywhere in the shaft. Furthermore we see that the stress potential and displacements are independent of the interfacial tangential stiffness α. A straight forward calculation shows that the torsional rigidity for this configuration is given by (1.2). When this configuration is perfectly bonded calculation shows that

$$T_\infty(D_R/D_a, D_R) = \frac{\pi G_r}{2}(R^4 - a^4) + \frac{\pi G_m}{2}a^4. \tag{3.8}$$

Collecting our results we see for this configuration that

$$\begin{aligned} T_\infty(D_R/D_a, D_R) &= T_\alpha(D_R/D_a, D_R) \\ &= \frac{\pi G_r}{2}(R^4 - a^4) + \frac{\pi G_m}{2}a^4. \end{aligned} \tag{3.9}$$

To finish the proof we show that

$$T_\infty(A_r, \Omega) \leqslant \frac{\pi G_r}{2}(R^4 - a^4) + \frac{\pi G_m}{2}a^4, \tag{3.10}$$

from which the isoperimetric inequality follows immediately from inequality (3.3) and the identity (3.9). To establish (3.10) we appeal to Theorem 1 of Alvino and Trombetti [2] (or Theorem 1.1 of Voas and Yaniro [10]). In order to state Theorem 1 of Alvino and Trombetti [2] we denote any symmetric bounded measurable matrix function defined on Ω by $a_{ij}(x_1, x_2)$ and suppose that for every vector $\eta = (\eta_1, \eta_2)$ that

$$a_{ij}(x_1, x_2)\eta_i\eta_j \geqslant \nu(x_1, x_2)|\eta|^2, \tag{3.11}$$

where $0 < \nu$ and $\int_\Omega \nu^{-p}\,\mathrm{d}x \leqslant \infty$ for some $p > 1$. We let Ω^* be the disk with the same area as Ω. The torsional rigidity for a cylindrical cross section Ω with local compliance specified by $a_{ij}(x_1, x_2)$ is denoted by P. We let $\nu^\sharp$ denote the decreasing spherically symmetric rearrangement of ν. The associated torsional rigidity for Ω^* with local compliance given by $\nu^\sharp$ is denoted by P^*. Theorem 1 of Alvino and Trombetti [2] asserts that

$$P \leqslant P^* \tag{3.12}$$

Inequality (3.10) follows immediately from (3.12) upon setting

$$a_{ij}(x_1, x_2) = \nu(x_1, x_2)I_{ij} = ((2G_r)^{-1}\chi_r + (2G_m)^{-1}\chi_m)I_{ij},$$

where χ_r and χ_m are the characteristic functions of A_r and A_m respectively.

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