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Optimal design and relaxation for reinforced plates subject to random transverse loads

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We consider a Kirchhoff plate subject to a random transverse load. We reinforce the plate with stiffeners. For a prescribed area fraction of stiffeners we seek their optimal layout so that on average, the plate is as stiff as possible with respect to the random load. We provide a mathematical formulation of this problem. To ensure the existence of a solution we relax the problem to include generalized designs. The relaxation procedure rests upon the derivation of new optimal lower bounds on the compliance energy for the effective elasticity of composites made from stiffeners on multiple scales. Our bounding method follows the program of Hashin and Shtrikman. However our method is novel as no fictitious *comparison material* is used in the derivation. This relaxation is new and is the extension to the two-dimensional setting of the relaxation given by Cheng and Olhoff (*Int. J. Solids Struct.*, **17** (1981) 305–23, 795–810) for one-dimensional plate problems, when the plate thickness is allowed to take two values. The relaxed formulation of the problem can be solved numerically.

1 INTRODUCTION

We suppose that a plate of midplane thickness h_1 is reinforced using ribs or stiffeners of thickness $h_2 > h_1$. The resulting structure is subjected to a stochastic transverse loading. For a prescribed area fraction of stiffeners we seek the optimal layout that minimizes the average compliance. For example, if we suppose that the loading is Gaussian with prescribed mean and covariance the idea is to build a structure that on average is stiffest subject to the load. What makes this case distinct from the deterministic one is that we must design for an ensemble of loads for which there exists infinite fluctuation in the loadings.

This problem is one of distributed parameter optimal control. Here the objective function is the average compliance, i.e. average work done by the load. Let the plate domain be given by R, then for a random load $f(x,\omega)$ (where the realization ω is taken from some probability space Ω) the midplane deflection w and the bending moment σ satisfy

$$\sigma_{ij}(x,\omega) = \mathbf{M}_{ijkl}\partial^2_{x_k x_l} w(x,\omega);$$
$$\partial^2_{x_i x_j}\sigma_{ij}(x,\omega) = f(x,\omega) \text{ on } R \tag{1}$$

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We suppose that the plate is clamped at the edges, so that

$$w(x,\omega) = \partial_n w(x,\omega) = 0 \text{ on } \partial R \tag{2}$$

Here ∂_n represents the outward normal derivative. The tensor $\mathbf{M}$ introduced in (1) is the bending rigidity of the plate and relates the bending moment σ_{ij} to the midplane curvature $\partial^2_{ij} w$. We shall assume that both plate and stiffeners are made from the same isotropic elastic material with Young's modulus and Poisson's ratio E and ν respectively. Let $\mathbb{P}_h$ and $\mathbb{P}_s$ denote the projections onto the spaces of hydrostatic and shear strains, respectively, then the rigidity tensors of the plate and stiffeners are given by

$$\mathbf{M}_1 = \tfrac{2}{3}h_1^3(2\mu\mathbb{P}_s + 2\kappa\mathbb{P}_h) \tag{3}$$

and

$$\mathbf{M}_2 = \tfrac{2}{3}h_2^3(2\mu\mathbb{P}_s + 2\kappa\mathbb{P}_h) \tag{4}$$

respectively. Here $\mu = E/2(1+\nu)$ is the shear modulus and $\kappa = E/2(1-\nu)$. Thus, for a reinforced plate the rigidity $\mathbf{M}$ is given by

$$\mathbf{M} = \chi_1\mathbf{M}_1 + \chi_2\mathbf{M}_2 \tag{5}$$

where χ_1 and χ_2 are the indicator functions of plate and stiffeners respectively and $\chi_1 = 1 - \chi_2$. The area fraction of stiffeners is given by

$$\int_R \chi_2(x)\,\mathrm{d}x \tag{6}$$

The average compliance or work done by the load $f(x,\omega)$ is given by

$$J = \left\langle \int_R w(x,\omega) f(x,\omega)\,\mathrm{d}x \right\rangle \tag{7}$$

Here $\langle\cdot\rangle$ denotes the average over the ensemble.

The control variable for this problem is the rigidity $\mathbf{M}$

$$\mathbf{M} = \begin{cases} \mathbf{M}_1 & \text{in the plate} \\ \mathbf{M}_2 & \text{in the stiffeners} \end{cases} \tag{8}$$

We indicate the functional dependence of the average compliance J on the rigidity $\mathbf{M}$ by writing $J = J(\mathbf{M})$.

The constrained optimization problem is to minimize the average compliance subject to a constraint on the area fraction of the stiffeners. Introducing a Lagrange multiplier $\lambda > 0$ for the area fraction constraint the optimization problem is given by

$$\min_{\mathbf{M}} \left\{ J(\mathbf{M}) + \lambda \int_R \chi_2(x)\,\mathrm{d}x \right\}. \tag{9}$$

From a theoretical standpoint it is known that a problem of the type given by (9) is not well posed.[1–5] It is well understood that the regularization is accomplished by extending the space of controls to include composite plates. It is within this class of designs that a global optimum can always be found.[1,2,4,6,7] In the context of our problem a composite plate exhibits an effective rigidity tensor; this tensor captures the overall limiting behavior of a sequence of plates reinforced with increasingly oscillatory arrangements of stiffeners. Indeed, as the length scale of the local geometry of the stiffeners goes to zero we may replace the detailed local elastic behavior with that of an effective rigidity tensor. This tensor may be anisotropic and depends upon the local microgeometry of the minimizing sequence of stiffeners. The notion of an effective tensor is a natural one for this problem as optimal designs are frequently approached by such sequences.[1,2,5] We emphasize here that the extension of the design space to include effective rigidity tensors is not equivalent to introducing isotropic plates with an effective thickness. Indeed, introduction of isotropic plates with an effective or averaged thickness would constrain the control $\mathbf{M}$ to be in the space of *isotropic* tensors. This is clearly inadequate for the purposes of relaxation as minimizing sequences are known to possess spatial anisotropy for asymptotically small length scales.[1,2,5]

The regularization in the context of reinforced plates was studied by several investigators.[1,2,5] These investigators treated the one dimensional problem of an annular plate reinforced with circumferential stiffeners. Their regularization amounted to understanding all possible effective rigidities. We remark that at this time the set of all effective rigidities for the general two dimensional reinforced plate is unknown. Fortunately, for the problem at hand it is sufficient to use only an extremal subset. The use of such subsets in the regularization of optimal design problems is called partial relaxation (cf. Refs 1, 2, 5, 8). The set of effective rigidities necessary for the partial relaxation of our design problem is the analogue of the well known finite rank laminates. Such composites have been discussed by many authors[9–11] and have been applied to the partial relaxation of problems in conductivity and elasticity (cf. Refs 4, 7, 12). In the context of plate theory Gibianski & Cherkaev[13] have obtained the partial relaxation for plate problems involving one energy. For the random loading problem discussed here we find in Section 3 the partial relaxation for a weighted infinite sum of energies. This relaxation is new and is the extension to the two dimensional setting of the relaxation given by Cheng & Olhoff[1,2] for one dimensional plate problems, when the plate thickness is allowed to take two values.

In physical terms the composites used in the partial relaxation are those whose effective rigidity provides the stiffest compliance in response to the local midplane curvature. This is illustrated in Section 2.

We note here that deterministic optimal compliance design problems have received much recent attention and the reader is referred to a review.[12] The relaxation provided here for the random problem also applies to the deterministic case. This is illustrated in Section 2 where it is seen that the weighted infinite sum of energies associated with the random case can be written as a sum of just three energies.

In the following sections we provide the relaxation of the design problem for a general random load $f(x,\omega)$. However, as most physically measurable processes are of the second order type it is practical to consider the loading as a second order random process. For this case the load $f(x,\omega)$ can be written in terms of its average $\bar{f}(x)$ and a mean zero random fluctuation $\tilde{f}(x,\omega)$ in the form

$$f(x,\omega) = \bar{f}(x) + \tilde{f}(x,\omega) \tag{10}$$

The mean zero fluctuation $\tilde{f}$ can be expanded in a Karhunan–Loeve expansion (cf. Refs 14, 15) given by

$$\tilde{f}(x,\omega) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\xi_n(\omega)\hat{f}_n(x) \tag{11}$$

Here, $\tilde{f}$ is determined by its covariance function $\Gamma(x_1,x_2)$. The deterministic functions $\hat{f}_n$ and the numbers are the eigenvectors and eigenvalues of the integral equation

$$\int_R \Gamma(x_1,x_2)\hat{f}_n(x_2)\,\mathrm{d}x_2 = \hat{f}_n(x_1) \tag{12}$$

The mean zero random variables $\xi_n(\omega)$ are determined by

$$\xi_n(\omega) = \frac{1}{\lambda_n}\int_R \hat{f}(x,\omega)\hat{f}_n(x)\,\mathrm{d}x \tag{13}$$

The functions $\hat{f}_n(x)$ and $\xi_n(\omega)$ satisfy the orthonormality conditions

$$\langle \xi_n(\omega)\xi_m(\omega)\rangle = \delta_{nm} \tag{14}$$

and

$$\int_R \hat{f}_n(x)\hat{f}_m(x)\,\mathrm{d}x = \delta_{nm} \tag{15}$$

It follows immediately from (1) and (2) and the orthonormality relations (14) and (15) that the mid-plane deflection w is of the form

$$w(x,\omega) = \bar{w}(x) + \sum_{n=1}^{\infty} \xi_n(\omega)a_n(x) \tag{16}$$

where

$$\partial^2_{x_i x_j}\mathbf{M}_{ijkl}\partial^2_{x_x x_l}\bar{w}(x) = \bar{f}(x) \tag{17}$$

and

$$\partial^2_{x_i x_j}\mathbf{M}_{ijkl}\partial^2_{x_k x_l}a_n(x) = \hat{f}_n(x), \quad (n = 1, 2, \ldots) \tag{18}$$

and $\bar{w}$, a_1, $a_2, \ldots$ satisfy the clamped boundary conditions (2). We see that $\bar{w}$ is the deflection due to the average loading and $\sum_{n=1}^{\infty}\xi_n(\omega)a_n$ is the response to the random fluctuations in the load.

We approximate the random load by truncating the series (11) at the Nth term. The associated midplane deflection due to the truncated load is given by

$$\tilde{w}(x,w) = \bar{w}(x) + \sum_{n=1}^{N} \xi_n(w)a_n(x) \tag{19}$$

The average compliance (7) is then approximated by $\tilde{J}(\mathbf{M})$, where

$$\tilde{J}(\mathbf{M}) = \int_R \left(\bar{w}(x)\bar{f}(x) + \sum_{n=1}^{N} \sqrt{\lambda_n}a_n(x)\hat{f}_n(x)\right)\mathrm{d}x \tag{20}$$

For problems of practical interest we may consider the optimization problem for the approximate average compliance (20). This problem may be interpreted as an optimal control problem for multiple independent *deterministic* loads. In terms of the approximate average compliance, problem (9) becomes

$$\min_{\mathbf{M}}\left\{\tilde{J}(\mathbf{M}) + \lambda\int_R \chi_2(x)\,\mathrm{d}x\right\} \tag{21}$$

The relaxation of the approximate problem follows immediately from the methods developed for the general problem given in Sections 2, 3 and 4. To illustrate minimum compliance design we include a numerical example, see Section 5. We remark that the plate model treated in this example is not a stiffener reinforced Kirchhoff plate, but instead, a stiffener reinforced Mindlin plate model introduced by Soto & Diaz.[16] We choose this model as there exist numerical codes for the solution of minimum compliance problems for this plate model. The numerical calculations were carried out by Diaz & Soto using the structural optimization code developed in Refs 16 and 17. We remark that the model of Diaz and Soto has not yet been proven to be a rigorous relaxation for the random loading case. However it enables the layout problem to be formulated as a sizing problem that can be solved using the optimality criteria method, see Refs 1, 2 and 17. The numerical implementation of the rigorous relaxation of problem (9) for Kirchhoff plate theory given by (36) follows the same lines as the example given in Section 5 and is the focus of joint work with Diaz and the author.

2 RELAXATION OF THE DESIGN PROBLEM

We formulate the relaxed version of the problem (9). To relax the problem we allow the control variable $\mathbf{M}$ to assume as before the values $\mathbf{M}_1$ and $\mathbf{M}_2$ associated with the plate and stiffener; and in addition we allow $\mathbf{M}$ to take values in the set of effective rigidities. The set of effective rigidities corresponds to a 'composite' material with local area fraction θ_2 of stiffeners. The optimal design now allows for regions of stiffeners, for which, $\theta_2 = 1$; regions of pure plate, for which $\theta_2 = 0$; and regions of 'composite' associated with the intermediate values, $0 < \theta_2 < 1$. The total area fraction of stiffeners is given by

$$\int_R \theta_2(x)\,\mathrm{d}x \tag{22}$$

We denote the set of all effective rigidities associated with local area fraction θ_2 of stiffeners by G_{θ_2}. The extended set of controls is then described by the pair $(\theta_2, \mathbf{M})$, where $\mathbf{M} \in G_{\theta_2}$. We note for $\theta_2 = 1$ and for $\theta_2 = 0$ that the G_{θ_2} set reduces to $\mathbf{M}_2$ and $\mathbf{M}_1$ respectively. In terms of the extended set of controls the relaxed problem becomes

$$\min_{\substack{\theta_2(x)\\ \mathbf{M}\in G_{\theta_2(x)}}}\left\{J(\mathbf{M}) + \lambda\int_R \theta_2(x)\,\mathrm{d}x\right\} \tag{23}$$

As was mentioned earlier, a complete characterization of the set G_{θ_2} is not available, however, for this problem we do not need to know the full G_{θ_2} set. To see what we need we develop a minimum complementary energy principle for the bending moment appearing in the composite plate. We introduce the following class of moment tensors.

$$\mathscr{C} = \{\tau_{ij} \mid \tau_{ij} = \tau_{ji};\, \partial^2_{ij}\tau_{ij}(x,\omega) = f(x,\omega);$$

(1) τ_{ij} in $L^2(R)$ almost everywhere in Ω

(2) τ_{ij} in $L^2(\Omega)$ almost everywhere in R and along curves 'S' of discontinuity (in the x variable) of τ:

(3) $[Q]=0$
(4) $[M_B]=0$
(5) $[\tau_{ij}t_i^+n_j^+ - \tau_{ij}t_i^-n_j^-]=0$ at the corner points 'x_k' of the curve $S\}$ (24)

Here Q is the Kirchhoff shear force

$$Q = -\partial_{x_i}\tau_{ij}n_j - \partial_s(\tau_{ij}t_in_j) \tag{25}$$

and M_B is the bending moment

$$M_B = \tau_{ij}n_in_j \tag{26}$$

where t and n are the tangent and normal vectors to the curve respectively and ∂_s is the tangential derivative along the curve. At the corner points x_k, the vectors t and n change discontinuously from t^- and n^- to t^+ and n^+. One easily obtains after integration by parts (cf. Ref. 2) and standard arguments that

$$\int_R f(x,\omega)w(x,\omega)\,dx = \min_{\tau\in\mathscr{C}}\int_R \mathbf{M}^{-1}\tau(x,\omega)\cdot\tau(x,\omega)\,dx \tag{27}$$

Ensemble averaging (27) gives

$$J(\mathbf{M}) = \min_{\tau_{ij}\in\mathscr{C}}\left\langle\int_R \mathbf{M}^{-1}\tau\cdot\tau\,dx\right\rangle \tag{28}$$

Upon interchanging operations of averaging and integration we obtain

$$J(\mathbf{M}) = \min_{\tau\in\mathscr{C}}\int_R \langle\mathbf{M}^{-1}\tau\cdot\tau\rangle\,dx \tag{29}$$

Since $\mathbf{M}^{-1}$ is deterministic we have $\langle\mathbf{M}^{-1}\tau\cdot\tau\rangle = \mathbf{M}^{-1}_{ijkl}\langle\tau_{ij}\tau_{kl}\rangle$ and defining

$$\mathbf{M}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle \equiv \mathbf{M}^{-1}_{ijkl}\langle\tau_{ij}\tau_{kl}\rangle \tag{30}$$

gives

$$J(\mathbf{M}) = \min_{\tau\in C}\int_R(\mathbf{M}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle)\,dx \tag{31}$$

Here is it easily checked that $\langle\tau\otimes\tau\rangle$ is a positive definite fourth order tensor. Substitution of (31) into (23) and rearrangement shows that the relaxed design problem may be written as

$$\min_{\tau\in\mathscr{C}}\min_{\substack{\theta_2(x)\\ \mathbf{M}\in G_{\theta_2(x)}}}\left\{\int_R[(\mathbf{M}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle)+\lambda\theta_2]\,dx\right\} \tag{32}$$

We denote the minimizer of the variational principle (31) by τ^*, so that

$$J(\mathbf{M}) = \int_R(\mathbf{M}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle)\,dx \tag{33}$$

Then from (33) we are able to argue as in Ref. 12 that the best design is one for which the form

$$\mathbf{M}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle \tag{34}$$

is smallest at each point in the design. Indeed, suppose that the 'best' design $\mathbf{M}$ did not minimize (33) at each point then there would exist a design $\mathbf{N}$ in $G_{\theta_2(x)}$ such that

$$\mathbf{M}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle > \mathbf{N}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle \tag{35}$$

from which we see that

$$J(\mathbf{M}) = \int_R \mathbf{M}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle\,dx > \int_R \mathbf{N}^{-1}\cdot\cdot\langle\overset{*}{\tau}\otimes\overset{*}{\tau}\rangle\,dx$$
$$> \min_{\tau\in\mathscr{C}}\int_R \mathbf{N}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle\,dx = J(\mathbf{N})$$

contradicting the optimality of $\mathbf{M}$.

From these arguments it is evident that we need only consider an extremal subset of controls in G_{θ_2} for which the local compliance form (34) is minimized. It is now well known for two phase composites that finite rank laminates have effective tensors that minimize one or more compliance energies.[18–20]

In this treatment we shall introduce an extremal set of effective rigities that minimize the form (34). These effective tensors are analogous to the finite rank laminates used in two phase elasticity. We shall denote this set of controls associated with the area fraction θ_2 by GL_{θ_2}. In view of the fact that the optimal design minimizes the local energy form (34) at every point we see that the relaxed optimal design problem can be written as

$$\min_{\tau\in C}\int_R H_\lambda(\langle\tau\otimes\tau\rangle)\,dx \tag{36}$$

where

$$H_\lambda(\langle\tau\otimes\tau\rangle) = \min_{0\le\theta_2\le 1}\left(\min_{\mathbf{M}\in GL_{\theta_2}}(\mathbf{M}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle)+\lambda\theta_2\right) \tag{37}$$

It is evident from (37), that the effect of randomness in the loading dictates the optimal choice of control in GL_{θ_2} for each point in the plate. We observe that since the tensor $\langle\tau\otimes\tau\rangle$ appearing in (37) is positive definite it has the spectral representation given by

$$\langle\tau\otimes\tau\rangle = \sum_{s=1}^{3}\rho_S\eta^s\otimes\eta^s \tag{38}$$

where η^s are the eigentensors associated with the eigenvalues $\rho^s \ge 0$, therefore,

$$\min_{\mathbf{M}\in GL_{\theta_2}}(\mathbf{M}^{-1}\cdot\cdot\langle\tau\otimes\tau\rangle) = \min_{\mathbf{M}\in GL_{\theta_2}}\sum_{s=1}^{3}\rho^s\mathbf{M}^{-1}\eta^s\cdot\eta^s \tag{39}$$

It now follows that the problem of minimizing the local energy from (34) for the random case reduces to minimizing the sum of three compliance energies.

In order to complete the relaxation we will fully disclose the extremal set $GL_{\theta 2}$ for reinforced plates. To this end we introduce a class of effective rigidity tensors associated with geometries that we call finite rank laminates. It is shown in the sequel that effective tensors in this class minimize sums of compliance energies. These effective rigidities constitute the GL_{θ_2} set. To prove this, we derive in Section 3 Hashin–Shtrikman[21] type variational principles for reinforced plates and obtain a new lower bound on the sum of compliance energies over the set of effective rigidity tensors with prescribed area fraction θ_2 of stiffeners (i.e. G_{θ_2}) (see Theorem 3.1). We remark that we provide here a new way of deriving Hashin–Shtrikman variational principles that make no use of comparison materials. Our method uses integral operators associated with response functions. In Section 4 we introduce the set of effective rigidities associated with finite rank laminates (see Definition 4.1). We show that the lower bound on sums of compliance energies is always attained by an effective rigidity in this class (see Section 4.2, Theorem 4.1). In Section 5 we provide an illustrative numerical example for the plate model of Soto & Diaz.[16]

3 BOUNDS OF HASHIN–SHTRIKMAN TYPE FOR REINFORCED PLATES

In this section we obtain Hashin–Shtrikman bounds on the effective compliance for plates with periodic arrangements of stiffeners. The assumption of periodicity is general provided that the variation in plate thickness is small with respect to the period cell.[23] We remark that our derivation is new as it makes no use of comparison materials. For our purposes we consider a unit period cell Q in $\mathbb{R}$. The periodic bending rigidity is then given by $\mathbf{M} = \chi_1\mathbf{M}_1 + \chi_2\mathbf{M}_2$, where χ_2 is the indicator function of the stiffeners and $\chi_1 = 1 - \chi_2$. Here $\mathbf{M}_1$ and $\mathbf{M}_2$ are the bending rigidities for the plate and stiffeners respectively.

We denote the average bending moment over the unit period cell by the constant symmetric 2×2 matrix ϵ. The local curvature is given by $\partial^2_{kl} w + \epsilon_{kl}$ where w is the Q periodic vertical displacement field. The local bending moment solves

$$\partial^2_{x_i x_j} M_{ijkl}[\partial^2_{x_k x_l} w + \epsilon_{kl}] = 0, \quad \text{in } Q \tag{40}$$

and the effective bending rigidity $\mathbf{M}^e$ is defined by

$$\mathbf{M}^e\epsilon \cdot \epsilon = \int_Q \mathbf{M}(\overset{*}{E} + \epsilon) \cdot (\overset{*}{E} + \epsilon)\,\mathrm{d}x \tag{41}$$

where $\overset{*}{E} = \partial^2_{ij} w$ denotes the oscillatory part of the midplane curvature.

For the purposes of bounding effective moduli it is convenient to work in a Hilbert space setting. Let $\mathscr{H}$ denote the space of all square integrable Q-periodic symmetric 2×2 matrix fields with the standard inner product, $[P, R]$ given by

$$[P, R] - \int_Q P(x) \cdot R(x)\,\mathrm{d}s \tag{42}$$

it can be decomposed into the orthogonal direct sum

$$\mathscr{H} = \mathscr{U} \oplus \mathscr{E} \oplus \mathscr{J} \tag{43}$$

of three subspaces: the three dimensional space $\mathscr{U}$ of constant 2×2 matrices; the infinite dimensional space $\mathscr{E}$ of all Q-periodic fields $E(x)$ characterized by

$$E_{ij}(x) = \partial^2_{ij}\varphi, \int_Q E\,\mathrm{d}x = 0 \tag{44}$$

for some Q-periodic function φ; and the infinite dimensional space $\mathscr{J}$ of all Q-periodic fields $J(x)$ characterized by

$$\partial^2_{ij} J_{ij} = 0, \int_Q J\,\mathrm{d}x = 0 \tag{45}$$

We remark that in (44) the relation $\int_Q E\,\mathrm{d}x = 0$ follows immediately from $E_{ij} = \partial_{ij}\varphi$ and by integration by parts.

We shall denote by Γ_0, Γ_1 and Γ_2 the orthogonal projections of $\mathscr{H}$ into $\mathscr{E}$ and $\mathscr{J}$ respectively. In Fourier space the operators Γ_1, Γ_2 are local. Indeed given a square integrable Q-periodic matrix field $\sigma = \sum_k \mathrm{e}^{\mathrm{i}2\pi k\cdot x}\hat{\sigma}(k)$, Γ_1 and Γ_2 are defined by

$$\Gamma_i = \sum_{k\neq 0} \mathrm{e}^{\mathrm{i}2\pi k\cdot x}\hat{\Gamma}_i(\hat{k})\hat{\sigma}(k) \quad i = 1, 2 \tag{46}$$

where $\hat{k} = k/|k|$, and

$$\hat{\Gamma}_1(k) = \hat{k} \otimes \hat{k} \otimes \hat{k} \otimes \hat{k} \tag{47}$$

and

$$\hat{\Gamma}_2(k) = \mathbf{I} - \hat{k} \otimes \hat{k} \otimes \hat{k} \otimes \hat{k} \tag{48}$$

Here $\mathbf{I}$ denotes the identity operator on 2×2 symmetric matrices.

For the optimal design problem at hand we are interested in a lower bound on the compliance $(\mathbf{M}^e)^{-1}$. To this end it is convenient to obtain an upper bound on the rigidity $\mathbf{M}^e$. The lower bound on the compliance $(\mathbf{M}^e)^{-1}$ then follows from algebraic manipulation.

To obtain an upper bound on the effective rigidity we note that oscillatory part of the midplane curvature satisfies the following integral equation

$$\overset{*}{E} = -\frac{1}{\frac{2}{3}h_2^3(\mu + k)}\Gamma_1\chi_1(\mathbf{M}_2 - \mathbf{M}_1)(\overset{*}{E} + \epsilon) \tag{49}$$

From which it follows that

$$(\mathbf{M}_2 - \mathbf{M}_1)(\overset{*}{E} + \epsilon) = \mathbf{A}\epsilon \tag{50}$$

where $\mathbf{A}$ is the positive definite operator given by

$$\mathbf{A} = \left((\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{1}{\frac{2}{3}h_2^3(\mu + k)}\Gamma_1\chi_1\right)^{-1} \tag{51}$$

Here, the operator $\mathbf{A}$ relates the polarization field inside the plate to the average midplane curvature ϵ.

Expansion of $\mathbf{M}$ and substitution of eqn (51) into eqn (41) for the effective rigidity gives

$$(\mathbf{M}_2 - \mathbf{M}^e)\epsilon \cdot \epsilon = [\chi_1 \mathbf{A}\epsilon, \epsilon] \tag{52}$$

For any Q in $\mathscr{H}$ we see that

$$[\chi_1 \mathbf{A}\epsilon, \epsilon] \geq 2[\chi_1 Q, \epsilon] - [\chi_1 \mathbf{A}^{-1} Q, Q] \tag{53}$$

Here (53) follows immediately from expansion of the left-hand side of the inequality

$$[\chi_1 \mathbf{A}(\epsilon - \mathbf{A}^{-1}Q), (\epsilon - \mathbf{A}^{-1}Q)] \geq 0 \tag{54}$$

Substitution of $Q = \eta$ where η is any 2×2 symmetric matrix into (53) yields

$$(\mathbf{M}_2 - \mathbf{M}^e)\epsilon : \epsilon \geq 2\theta_1 \epsilon \cdot \eta - [\chi_1 \mathbf{A}^{-1}\eta, \eta] \tag{55}$$

From (51) we expand $\mathbf{A}^{-1}$ in (54) to obtain

$$(\mathbf{M}_2 - \mathbf{M}^e)\epsilon \cdot \epsilon \geq 2\theta_1 \eta \cdot \epsilon - \theta_1 (\mathbf{M}_2 - \mathbf{M}_1)^{-1}\eta \cdot \eta + \frac{1}{\frac{2}{3}h_2^3(\mu + k)} \int_Q \chi_1 \Gamma_1 \chi_1 \, dx \eta \cdot \eta \tag{56}$$

Inequality (56) is the Hashin–Shtrikman type bound on the effective bending rigidity for plates with stiffeners. We remark that this derivation makes no use of a comparison medium as is customarily done. It is based solely upon the positive definiteness of the operator $\mathbf{A}$. This method of deriving Hashin–Shtrikman bounds for anisotropic composites can be immediately applied to two and three dimensional elastic composites.

For our purposes we write (56) in a more convenient form

$$(\mathbf{M}_2 - \mathbf{M}^e)\epsilon \cdot \epsilon \geq \sup\{2\theta_1 \eta \cdot \epsilon - \theta_1^2 (\mathbf{M}_2 - \bar{\mathbf{M}})^{-1}\eta \cdot \eta\} \tag{57}$$

where $\bar{\mathbf{M}}$ is defined by

$$(\mathbf{M}_2 - \bar{\mathbf{M}})^{-1} = \theta_1^{-1}(\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{1}{\frac{2}{3}h_2^3(\mu + \kappa)} \frac{\int_Q \chi_1 \Gamma_1 \chi_1 \, dx}{\theta_1^2} \tag{58}$$

It now follows from (57) and the Legendre transform that for any 2×2 symmetric constant matrix ϵ

$$(\mathbf{M}_2 - \mathbf{M}^e)\epsilon \cdot \epsilon \geq (\mathbf{M}_2 - \bar{\mathbf{M}})\epsilon \cdot \epsilon$$

or

$$\mathbf{M}^e \epsilon \cdot \epsilon \leq \bar{\mathbf{M}}\epsilon \cdot \epsilon \tag{59}$$

From (59) it follows that the lower bound on the effective compliance $(\mathbf{M}^e)^{-1}$ is given by

$$\bar{\mathbf{M}}^{-1}\sigma \cdot \sigma \leq (\mathbf{M}^e)^{-1}\sigma \cdot \sigma \tag{60}$$

Similarly for a sum of compliance energies we have the lower bound

$$\sum_{i=1}^{j} \bar{\mathbf{M}}^{-1}\sigma^i \cdot \sigma^i \leq \sum_{i=1}^{j} (\mathbf{M}^e)^{-1}\sigma^i \cdot \sigma^i \tag{61}$$

We see from (58) that the tensor $\bar{\mathbf{M}}$ encodes partial information on the microstructure through the tensor $\int_Q \chi_1 \Gamma_1 \chi_1 \, dx$. Expanding this tensor using Plancherel's equality we obtain

$$\int_Q \chi_1 \Gamma_1 \chi_1 \, dx = \sum_{k \neq 0} |\hat{\chi}_1(\kappa)|^2 \hat{\Gamma}_1(\hat{\kappa}) \tag{62}$$

where

$$\sum_{k \neq 0} |\hat{\chi}_1(\kappa)|^2 = \theta_1 \theta_2 \tag{63}$$

Following Ref. 24 it is convenient to make use of (63) and write the sum in (62) as

$$\int_Q \chi_1 \Gamma_1 \chi_1 \, dx = \theta_1 \theta_2 \int_{S^1} \hat{\Gamma}(n) P(dn) \tag{64}$$

where n is a unit vector on S^1 and the positive measure μ is given by

$$P(dn) = \sum_{|l|=1} \frac{1}{\theta_1 \theta_2} \left(\sum_{\hat{k}=n} |\hat{\chi}_1(k)|^2 \delta(l - n) dn \right) \tag{65}$$

It is easily seen that μ is a probability measure.

We indicate the dependence of the tensor on the measure P by writing $\bar{M} = \bar{M}(P)$ and (58) becomes

$$(\mathbf{M}_2 - \bar{\mathbf{M}}(P))^{-1} = \theta_1^{-1}(\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{\theta_2/\theta_1}{\frac{2}{3}h_2^3(\mu + \kappa)} \int_{S^1} \hat{\Gamma}(n) P(dn) \tag{66}$$

Collecting the results above we have the following.

Theorem 3.1

The lower bound on the sum of compliance energies $\sum_{i=1}^{j} (\mathbf{M}^e)^{-1}\sigma^i \cdot \sigma^i$ over all composites with prescribed area fraction θ_2 of stiffeners is given by

$$\min_P \left\{ \sum_{i=1}^{j} (\bar{\mathbf{M}}(P))^{-1}\sigma^i \cdot \sigma^i \right\} \leq \sum_{i=1}^{j} (\mathbf{M}^e)^{-1}\sigma^i \cdot \sigma^i \tag{67}$$

where P is any measure defined by (65) and $\bar{\mathbf{M}}(P)$ is given by (66).

4 OPTIMALITY OF THE LOWER BOUND ON THE COMPLIANCE ENERGIES BY FINITE RANK LAMINATES

Attainability of the bound given in Theorem 3.1 will be established with the aid of microstructures analogous to the finite rank laminates used in two and three dimensional elasticity. There are several different attainability proofs for the Hashin–Shtrikman bounds on anisotropic elastic composites using laminates, see Refs 18–20. In this presentation, as we seek to optimize bounds on sums of energies, our arguments are motivated by those given in Ref. 19.

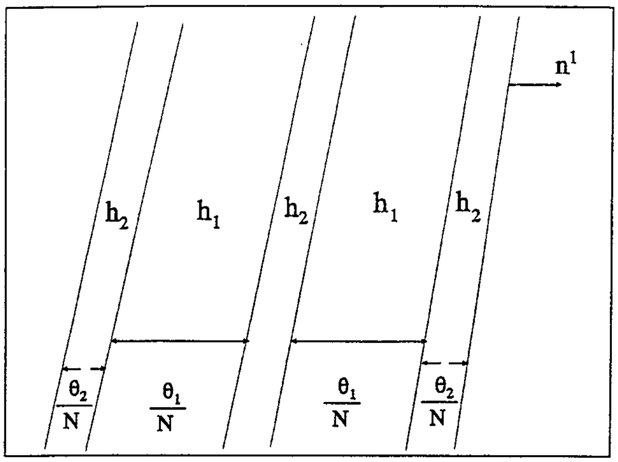

Fig. 1. A rank 1 reinforced plate. The stiffeners are of width θ_2/N and equally spaced by θ_1/N. Here n^1 is the direction perpendicular to the stiffeners and h_2 and h_1 denote the midplane thickness of stiffeners and plate respectively.

4.1 The effective rigidity of finite rank laminates

We start by deriving an explicit formula for the effective rigidity tensor of a square plate of unit length reinforced with parallel uniformly spaced identical stiffeners. We suppose that the area fraction occupied by the stiffeners is given by θ_2 and consider N stiffeners of width θ_2/N equally spaced by the distance θ_1/N. We shall denote the unit vector perpendicular to the orientation of the stiffeners by $n^{(1)}$ (see Fig. 1). The rigidity tensor for the reinforced plate is given by

$$\mathbf{M} = \chi_1 \mathbf{M}_1 + \chi_2 \mathbf{M}_2 \tag{68}$$

where $\chi_2(x) = \chi_2(n \cdot x)$ and $\chi_1 = 1 - \chi_2$. The associated local displacement solves (40) with $\mathbf{M}$ given by (68) and the effective rigidity for this geometry is denoted by $\mathbf{M}^{\mathrm{L}}$ where

$$\mathbf{M}^{\mathrm{L}}\epsilon = \int_Q \mathbf{M}(\overset{*}{E} + \epsilon)\, \mathrm{d}x \tag{70}$$

and $\overset{*}{E}_{ij} = \partial^2_{ij} w$. Due to the one dimensional nature of the thickness variation we argue that the curvatures in the plate and stiffeners are constant and given by ϵ_1 and ϵ_2 respectively. Thus the problem of computing the effective rigidity reduces to an algebraic one. The system of algebraic equations is given by

$$\mathbf{M}^{\mathrm{L}}\epsilon = \theta_1 \mathbf{M}_1 \epsilon_1 + \theta_2 \mathbf{M}_2 \epsilon_2 \tag{71}$$

$$\epsilon = \theta_1 \epsilon_1 + \theta_2 \epsilon_2 \tag{72}$$

$$(\mathbf{M}_{1_{ijkl}} \epsilon_{1_{kl}} - \mathbf{M}_{2_{ijkl}} \epsilon_{2_{kl}}) n_i n_j = 0 \tag{73}$$

and

$$\epsilon_1 - \epsilon_2 = \alpha n \otimes n \tag{74}$$

Here, n is the unit vector perpendicular to the stiffeners, eqns (71) and (72) represent the average stress and strain respectively, eqn (73) is the continuity of the effective bending moment between plate and stiffener, and equation (74) is the compatibility condition requiring that the jump in strain be a scalar multiple of the dyadic $n \otimes n$.

We now solve the system (71)–(74) to obtain a formula for $\mathbf{M}^{\mathrm{L}}$ in terms of the layer normal n. Eliminating ϵ_2 in (71) using (72) yields

$$(\mathbf{M}_2 - \mathbf{M}^{\mathrm{L}})\epsilon = \theta_1 (\mathbf{M}_2 - \mathbf{M}_1)\epsilon_1 = \theta_1 \mathbf{r} \tag{75}$$

Here we have introduced the 'polarization matrix' $\mathbf{r}$. We use (74) to write $\epsilon_2 = \epsilon_1 - \alpha n \otimes n$ and substitution of this expression into (73) gives

$$\alpha = \frac{(\mathbf{M}_2 - \mathbf{M}_1)\epsilon_1 n \cdot n}{\frac{2}{3}h_2^3(\mu + \kappa)} \tag{76}$$

From (74), (76), and (72) it is evident that

$$\epsilon = \epsilon_1 - \frac{\{\theta_2(\mathbf{M}_2 - \mathbf{M}_1)\epsilon_1 n \cdot n\} n \otimes n}{\frac{2}{3}h_2^3(\mu + \kappa)} \tag{77}$$

We use (75) to write $\epsilon = \theta_1(\mathbf{M}_2 - \mathbf{M}^{\mathrm{L}})^{-1}\mathbf{r}$ and $\epsilon_1 = (\mathbf{M}_2 - \mathbf{M}_1)^{-1}\mathbf{r}$ and from (77) it follows that

$$\theta_1(\mathbf{M}_2 - \mathbf{M}^{\mathrm{L}})^{-1}\mathbf{r} = (\mathbf{M}_2 - \mathbf{M}_1)^{-1}\mathbf{r} - \frac{\theta_2(\mathbf{r} n \cdot n\} n \otimes n}{\frac{2}{3}h_2^3(\mu + \kappa)} \tag{78}$$

We observe from (47) that $n \otimes n \otimes n \otimes n$ is precisely the symbol for the operator Γ_1 and collecting our results, we see that the effective rigidity M^{L} for this microstructure is given by

$$\theta_1(\mathbf{M}_2 - \mathbf{M}^{\mathrm{L}})^{-1} = (\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{\theta_2}{\frac{2}{3}h_2^3(\mu + \kappa)} \hat{\Gamma}_1(n^{(1)}) \tag{79}$$

At this point we could further reinforce the plate using parallel stiffeners perpendicular to a second direction $n^{(2)}$ (see Fig. 2), the idea being that the additional reinforcement will stiffen the plate in two

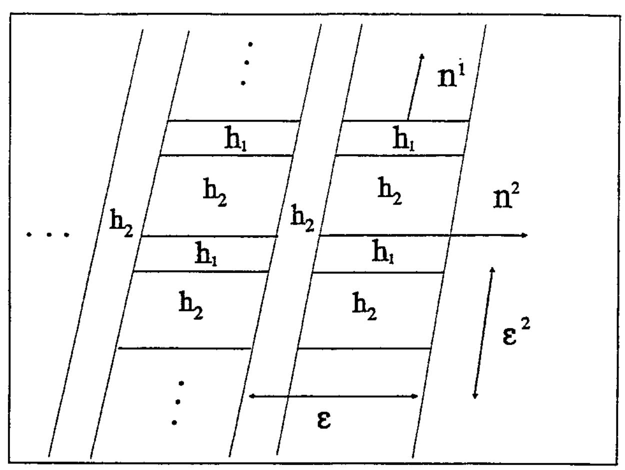

Fig. 2. A rank 2 reinforced plate. The stiffeners of width ϵ^2 are aligned perpendicular to direction n^1 the stiffeners of width ϵ are aligned perpendicular to direction n^2.

directions. In what follows, we shall obtain a mathematical formula for the effective rigidity for such geometries. However, implicit in the formula is the supposition that families of stiffeners associated with different directions oscillate on widely different length scales. The motivation for considering these structures is that they turn out to be a class of microgeometries that achieve the lower bound on the sum of compliance energies. We note that it is of great practical significance to see if there exist other simpler microgeometries for which the lower bound on the compliance energy is attained. We remark that it follows from (56) that a necessary condition for attainability is that the curvature tensor be homogeneous in the unreinforced plate regions of the composite.

We shall call a plate reinforced with j families of parallel stiffeners such that each family is perpendicular to a direction $n^{(i)} (i = 1, \dots, j)$, a finite rank j laminate. A finite rank j laminate is defined iteratively. To fix ideas we show how to construct a rank 2 laminate. Let $0 < \rho_1 < 1$ and consider a plate of midplane thickness h_1 reinforced with stiffeners of thickness h_2 and width $\rho_1 \epsilon^2$ separated by $(1 - \rho_1)\epsilon^2$. We suppose the stiffeners are perpendicular to a prescribed direction $n^{(1)}$. We then take this finely ribbed plate and along strips of width $\rho_2 \epsilon$ separated by $(1 - \rho_2)\epsilon$ we replace the finely ribbed plate with stiffeners of thickness h_2. We suppose that these stiffeners are perpendicular to a second direction n^2. The $\epsilon = 0$ limit of this geometry is called a rank 2 laminate (see Fig. 2). Finite rank laminates of higher rank are constructed in the same way.

To derive the formula for the effective rigidity for a rank j laminate we first consider an anisotropic elastic plate with rigidity $\overset{*}{\mathbf{M}} < \mathbf{M}_2$ reinforced with stiffeners of rigidity $\mathbf{M}_2$. If we assume that there are N stiffeners of thickness θ_2/N separated by a distance θ_1/N perpendicular to a direction $n^{(2)}$, then we obtain using the same analysis as before that the associated effective rigidity $\mathbf{M}^e$ is given by

$$\theta_1(\mathbf{M}_2 - \mathbf{M}^e)^{-1} = (\mathbf{M}_2 - \overset{*}{\mathbf{M}})^{-1} - \frac{\theta_2}{\frac{2}{3}h_2^3(\mu + \kappa)} \hat{\Gamma}_1(n^2) \tag{80}$$

We see from formulas (79) and (80) that by choosing $\overset{*}{\mathbf{M}} + \mathbf{M}^L$ in (80) we obtain the formula for the effective rigidity $\mathbf{M}^{L_2}$ of a rank 2 laminate with two families of stiffeners perpendicular to $n^{(1)}$ and $n^{(2)}$ given by

$$\theta_1(\mathbf{M}_2 - \mathbf{M}^{L_2})^{-1} = (\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{\theta_2(p_1\hat{\Gamma}_1(n^1) + p_2\hat{\Gamma}_1(n^2))}{\frac{2}{3}h_2^3(\mu + \kappa)} \tag{81}$$

Here $p_1 + p_2 = 1, p_1, p_2 \geq 0$ and $\theta_2 p_1$ and $\theta_2 p_2$ represent the increase in the area fraction of stiffeners as we first reinforce the plate with stiffeners of width $O(\epsilon^2)$ and then with stiffeners of width $O(\epsilon)$ respectively.

For a rank j laminate the formula for the effective rigidity $\mathbf{M}^{L_j}$ associated with a microstructure of j families of stiffeners each of width $O(\epsilon^i)$ $i = 1, \dots, j$ perpendicular to the directions n^i $i = 1, \dots, j$ is given by

$$\theta_1(\mathbf{M}_2 - \mathbf{M}^{L_j})^{-1} = (\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{\theta_2 \sum_{i=1}^{j} p_i \hat{\Gamma}(n_i)}{\frac{2}{3}h_2^3(\mu + \kappa)} \tag{82}$$

Here, θ_2 is the total area fraction of stiffeners and $\theta_2 p_i$ is the increase in area fraction of stiffeners due to the stiffeners of width $O(\epsilon^i)$.

We conclude with the definition:

Definition 4.1. The set of all compliance tensors with effective rigidity tensors associated with rank j laminates given by (82) is denoted by GL_{θ_2}.

4.2 Optimality using finite rank laminates

The key observation in proving optimality of the compliance bound given in Theorem 1 is to notice the connection between the effective rigidity tensor of a rank j laminate and the tensor $\bar{M}(P)$ appearing in the bound. To see this we rewrite the convex sum in the last term of (82) as

$$\sum_{i=1}^{j} p_i \hat{\Gamma}(n_i) = \int_{S^1} \hat{\Gamma}(n) P^j(\mathrm{d}n) \tag{83}$$

where $P^j(\mathrm{d}n)$ is a probability measure on the unit sphere defined by

$$P^j(\mathrm{d}n) = \sum_{i=1}^{j} p_i \delta(n - n^i)\,\mathrm{d}n \tag{84}$$

Thus the effective rigidity is written

$$(\mathbf{M}_2 - \mathbf{M}^{L_j})^{-1} = \theta_1^{-1}(\mathbf{M}_2 - \mathbf{M}_1)^{-1} - \frac{\theta_2/\theta_1}{\frac{2}{3}h_2^3(\mu + \kappa)} \int_{S^1} \hat{\Gamma}_1(n) P^j(\mathrm{d}n) \tag{85}$$

Now we consider the set of all tensors of the form

$$\int_{S^1} \hat{\Gamma}_1(n) P(\mathrm{d}n) \tag{86}$$

appearing in (66). This set is a convex four dimensional set. The tensors $\hat{\Gamma}(n)$ are extreme points for the set, therefore from Catheadory's theorem all tensors of the form (86) can be represented as a convex combination of at most five extreme points. It now follows immediately from (83), (85) and (86) that the set of all tensors $\bar{\mathbf{M}}(P)$ given by (66) corresponds to the set of all effective rigidity tensors for finite rank laminates and attainability of the lower bound in Theorem 3.1 is established. Summing up we have Theorem 4.1.

Theorem 4.1. Given the j bending moments σ^i, $i + 1, \dots, j$, the lower bound on the sum of effective compliance energies (66) is always attainable by finite rank laminates. Moreover, it follows that for any effective compliance tensor $\mathbf{M}^{e^{-1}}$ in G_{θ_2} there exists an effective compliance tensor $\bar{\mathbf{M}}^{-1}$ of a finite rank laminate in GL_{θ_2} such that for any set of bending moments $\sigma_1, \sigma_2, \ldots, \sigma^i$ one has

$$\sum_{i=1}^{j} \bar{\mathbf{M}}^{-1}\sigma^i \cdot \sigma^i \leq \sum_{i=1}^{j} \mathbf{M}^{e^{-1}}\sigma^i \cdot \sigma^i \tag{87}$$

5 NUMERICAL EXAMPLE

To illustrate optimal compliance design we give a numerical example. Instead of a Kirchhoff plate we consider here the stiffener reinforced Mindlin plate model of Soto & Diaz.[16] The plate and stiffeners are made from isotropic elastic material with Young's modulus $E_1 = 100$ and Poisson's ratio $\nu_1 = 0{\cdot}3$. The plate thickness is 0·05 and the stiffener reinformed plate is of thickness 0·1. The stiffened plate possesses effective elasticities associated with rank 2 laminates, with two orthogonal layer directions. The design variables for the layout are the relative volume fractions of stiffeners in each laminate and the orientation of the composite material.

The plate is clamped at the edges and the plate domain is the rectangle $-1 \leq x \leq 1$, $-\frac{1}{2} \leq y \leq \frac{1}{2}$. The area fraction occupied by the strong elastic material in the two outer plies is 35·0% and we consider the problem of minimizing the mean compliance of the plate subject to this constraint.

The plate is subjected to a random transverse load defined by a second order random process with two point correlation given by the Markovian kernel

$$\begin{aligned}\langle f(x_1) f(x_2)\rangle &= \Gamma(x_1, x_2)\\ &= \exp\left\{-\frac{|x_1^{(1)} - x_2^{(1)}|}{4}\frac{|x_1^{(2)} - x_2^{(2)}|}{3}\right\}\end{aligned} \tag{88}$$

Here (x_1^1, x_2^1) and (x_1^2, x_2^2) are the coordinate vectors of the points x_1 and x_2 and the correlation lengths are 4 and 3 along the $(1,0)$ and $(0,1)$ directions respectively. The mean load is a point of magnitude b applied at the origin

$$\bar{f}(x) = b\delta(x) \tag{89}$$

The mean zero fluctuations $g_n(x)$ and eigenvalues λ_n are found by solving the integral equation (12) with the kernel given by (88). We calculate the first five terms in the expansion for the mean zero random fluctuation and solve the approximate minimum compliance problem given by:

$$\min_{\text{Layouts}} \int_R [w(x) b\delta(x) + \sum_{n=1}^{5} \sqrt{\lambda_n} g_n(x) a_n(x)]\, dx \tag{90}$$

subject to the area constraint on the amount of strong elastic material used in the two outer plies.

Here $w(x)$, $a_1(x), \ldots, a_5(x)$ are the midplane deflections associated with the loads $\bar{f}$, g_1, $g_2, \ldots, g_5$. The mean zero fluctuations are given by

$$\begin{aligned} g_1(x) &= \sin(\omega_1^{(1)} x^{(1)}) \sin(\omega_1^{(2)} x^{(2)})/d_{11} e_{12}\\ g_2(x) &= \cos(\omega_2^{(1)} x^{(1)}) \sin(\omega_1^{(2)} x^{(2)})/d_{12} e_{12}\\ g_3(x) &= \sin(\omega_3^{(1)} x^{(1)}) \sin(\omega_1^{(2)} x^{(2)})/d_{13} e_{12}\\ g_4(x) &= \sin(\omega_1^{(1)} x^{(1)}) \cos(\omega_2^{(2)} x^{(2)})/d_{11} e_{22}\\ g_5(x) &= \cos(\omega_4^{(1)} x^{(1)}) \sin(\omega_1^{(2)} x^{(2)})/d_{14} e_{12}\end{aligned} \tag{91}$$

where

$$\begin{aligned} d_{11} &= (1 - \sin(2\omega_1^{(1)})/2\omega_1^{(1)})^{1/2}\\ d_{12} &= (1 + \sin(2\omega_2^{(1)})/2\omega_2^{(1)})^{1/2}\\ d_{13} &= (1 - \sin(2\omega_3^{(1)})/2\omega_3^{(1)})^{1/2}\\ d_{14} &+ (1 + \sin(2\omega_4^{(1)})/2\omega_4^{(1)})^{1/2}\\ e_{12} &= [1/2 - \sin(2\omega_1^{(2)})/2\omega_1^{(2)}]^{1/2}\\ e_{22} &= [1/2 + \sin(2\omega_2^{(2)})/2\omega_2^{(2)}]^{1/2}\end{aligned} \tag{92}$$

and

$$\omega_1^{(1)} = 2{\cdot}57043, \quad \omega_2^{(1)} = 3{\cdot}93516, \quad \omega_3^{(1)} = 5{\cdot}35403$$
$$\omega_4^{(1)} = 6{\cdot}81401, \quad \omega_1^{(2)} = 4{\cdot}34925, \quad \omega_2^{(2)} = 7{\cdot}08433$$

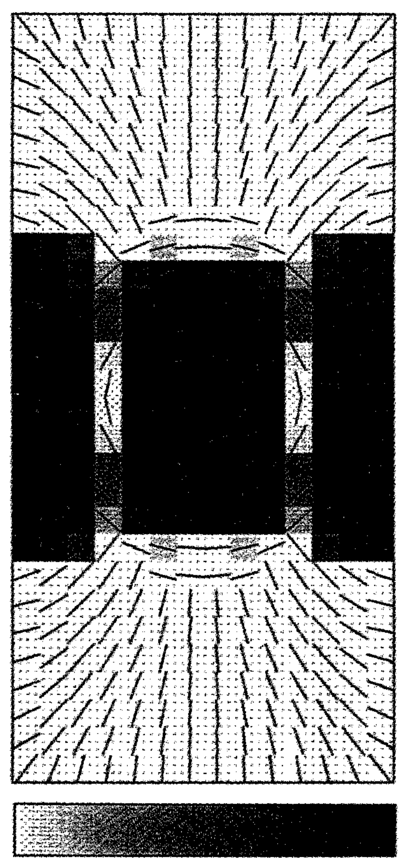

Fig. 3. Deterministic load case; point load of unit magnitude at origin.

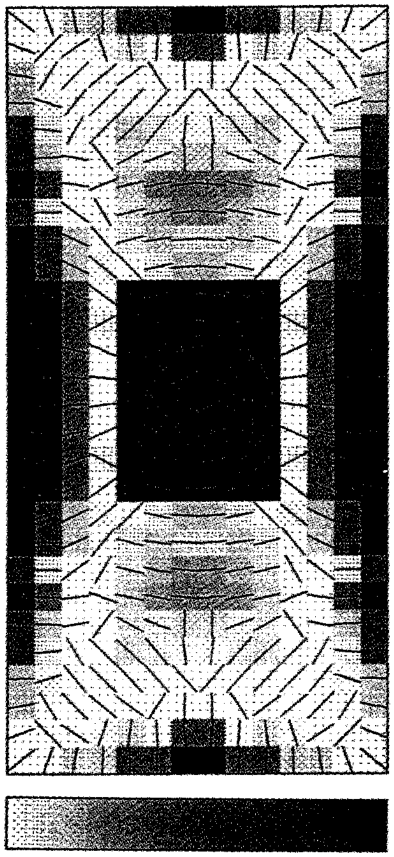

Fig. 4. Random load case (1); mean load of 0·01 at origin with fluctuations given by (91).

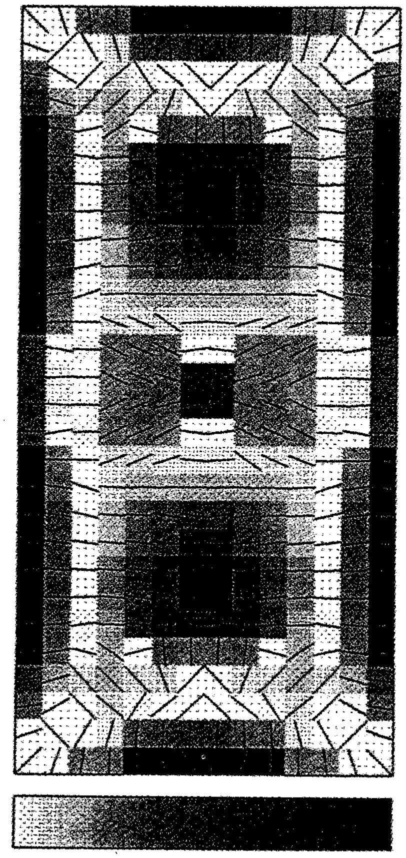

Fig. 5. Random load case (2); mean load of 0·001 at origin with fluctuations given by (91).

The associated parameters $\sqrt{\lambda_n}$ are given by

$$\sqrt{\lambda_1} = 0{\cdot}27578$$

$$\sqrt{\lambda_2} = 0{\cdot}23368$$

$$\sqrt{\lambda_3} = 0{\cdot}19620$$

$$\sqrt{\lambda_4} = 0{\cdot}18940$$

$$\sqrt{\lambda_5} = 0{\cdot}16595$$

We illustrate the associated minimum compliance layouts for three different values of the mean load. We display the density of strong elastic material in the outer two plies. Regions of pure strong material are shown as black, regions of pure weak material are shown as white. and composite regions are colored grey. The darkness of the grey regions reflects the local density of strong material in the design. Figure 3 illustrates a purely deterministic load case. Here a point load of magnitude 1 is applied to the origin. Figure 4 illustrates a random loading. Here the mean load is a point load of magnitude 0·01 at the origin and the mean zero fluctuations are given by (91). Figure 5 illustrates mean load of magnitude 0·001 at the origin with mean zero fluctuations given by (91).

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