Peculiar Ways of Ordering Commutative Rings

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Abstract. Henriksen and Isbell showed in 1962 that commutative rings that have nilpotent elements sometimes admit total orderings that violate equational laws (in the language of lattice-ordered rings) that are satisfied by all totally-ordered fields. In this paper, we construct and classify some instances of this peculiar phenomenon. We also introduce a certain construction reminiscent of algebra cohomology and show how in certain cases it detects peculiar orders.

0. Introduction

In the early 20th century, ordered fields appeared in the work of Hilbert (*Grundlagen der Geometrie*), in the work of Hahn (representations of ordered fields) and in the work of Artin and Schreier (on Hilbert's 17^{th} Problem). Since the middle of the 20th century, the study of ordered rings has been strongly tied with real algebraic geometry. Reduced rings (*i.e.*, rings without nilpotents) have been the most important algebraic objects in this context. Standard (but by no means trivial) abstract geometric methods allow one systematically to generalize the theory of ordered fields to build up a theory of orderings for reduced rings; see, *e.g.*, [SM].

Rings with nilpotents present a whole different set of problems. Important work related to the ordering of such rings seems to begin in the middle of the last century, when ordered structures were studied abstractly for their own sake. Some landmarks include: Birkhoff and Pierce's 1956 work [BP]—which defined and studied lattice-ordered rings from the perspective of universal algebra, Hion's 1957 work on generalized valuations [Hi], and the 1962 work of Henriksen and Isbell [HI] on so-called "formally real" f-rings. Henriksen and Isbell greatly deepened the connections to universal algebra, and Isbell developed the theory further in the remarkably original paper [I].

The phenomena that concern us in the present paper were first noticed by Henriksen and Isbell. Sections 1 through 4 trace ideas relevant to our central theme through the three sources just listed. At the end of section 4, we give an example of a totally-ordered finite-dimensional algebra over $\mathbb{R}$ that violates an order-theoretic law that is satisfied by all totally-ordered fields. Most of the rest of this paper is concerned with understanding how and why this example works and finding a general setting that might make sense of a large class of similar examples. In sections 5 and 6, we examine the example and its generalizations from the perspective of semigroup rings and we exhibit a twisted semigroup ring that shows that the canonical valuation of Hion may fail to detect peculiar orders. In sections 7 and 8, we examine twistings of semigroup rings explicitly and make some attempts to establish a general framework for considering peculiar orders by using a cohomology group that classifies twists up to graded isomorphism.

Rings with nilpotents arise naturally in geometric settings, as stressed by Brumfiel [B], but the connections have not yet been examined deeply. One area of current research where a better understanding is needed occurs in relation to the Pierce-Birkhoff Conjecture (PBC). This challenging problem was first stated in [BP]; a more recent source is [M]. The approach that seems most promising (to me) requires one to consider the set of all the orderings of a polynomial ring A that agree up to a certain order of vanishing with a given order. To understand the structure of this set, it would be useful to have an explicit statement of the conditions that allow an order on A/I to lift to a A, where I is a given ideal that is convex for some given order. I, of course, need not be prime. I hope to develop the connections to the PBC in a future paper.

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1. Basic facts

All rings in this paper are commutative and have a unit.

Definition. *A totally ordered ring—or* "toring" *for short—is a ring A equipped with a total order $\leq$ that satisfies the following conditions:*

- *for all $a, b, c \in A$, if $a \leq b$ then $a + c \leq b + c$, and*
- *for all $a, b, c \in A$, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$.*

A toring morphism *is an order-preserving ring homomorphism between torings. An ideal I in a toring A is said to be* "convex" *if for all $x, y \in A$,*

$$(0 \leq x \leq y \;\&\; y \in I) \Rightarrow x \in I.$$

Fact. *The kernel of any toring morphism is convex. If I is convex, then*

$$x + I \leq y + I :\Leftrightarrow x \leq y$$

unambiguously defines toring order on A/I.

Fact. *Any toring that is not a domain contains nilpotent elements.*

Proof. If $0 \leq x \leq y$ and $xy = 0$, then $x^2 = 0$. ∎

We use the word "tomonoid" to mean a totally-ordered commutative monoid, *i.e.*, a monoid with total order satisfying $x \leq y \Rightarrow x + z \leq y + z$.

2. The work of Hion (1957)

Hion's paper [Hi] defines and studies a natural "generalized valuation" on any toring. We say "generalized" because the Hion valuation takes values in a tomonoid rather than in a totally-ordered group. In this section, we shall summarize the parts of Hion's work that are relevant to our present purposes.

Definition. *Let $(A, \leq)$ be a toring and let $a, b \in A$. We say that a and b are archimedean equivalent if there are natural numbers m and n with $|b| \leq m|a|$ and $|a| \leq n|b|$.*

We leave it to the reader to verify that archimedean equivalence is an equivalence relation on A. The equivalence class of a is denoted $h(a)$. It is called the *archimedean class of* a. Let $\mathcal{H}(A)$ denote the set of all archimedean classes of A. In $\mathcal{H}(A)$, the following rules define a monoid operation and an order.

- $h(a) + h(b) := h(ab)$
- $h(a) \leq h(b)$: iff $|b| \leq |a|$

Note that h reverses order; this conforms to certain notational customs in valuation theory. We leave to the reader the straightforward verification that the definitions are independent of the representatives chosen. Thus, $\mathcal{H}(A)$ has a natural tomonoid structure. Note that $\infty := h(0)$ is the largest element of $\mathcal{H}(A)$, and it is absorbing: $\infty + h(a) = \infty$ for all a. (A monoid can have at most one absorbing element, for if x and y are both absorbing, $x = x + y = y$.)

In addition to the properties already mentioned, the reader may easily verify that $\mathcal{H}(A)$ satisfies

- $h(a + b) \geq \min\{\, h(a), h(b)\,\}$, with equality whenever $h(a) \neq h(b)$.

Thus, h possesses all the properties of a valuation, except that the target is only a monoid, not a group with an absorbing element adjoined, as in a typical valuation.

Next, we identify a property that characterizes the tomonoids that arise as $\mathcal{H}(A)$.

Definition. *Let H be a tomonoid. We call H a* "Hion tomonoid" *if it has a largest element ∞, this element is absorbing and H satisfies the following weak cancelation law: for all $x, y, z \in H$:*

$$x + z = y + z \neq \infty \quad \Rightarrow \quad x = y.$$

Theorem (Hion). *1) If A is a toring, then $\mathcal{H}(A)$ is a Hion tomonoid. 2) Moreover, for any Hion tomonoid H, there is a toring A such that $\mathcal{H}(A)$ is isomorphic to H.*

Proof of 1). Suppose $a, b, c \in A$. We prove the contrapositive. Suppose $h(a) \neq h(b)$. Without loss of generality, $h(a) < h(b)$. Then $n|b| \leq |a|$ for all $n \in \mathbb{N}$, and therefore $n|bc| \leq |ac|$ for all $n \in \mathbb{N}$. Suppose $h(bc) = h(ac)$. Pick $m \in \mathbb{N}$ such that $|ac| \leq m|bc|$. Then we get $n|bc| \leq m|bc|$ for all $n \in \mathbb{N}$, so $|bc| = 0$, so $h(b) + h(c) = \infty$.

Proof of 2). If R is a ring and S is a monoid, the monoid ring $R[S]$ is the set of all finite formal sums $r_1 X^{s_1} + \cdots + r_n X^{s_n}$, where X is an indeterminate, $r_i \in R$ and $s_i \in S$. Multiplication defined by the rule $X^s X^t = X^{s+t}$ and distributivity. If S is a tomonoid, then we say $g \in R[S]$ is in *normal form* when it is written with exponents of ascending order:

$$g = r_1 X^{s_1} + \cdots + r_n X^{s_n},$$

with $r_i \neq 0$ and $s_1 < s_2 < \cdots < s_n$. Now suppose S is a Hion tomonoid and R is a toring with no zero-divisors. Let $R[S]^*$ denote the quotient of $R[S]$ obtained by identifying X^∞ with 0, ordered in such a way that an element $a_1 X^{h_1} + \cdots$ in normal form is positive iff $a_1 > 0_R$. The Hion condition suffices to show that products of non-negative elements are non-negative (as the reader is invited to check). ∎

Definition. *The tomonoid algebra $R[S]^* := R[S]/(X^\infty)$ introduced in the proof will be called* "the Hion algebra of S over R"*.*

3. Work of Birkhoff and Pierce (1956)

Birkhoff and Pierce initiated the study of f-rings in [BP]. An f-ring is a member of the equational class of ring-with-binary-operation $\vee$, whose laws are the laws of commutative rings together with the following laws for $\vee$:

$$\begin{aligned} x \vee (y \vee z) &= (x \vee y) \vee z, \\ x \vee y &= y \vee x, \\ x \vee x &= x, \\ (x \vee y) + z &= (x + z) \vee (y + z), \\ (0 \vee z)(x \vee y) &= (0 \vee z)\, x \vee (0 \vee z)\, y. \end{aligned}$$

(In [BP], f-rings that failed to have identity were also considered. The defining identities for non-unital f-rings differ in non-obvious ways from those given above, but these subtleties are not important in the present context.)

Any toring is an f-ring with respect to the operation $x \vee y := \max\{x, y\}$, but in general an f-ring need not be totally-ordered. Besides the torings, the most extensively studied examples of f-rings are the function rings, *e.g.*, $C(X)$, the ring of all continuous real-valued functions on a topological space X. In fact, the "f" in "f-ring" stands for "function".

An ℓ-ring is a ring endowed with a binary operation $\vee$ that satisfies all but the last of the f-ring identities, above. In [BP], Birkhoff and Pierce showed that any f-ring—but not any ℓ-ring—is both a subring *and* sub-$\vee$-lattice of a product of totally ordered rings. In particular, a lattice-ring identity that is violated by an f-ring is violated by a toring. This provides a way to use the tools of universal algebra to treat questions about torings by rephrasing them as questions about f-rings.

4. Work of Henriksen and Isbell (1962)

In [HI], Henriksen and Isbell undertook a deep study of the equational theory of f-rings. We summarize the relevant parts.

Theorem 1 [HI]. *All totally-ordered fields satisfy the same lattice-ring identities, and not all of these identities are implied by the f-ring identities.*

The reader is referred to [HI] for the proof of the first assertion. Later in this section, we will present an explicit example that proves the second. Henriksen and Isbell called a toring that satisfies all lattice-ring identities satisfied by a totally-ordered field "formally real". They described a 9-generator toring that is *not* formally real and that has the additional property that all 8-generator sub-torings are formally real, thus showing that at least 9 variables are required to axiomatize the equational theory of formally real f-rings. Their example was a Hion algebra of a tomonoid with 80 elements. In 1972 Isbell [I] showed, by generalizing this example, that the equational theory of formally real f-rings does not have a base with a finite number of variables. Henriksen and Isbell also proved the following characterization of formally real torings.

Theorem 2. *A toring is formally real (i.e., satisfies all the lattice-ring identities that are true in a totally-ordered field) if and only if it is a quotient (by a convex ideal) of a totally ordered toring without zero-divisors.*

The "if" direction of the proof is easy: any toring without zero-divisors is a subtoring of a tofield, and equational laws are preserved by subalgebras and quotient algebras. We sketch the ideas in the "only if" part. Suppose A is a formally real toring. It follows from Theorem 1 that the free formally real f-ring $F(E)$ on E generators is the *sub-f-ring* of the f-ring of all $\mathbb{Q}$-valued functions on $\mathbb{Q}^E$ that is generated by the projections. We can construct a surjection $\phi : F(E) \to A$. The kernel is an ℓ-prime ℓ-group ideal, hence is contained in a minimal ℓ-prime ℓ-group ideal, J, say. But any such ℓ-group ideal is also a ring ideal, as shown in standard references on ordered algebra, such as [BKW]. The proof is completed by showing that J is in fact disjoint from the sub-*ring* $\mathbb{Z}[\,\pi_e \mid e \in E\,] \subseteq F(E)$ generated by the projections, so the quotient $F(E)/J$ is in fact a total ordering of this domain. See [HI] for details.

Example. Here is a ring-lattice identity true in all totally-ordered fields, but violated in a toring:

$$0 \;=\; 0 \vee \big(x \wedge y \wedge z \wedge (x^3 - yz) \wedge (y^2 - xz) \wedge (z^2 - x^2y)\big). \tag{1}$$

This is true in every totally-ordered field, for in a totally-ordered field it is impossible for x, y, z as well as all the binomials to be positive all at once. If the variables and the first two binomials are positive, then $x^3 > yz$ and $y^2 > xz$, so $x^3y^2 > xyz^2$. Multiplying by $x^{-1}y^{-1} > 0$, we get $x^2y > z^2$. This makes the last binomial negative. Yet identity (1) is *not* an f-ring identity. Here is a toring in which it fails. Let S be the monoid whose elements are the integers

$$0, 9, 12, 16, 18, 21, 24, 25, 27, 28, 30, 32$$

together with an absorbing element ∞. The monoid operation is standard integer addition, unless $a + b > 32$, in which case we take $a + b = \infty$. Now, order S so that

$$0 < 9 < 12 < 16 < 18 < 21 < 24 < 25 < 27 < 28 < 32 < 30 < \infty.$$

Note that 32 is not in its "usual" place. It is easy—if tedious—to check that with this order S is a Hion tomonoid. Hence we may form the Hion algebra $\mathbb{R}[S]^*$. If we set $x = X^9$, $y = X^{12}$ and $z = X^{16}$, we get a counterexample to the identity. All the strict inequalities $x^3 > yz$, $y^2 > xz$, and $z^2 > x^2y$ hold, and the right hand side of (1) is equal to $z^2 > x^2y = X^{32} - X^{30} \neq 0$. (Recall that the order in $\mathbb{R}[A]^*$ reverses the order in S.) (I found this example around 1997 and discussed it with Isbell, who was unaware at the time of an example with only three variables, though he supplied examples with four. It appears to be one of the simplest examples possible.)

I will say "the tofield identities" as a synonym for "the equational theory of formally real f-rings." Already, it is possible to state two significant conjectures. The rationale for these conjectures will be explained in the course of the remainder of this paper.

Conjecture A. *Every 2-variable tofield identity is a consequence of the f-ring identities.*

Conjecture B. *The 3-variable tofield identities do not have a finite base.*

5. Peculiarities detectable by the Hion valuation

The example we constructed in the last section depended on the creation of a Hion tomonoid with deviant properties. This raises several questions which we examine in the present section and the next: *What properties of a Hion tomonoid S are necessary and sufficient for $R[S]^*$ to be formally real? What special properties do the Hion tomonoids of formally real torings posses? If a toring fails to be formally real, must this be manifest in the Hion tomonoid of A?*

Definition. *Suppose S is a monoid and $K \subseteq S$. K is called a* "monoid ideal" *if $k \in K$ & $s \in S \Rightarrow k + s \in K$. If S is a tomonoid and K is an ideal, we say that K is* "convex" *if $x \geq y \in K \Rightarrow x \in K$.*

Lemma. *Let S be a tomonoid with convex ideal K. On the set $(S \setminus K) \cup \{\infty\}$, there is a tomonoid operation + defined by*

$$a + b := \begin{cases} a +_S b, & \text{if } a +_S b \notin K; \\ \infty & \text{if } a = \infty \text{ or } b = \infty \text{ or } a +_S b \in K. \end{cases}$$

This tomonoid is denoted S/K and it is called a truncation *of S at K.*

In case $K = \{ x \mid x \geq a \}$, we denote the resulting tomonoid S/a. Note that K may be empty, in which case the construction adjoins to S a new element that is absorbing. If S already contains an absorbing element a and K is empty, then in S/K, a is no longer absorbing since $a + \infty = \infty$.

Definition. *A* "formally real tomonoid" *is a tomonoid that is isomorphic (as a tomonoid) to a truncation of a subtomonoid of a totally ordered abelian group.*

Observe that every formally real tomonoid is Hion. The following propositions explain the connection to formally real torings.

Proposition. *If S is a formally real tomonoid and R is a totally-ordered domain, then $R[S]^*$ is a formally real toring.*

Proof. If S were a counterexample, $R[S]$ would contain finitely many elements violating a tofield identity. These elements would be contained in a subalgebra of $R[S]$ of the form

$R[S']$ with $S' \subseteq S$ finitely generated, providing a finitely generated counterexample. Thus, it suffices to prove the assertion when S is finitely generated. In this case, $S = T/K$ is a truncation of a subtomonoid T of a totally ordered group with underlying group $\mathbb{Z}^n$. Now, $R[T]$ is a totally-ordered domain since it is a subring of the ring of Laurant polynomials $R[X_1^{\pm 1}, \ldots X_n^{\pm i}]$. Also, $R[T]$ has a natural toring order induced by the order on T. Finally there is a natural, order-preserving surjection $R[T] \to R[S]^*$. ▪

Proposition. *If A is a formally real toring then $\mathcal{H}(A)$ is formally real.*

Remark. The converse is false. An example will be given in the next section.

Proof. There a toring surjection $\phi : D \to A$, where D is a totally ordered domain. Let F be the ordered field of fractions of D. Then $\mathcal{H}(D) \subseteq \mathcal{H}(F)$, and the latter (with ∞ removed) is a totally ordered group. Now it suffices to show that if $h(\phi(x)) = h(\phi(y))$, then either $h(x) = h(y)$ or $h(\phi(y)) = \infty$. Suppose $x, y \in D^+$, $h(x) < h(y)$ and $\phi(y) \neq 0$. Then $x > ny$ for all $n \in \mathbb{Z}$, and $x > z$ for all $z \in \ker \phi$. It follows that $h(\phi(x)) < h(\phi(y))$. ▪

Corollary. *Let S be a Hion tomonoid. Then S is formally real as a tomonoid if and only if $\mathbb{R}[S]^*$ is formally real as a toring.* ▪

There is an important class of tomonoids that is closely related to the formally real tomonoids but is properly larger. A tomonoid is *formally integral* if its order lifts to one—hence to any—free monoid of which it is an image. This property is explored at length in [E]. A formally real tomonoid is formally integral, but not very formally integral tomonoid is Hion and even if it is, it need not be formally real.

Here is an example of a formally integral Hion tomonoid that is not formally real. Let U be the quotient of $\langle 9, 12, 16 \rangle / 33$ obtained by identifying 30 and 32. Using a, b and c to denote 9, 12 and 16, respectively, we have:

$$U = \{\, 0 < a < b < c < 2a < a + b < 2b < a + c < 3a < b + c < 2a + b = 2c < \infty \,\}.$$

U is a formally integral Hion tomonoid but it is not an formally real, since $2a + b < 2c$ in any totally ordered group in which $2b < a + c$ and $3a < b + c$.

In her 1999 LSU Ph.D. dissertation, Gretchen Whipple showed that every positive 2-generator Hion tomonoid is formally real. The first step of the proof was to show that every 2-generator tomonoid is formally integral. This itself is a non-trivial result. A proof, based on a kind of Euclidean algorithm applied to lattice vectors, can be found in [E]. Whipple's work is the first piece of circumstantial evidence for Conjecture A. A weaker conjecture is:

Conjecture A.1. *The Hion tomonoid of any 2-generator toring is formally real.*

Note that the Hion tomonoid of a 2-generator ring may fail to be generated *as a monoid* by 2 elements. (For example, the Hion tomonoid of $\mathbb{R}[\,t^4, t^6 + t^7\,]$ $(0 < t << 1)$ requires three generators.) Nonetheless, it satisfies extremely strong conditions that have been extensively investigated in the study of plane curve singularities; see [C]. In particular, Theorem 6.1 of [E] seems to apply to these tomonoids, showing that they are at least formally integral.

6. A non-formally real toring whose Hion tomonoid is formally real.

The example is a twisted monoid algebra. Such objects have been studied recently under the name "binomial algebras" by Sturmfels and others; see [S]. Years ago, Anderson and Ohm pointed out that the ways of twisting a monoid algebra are classified by the second cohomology of the monoid with coefficients in the group of units of the ring of scalars; see [AO]. We will discuss the implications of this after the example. What seems interesting here is the connection between cohomology and the structure of orderings.

Example. Let $A = A_{a,b,c} := \mathbb{R}[X, Y, Z]/J$, where

$$J = \langle\, X^3 - aYZ,\ Y^2 - bXZ,\ Z^2 - cX^2Y,\ X^2Z,\ X^4,\ X^3Y \,\rangle.$$

Let x, y and z stand for the residues of X, Y and Z in A. Assigning the degrees 3, 4 and 5 to the variables X, Y and Z, respectively, A is graded by the tomonoid

$$H := \{\, 0, 3, 4, 5, 6, 7, 8, 9, 10, \infty \,\}.$$

We have

$$A = A_0 \oplus A_3 \oplus A_4 \oplus \cdots \oplus A_{10},$$

and $\dim_{\mathbb{R}} A_i = 1$ for $i = 0, 3, 4, \ldots, 10$. Assuming that $0 < a, b, c$, we can totally order A as a ring by requiring that x, y, z be positive (thus determining the order of each graded piece) and extending to A lexicographically. That is, we put

$$0 \le \lambda_0 + \lambda_3 x + \lambda_4 y + \lambda_5 z + \lambda_6 x^2 + \lambda_7 xy + \lambda_8 y^2 + \lambda_9 x^3 + \lambda_{10} x^2 y$$

if all the coefficients vanish or the first non-zero coefficient is positive. Note that the Hion tomonoid of A is H, and H is obviously formally real. (Indeed, $H \cong \langle 3, 4, 5\rangle/11$.)

Assertion. *With the order described above, $A_{a,b,c}$ is formally real if and only if $abc = 1$.*

Proof. If $abc = 1$, then A is isomorphic to the tomonoid algebra $\mathbb{R}[H]^*$, which is a quotient of $\mathbb{R}[t^3, t^4, t^5]$, ordered so that $0 < t << 1$. For the converse, suppose that $abc \neq 1$. Let $\phi : \mathbb{R}[X, Y, Z] \to A$ be defined by $\phi(f) := f + J$. It suffices to show that there is no total order $\le$ on $\mathbb{R}[X, Y, Z]$ such that $f \le g \Rightarrow \phi(f) \le \phi(g)$. We treat the case when $abc < 1$, the case $abc > 1$ admitting an analogous treatment. Pick $\delta > 1$ so that $abc\delta^2 < 1$. If $\le_0$ is a total order on $\mathbb{R}[X, Y, Z]$ preserved by ϕ, then

$$0 <_0 X,\ 0 <_0 Y,\ 0 <_0 Z,$$

$$X^3 <_0 a\delta YZ \text{ and } Y^2 <_0 b\delta XZ,$$

and hence

$$X^2Y <_0 ab\delta^2 Z^2.$$

Thus, $\phi(X^2Y) \le_A ab\delta^2\phi(Z^2) = abc\delta^2\phi(X^2Y) < \phi(X^2Y)$. But this is obviously impossible, so no order on $\mathbb{R}[X, Y, Z]$ preserved by ϕ exists. ∎

Conjecture B (stated at the end of §2) comes from considering the possible modifications of the example above. Suppose that σ_1, σ_2 and σ_3 form a minimal generating set for a submonoid $S \subseteq \mathbb{N}$. Let k be a field, and consider the monoid algebra $k[S] \cong k[X_1, X_2, X_3]/I$, graded by the weighting $w(X_1^{n_1} X_2^{n_2} X_3^{n_3}) := \sigma_1 n_1 + \sigma_2 n_2 + \sigma_3 n_3$. As shown by Herzog [H], I has either two or three minimal generators. In the former case, I is a *complete intersection*. The latter case is *generic* (see [MS], page 187) and in this case the generators of I are of the form

$$\begin{aligned} b_1 &:= X_1^{n_{11}} - X_2^{n_{12}} X_3^{n_{13}} \\ b_2 &:= X_2^{n_{22}} - X_1^{n_{21}} X_3^{n_{23}} \\ b_3 &:= X_3^{n_{33}} - X_1^{n_{31}} X_2^{n_{32}} \end{aligned}$$

where the $n_{ij} > 0$ and n_{ii} is the smallest exponent of X_i that occurs in an element of I with the same structure as b_i, *i.e.*, a difference between a power of X_i an a monomial in the other two variables. It can be shown (see [H]) that $n_{11} = n_{21} + n_{31}$, $n_{22} = n_{12} + n_{32}$ and $n_{33} = n_{13} + n_{23}$. Thus, the b_i satisfy the following syzygies:

$$\begin{aligned} 0 &= X_2^{n_{32}} b_1 + X_3^{n_{13}} b_2 + X_1^{n_{21}} b_3, \\ 0 &= X_3^{n_{23}} b_1 + X_1^{n_{31}} b_2 + X_2^{n_{12}} b_3, \end{aligned}$$

of weight $\sigma_2 n_{32} + \sigma_1 n_{11}$ and $\sigma_3 n_{23} + \sigma_1 n_{11}$, respectively. The weights of the syzygies are, of course, greater than the weights of any of the generators. Let J be the ideal:

$$\langle X_1^{n_{11}} - aX_2^{n_{12}} X_3^{n_{13}},\ X_2^{n_{22}} - bX_1^{n_{21}} X_3^{n_{23}},\ X_3^{n_{33}} - cX_1^{n_{31}} X_2^{n_{32}} \rangle \vee J_0,$$

where J_0 is the monomial ideal generated by the monomials whose weight is greater than or equal to the weight of a syzygy. Arguing as in the example, we see that $\mathbb{R}[X_1, X_2, X_3]/J$ has a total order and that if $abc \neq 1$, then this order violates the identity:

$$0 \;=\; 0 \vee \big(X_1 \wedge X_2 \wedge X_3 \wedge b_1 \wedge b_2 \wedge b_3\big). \qquad \mathrm{Id}\,(\sigma_1, \sigma_2, \sigma_3)$$

One conjectures that the identities $\mathrm{Id}\,(\sigma_1, \sigma_2, \sigma_3)$ are independent of one another. There is no apparent way that the violations of the tofield identities in these examples could all be deduced from some finite set of identities, hence Conjecture B.

7. Twisted monoid algebras and their graded equivalence.

Here, we extend the meanings of some standard ideas from homological algebra just slightly to be able to apply them to the constructions we have been making. The deeper meaning of some of this is not entirely clear, and finding the right setting and development for the theory of $H^n(S, I, A)$ (defined below) is an open problem. It is apparent that the syzygies of the ideal defining $R[S, I]$ are involved.

Monoid algebras. Let R be a commutative ring and S a monoid. $R[S]$ denotes the semigroup ring over S with coefficients from R. This is the set of formal polynomials $\sum_{i=1}^{n} \lambda_i X^{s_i}$, where $\lambda_i \in R$, $s_i \in S$ and X is a formal symbol. Multiplication in $R[S]$ is determined by allowing scalars to commute with X^a for all $a \in S$, the distributive law and the rule $X^a X^b = X^{a+b}$. In particular, $X^0 = 1_{R[S]}$. In fact, $R[\_]$ is a functor from the category of commutative monoids to commutative R-algebras. If $\phi : S \to T$ is a monoid homomorphism, then $R[\phi] : R[S] \to R[T]$ is given by: $R[\phi]\Big(\sum_{i=1}^n \lambda_i X^{s_i}\Big) = \sum_{i=1}^n \lambda_i X^{\phi(s_i)}$.

Cocycles, coboundaries and H^2 of a monoid. Let A be an abelian group (which we write multiplicatively), and S a monoid.

Definition. *A function $\tau : S \times S \to A$ is called an A-valued symmetric 2-cocycle if for all $a, b, c \in S$:*

$$\tau(a,b) = \tau(b,a), \quad \text{and}$$

$$\tau(a,b)\tau(a+b,c) = \tau(a,b+c)\tau(b,c).$$

It is obvious that the product of a pair of symmetric 2-cocycles is a symmetric 2-cocycle. Every symmetric cocycle clearly has an inverse and hence the collection of all symmetric 2-cocycles forms a group. It is denoted $Z^2(S,A)$.

Definition. *If $\gamma : S \to A$ is any function, then*

$$\beta(a,b) := \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}$$

is an element of $Z^2(S,A)$. The set of all β determined in this manner by functions $\gamma : S \to A$ is denoted $B^2(S,A)$. It is a subgroup of $Z^2(S,A)$.

Definition. $H^2(S,A) := Z^2(S,A)/B^2(S,A)$.

Twisted monoid algebras. Let R be a commutative ring with 1 and let $U(R)$ be the group of units of R.

Definition. *A $U(R)$-valued symmetric 2-cocycle is called a twist; see [AO]. If τ is a twist, the twisted monoid algebra $R[S,\tau]$ is the set of formal polynomials $\sum_{i=1}^n \lambda_i X^{s_i}$, where multiplication is determined by allowing scalars to commute with X^a for all $a \in S$, the distributive law and the rule*

$$X^a X^b = \tau(a,b) X^{a+b}.$$

The two defining conditions (i) and (ii) are necessary and sufficient for this rule to determine a commutative and associative multiplication, as the reader can easily verify. We say that $R[S,\tau_1]$ and $R[S,\tau_2]$ are *graded equivalent* if there is an R-algebra isomorphism $\phi : R[S,\tau_1] \to R[S,\tau_2]$ such that for each $s \in S$ there is $\gamma(s) \in U(R)$ with $\phi(X^s) = \gamma(s)X^s$.

Lemma 1. *Twisted monoid algebras $R[S,\tau_1]$ and $R[S,\tau_2]$ are graded equivalent if and only if $\tau_2/\tau_1 \in B^2(S,U(R))$. Thus, the elements of $H^2(S,U(R))$ are in one-to-one correspondence with graded equivalence classes.*

Lemma 2. *If $\tau \in Z^2(S,U(R))$, then $R[S,\tau]$ is isomorphic to $R[X_i : i \in I]/J$, where J is a proper binomial ideal, i.e., an ideal generated by polynomials of the form $X^\alpha - uX^\beta$, with $u \in U(R)$.*

The proofs of these lemmas are presented in §8.

Example. In this example, $F_3 = \{0, 1, 2\}$ is the 3-element field and $Z_2 = \{0, 1\}$ is the 2-element group.

$$F_3[Z_2] \cong F_3[X]/(X^2 - 1),$$

which is not a field. Let $\tau(0,0) = \tau(0,1) = \tau(1,0) = 1$ and let $\tau(1,1) = 2$. It is easy to check that this is a twist and that $F_3[Z_2, \tau]$ is isomorphic to $F_3[X]/(X^2+1)$, the 9-element field. Actually, it is not hard to write down all the functions from $Z_2 \times Z_2$ to $U(F_3)$, and to determine which are in $Z^2(Z_2, U(F_3))$ and $B^2(Z_2, U(F_3))$. When one does this, one finds that $H^2(Z_2, U(F_3)) \cong Z_2$, so we have found all the graded equivalence classes. ▪

Cohomology relative to an ideal. We generalize the results of the last sub-section so that we can apply them to algebras of the form $R[S]^*$. Recall that in this construction, which first appeared in the proof in §2, we "throw away" the terms corresponding to X^∞. This means that some of the information on a full cocycle on S is irrelevant. The best way of handling this is not clear, but here is an attempt to create a framework with reasonable generality.

Let S be a monoid and let $I \subseteq S$ be an ideal in S, *i.e.*, $i + s \in I$ whenever $i \in I$ and $s \in S$. We define an *A-valued symmetric 2-cocycle on* (S, I) to be a map

$$\tau : \{\, (a, b) \in S \times S \mid a + b \notin I \,\} \to A$$

that satisfies conditions (i) and (ii) of the previous section whenever all the values of τ in the statements of these conditions are defined. $Z^2(S, I, A)$ denotes the group of all symmetric 2-cocycles on (S, I), while $B^2(S, I, A)$ denotes the subgroup consisting of the symmetric 2-cocycles of the form

$$\beta(a, b) = \frac{\gamma(a + b)}{\gamma(a)\gamma(b)},$$

where $\gamma : S \setminus I \to A$ is any function. Finally , $H^2(S, I, A)$ denotes $Z^2(S, I, A)/B^2(S, I, A)$.

From monoids-with-ideal to rings. We make the following definitions:

1) $R[S, I] := R[S]/\langle\, X^s \mid s \in I \,\rangle$
2) $R[S, I, \tau]$ denotes $R[S, I]$ with the twisted multiplication induced by $\tau \in Z^2(S, I, A)$.
3) $R[S, I, \tau_1]$ and $R[S, I, \tau_2]$ are *graded equivalent* if there is an R-algebra isomorphism $\phi : R[S, I, \tau_1] \to R[S, I, \tau_2]$ such that for each $s \in S \setminus I$ there is $\gamma(s) \in U(R)$ with $\phi(X^s) = \gamma(s)X^s$.

Lemma 3. *$H^2(S, I, U(R))$ is in one-to-one correspondence with graded equivalence classes of algebras $R[S, I, \tau]$.*

Lemma 4. *$R[S, I, \tau]$ is isomorphic to $R[X_i : i \in I]/J$, where J is an ideal generated by monic monomials and polynomials of the form $X^\alpha - uX^\beta$, with $u \in U(R)$.*

The proofs of these lemmas are analogous to the proofs of Lemmas 1 and 2, given below.

Example. Let $S = \langle\, 3, 4, 5 \,\rangle = \{\, 0, 3, 4, 5, 6, \ldots\}$, the submonoid of the non-negative integers generated by 3, 4 and 5. Let $I = \{\, 11, 12, 13, \ldots\}$. If $\mathbf{Q}$ denotes the additive group of rational numbers, then an explicit calculation shows that $H^2(S,I,\mathbf{Q}) \cong \mathbf{Q}$. Thus, if $U(\mathbb{R})$ is the multiplicative group of the reals, $H^2(S,I,U(\mathbb{R})) \cong U(\mathbb{R})$. If τ is a (real-valued) twist on (S,I), then $\mathbb{R}[S,I,\tau] \cong \mathbb{R}[X,Y,Z]/J$, where

$$J = \langle\, X^3 - aYZ,\ Y^2 - bXZ,\ Z^2 - cX^2Y,\ X^2Z,\ X^4,\ X^3Y\,\rangle,$$

and $a = \frac{\tau(3,3)\,\tau(3,6)}{\tau(4,5)}$, $b = \frac{\tau(4,4)}{\tau(3,5)}$ and $c = \frac{\tau(5,5)}{\tau(3,3)\,\tau(6,4)}$. It is easy to show that abc is invariant under graded isomorphism, thus the cohomology class of $R[S,I,\tau]$ is determined by this number. ∎

This example obviously generalizes to other generic submonoids $\langle\sigma_1,\sigma_2,s_3\rangle$, analogous to the extensions of the example at the end of §6.

8. Proofs of Lemmas 1 and 2.

The following lemma and its proof are a restatement of [S], Lemma 4.1.

Lemma 0. *For any monoid homomorphism $\phi : S \to T$, $\ker R[\phi]$ is generated by the set*

$$B(\phi) := \{\, X^a - X^b \mid \phi(X^a) = \phi(X^b)\,\}.$$

Proof. It is clearly the case that $B(\phi) \subseteq \ker R[\phi]$. Let us prove the opposite inclusion first in the case that $S = \mathbb{N}^{(E)}$. Suppose $\ker R[\phi] \setminus \langle B(\phi)\rangle \neq \emptyset$. Fix a term order on $\mathbb{N}^{(I)}$, and pick $f = \lambda_1 X^{a_1} + \lambda_2 X^{a_2} + \lambda_\ell X^{a_\ell} \in \ker R[\phi] \setminus \langle B(\phi)\rangle$ ($\lambda_i \neq 0$) such that the leading term X^{a_1} of f is minimal among the leading terms of the elements of $\ker R[\phi] \setminus \langle B(\phi)\rangle$. Now, $R[\phi](f) = \lambda_1 X^{\phi(a_1)} + \lambda_2 X^{\phi(a_2)} + \cdots + \lambda_\ell X^{\phi(a_\ell)} = 0$. We may assume that $\phi(a_1) = \phi(a_i)$ if and only if $i = 1,\ldots,k$ (where $k > 1$). Then $f + \lambda_1(X^{a_2} - X^{a_1}) \in \ker R[\phi] \setminus \langle B(\phi)\rangle$, yet its leading term is less than that of f, a contradiction. For the general case, let $\pi : \mathbb{N}^{(E)} \to S$ be a surjection. The kernel of $R[\phi\circ\pi] = R[\phi]\circ R[\pi]$ is generated by the $X^a - X^b$ with $\phi(\pi(a)) = \phi(\pi(b))$. The kernel of $R[\phi]$, therefore, is generated by the images of these elements under $R[\pi]$, which is what we wanted to prove. ∎

Proof of Lemma 1. Suppose there is an R-algebra isomorphism

$$\phi : R[S,\tau_1] \to R[S,\tau_2]$$

such that for each $s \in S$ there is $\gamma(s) \in U(R)$ with $\phi(X^s) = \gamma(s)X^s$. Then

$$\phi(X^aX^b) = \phi(\tau_1(a,b)X^{a+b}) = \gamma(a+b)\tau_1(a,b)X^{a+b}$$

and

$$\phi(X^a)\phi(X^b) = \gamma(a)X^a\gamma(b)X^b = \gamma(a)\gamma(b)\tau_2(a,b)X^{a+b}.$$

Because the left-hand sides are equal,

$$\gamma(a+b)\tau_1(a,b) = \gamma(a)\gamma(b)\tau_2(a,b),$$

or $\frac{\tau_2(a,b)}{\tau_1(a,b)} = \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}$, so $\tau_2/\tau_1 \in B^2(S,U(R))$. For the converse, assume $\frac{\tau_2(a,b)}{\tau_1(a,b)} = \frac{\gamma(a+b)}{\gamma(a)\gamma(b)}$ for some $\gamma : S \to U(R)$. Define $\phi(X^s) := \gamma(s)X^s$ and extend by linearity to $R[S,\tau_1]$. Then ϕ is an isomorphism of R-modules. Moreover, in the equations above the right-hand sides are equal, and this shows that ϕ preserves multiplication—hence is a ring isomorphism. ∎

Proof of Lemma 2. Any constant function $S^2 \to U(R)$ is in $B^2(S,U(R))$. Thus every graded equivalence class is represented by a τ such that

$$\tau(0,0) = 1.$$

In the proof, therefore, we will assume $\tau(0,0) = 1$.

Let s_i, $i \in E$ be a set of generators for S, and let $\pi : \mathbb{N}^{(E)} \to S$ be the monoid morphism induced by sending the generator ϵ^i of $\mathbb{N}^{(E)}$ to s_i. If $\alpha \in \mathbb{N}^{(E)}$, we write X^α as shorthand for $\prod_{i\in E} X_i^{\alpha_i} \in R[\,X_i \mid i \in E\,]$. Let $\phi : R[\,X_i \mid i \in E\,] \to R[S,\tau]$ be the homomorphism induced by sending X^α to $X^{\pi(\alpha)}$.

Now, let $T : \mathbb{N}^{(E)} \to U(R)$ be defined as follows: First, set $T_0 = 1$ and set $T_{\epsilon^i} = 1$ for each generator ϵ^i. This defines T on all elements of degree 0 and 1 in $\mathbb{N}^{(E)}$ (where the degree of α is $\sum_{i\in E} \alpha_i$). Assuming T has been defined on elements of degree less than k, and $\alpha + \beta$ has degree k, we set

$$T_{\alpha+\beta} := \tau(\pi(\alpha), \pi(\beta))T_\alpha T_\beta.$$

That this is well-defined is guaranteed by the identities that τ is required to satisfy. In detail, note first that the second identity for τ gives directly that $T_{(\alpha+\beta)+\gamma} = T_{\alpha+(\beta+\gamma)}$, whenever $T_{(\alpha+\beta)}$, T_γ, T_α and $T_{(\beta+\gamma)}$ are defined. Now if $\gamma = \alpha + \beta = \alpha' + \beta'$, then let $\gamma_1 = \alpha \wedge \alpha'$, $\gamma_2 = 0 \vee (\alpha - \alpha')$ $\gamma_3 = 0 \vee (\alpha' - \alpha)$ and $\gamma_4 = \beta \wedge \beta'$. Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, $\alpha = \gamma_1 + \gamma_2$ and $\alpha' = \gamma_1 + \gamma_3$. Then $T_{\alpha+\beta} = T_{\gamma_1+(\gamma_2+\gamma_3+\gamma_4)} = T_{\gamma_1+(\gamma_3+\gamma_2+\gamma_4)} = T_{\alpha'+\beta'}$.

Now, note that

$$\phi(X^\alpha) = T_\alpha X^{\pi(\alpha)}.$$

Let $J \subseteq R[\,X_i \mid i \in E\,]$ be the ideal generated by $\{\, T_\alpha^{-1} X^\alpha - T_\beta^{-1} X^\beta \mid \pi(\alpha) = \pi(\beta)\,\}$. We claim that $J = \ker\phi$. If $\pi(\alpha) = \pi(\beta)$, then $\phi(T_\alpha^{-1} X^\alpha - T_\beta^{-1} X^\beta) = X^{\pi(\alpha)} - X^{\pi(\beta)} = 0$, so $J \subseteq \ker\phi$. On the other hand, suppose $\ker\phi \setminus J \neq \emptyset$. Fix a term order on $\mathbb{N}^{(E)}$, and pick $f = \lambda_1 X^{\alpha_1} + \lambda_2 X^{\alpha_2} + \cdots + \lambda_\ell X^{\alpha_\ell} \in \ker\phi\setminus J$ ($\lambda_i \neq 0$) such that the leading term X^{α_1} of f is minimal among the leading terms of the elements of $\ker\phi\setminus J$. Since $f \in \ker\phi$, we must have $\pi(\alpha_1) = \pi(\alpha_i)$ for some i. We may assume that $\pi(\alpha_1) = \pi(\alpha_i)$ if and only if $i = 1, \ldots, k$ (where $k > 1$). Now, $\phi(f) = \lambda_1 T_{\alpha_1} X^{\pi(\alpha_1)} + \lambda_2 T_{\alpha_2} X^{\pi(\alpha_2)} + \cdots + \lambda_\ell T_{\alpha_\ell} X^{\pi(\alpha_\ell)} = 0$. Then $f + \lambda_1(T_{\alpha_1} T_{\alpha_2}^{-1} X^{\alpha_2} - X^{\alpha_1}) \in \ker\phi \setminus J$, yet its leading term is less than that of f, a contradiction. ∎

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