Composition-closed ℓ-groups of almost-piecewise-linear functions

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Abstract. We construct some non-essential H-closed epireflections of $\mathbf{W}$, the category of archimedean ℓ-groups with weak order unit, that are not comparable with any known H-closed epireflections of $\mathbf{W}$ other than the divisible hull and the epicompletion. In preparation, we show that the free objects in any H-closed epireflective subcategory of $\mathbf{W}$ must be closed under composition (see section 2 for a precise definition), and that any epic extension of a finite- or countably-generated free $\mathbf{W}$-object that is closed under composition is actually a free object in some H-closed epireflective subcategory of $\mathbf{W}$. We apply these results to certain ℓ-groups of almost-piecewise-linear Baire functions on $\mathbb{R}$ to obtain the examples. By definition, a function $f : \mathbb{R} \to \mathbb{R}$ is *almost-piecewise-linear* if there is a finite point set $S \subset \mathbb{R}$ such that f is piecewise-linear on the complement of any neighborhood of S.

MSC classification: primary 06F20, secondary 06D22, 18B30, 26A30

Key words: epireflection, monoreflection, archimedean ℓ-group, Yosida Theorem, closed under composition

0. Introduction

Since the 1970s, many mathematicians have been investigating and classifying epireflective subcategories of $\mathbf{W}$, the category of archimedean ℓ-groups with weak order unit (see [H1], [BH2], [BH3], [MV]) and related categories (see [MS]). Examples in $\mathbf{W}$ include: divisible $\mathbf{W}$-objects, archimedean vector-lattices (with weak order unit), archimedean f-rings (see [HR]), CCC-rings (see [H1]), and epicomplete archimedean f-rings (see [MV] and [BH2]), while examples in the category of reduced partially-ordered rings include: f-rings, real-closed rings and epicomplete abstract semialgebraic function rings (see [MS]). The work in hand presents some new examples that illuminate the structure of the lattice of H-closed monoreflections of $\mathbf{W}$. In particular, we show that there are H-closed monoreflections of $\mathbf{W}$ that are properly stronger than the divisible hull, yet are not rings. Our work also uncovers some elementary—but interesting—facts concerning the algebra of function composition. In general, we know little about the structure of the ℓ-group we obtain if we adjoin an arbitrary almost-piecewise-linear function (defined in the abstract) to the ℓ-group of all piecewise-linear functions on $\mathbb{R}$ and then close under composition and the ℓ-group operations. Nevertheless, in this paper we succeed in identifying some classes of almost-piecewise-linear functions for which this operation yields an ℓ-group whose structure we can fully analyze.

1. Preliminaries

In this section, we present notation and background information. The subsection on the Yosida Theorem reformulates and systematizes some basic ideas that were first presented in [HM].

Reflective subcategories. Let $\mathbf{C}$ be a category and let $\mathbf{R}$ be a full, isomorphism-closed subcategory. Because in this paper $\mathbf{R}$ is always assumed to be full, we do not distinguish between the class of objects of $\mathbf{R}$ and the category itself. We say $\mathbf{R}$ is *reflective in* $\mathbf{C}$ if for every $\mathbf{C}$-object A, there is an $\mathbf{R}$-object rA and a morphism $\rho_A : A \to rA$ with the following universal mapping property: for any morphism β from A to an $\mathbf{R}$-object B, there is a unique morphism $\overline{\beta} : rA \to B$ such that $\beta = \overline{\beta}\rho_A$. If for all A in $\mathbf{C}$ the reflection morphism ρ_A is an epimorphism, i.e., right-cancellable, we say that $\mathbf{R}$ is an *epireflection*. The concept of a *monoreflection* is defined analogously. It is well-known that every monoreflection is an epireflection; see [HS]. If the objects of $\mathbf{C}$ are sets (with some additional structure), we say $\mathbf{R}$ is *H-closed* if B belongs to $\mathbf{R}$ whenever there is a surjective morphism $f : A \to B$ with A in $\mathbf{R}$. For a systematic exposition of the general theory of epireflections, see [HS].

Unital archimedean ℓ-groups. $\mathbf{W}$ denotes the category of archimedean ℓ-groups with weak order unit and unit-preserving ℓ-homomorphisms. In $\mathbf{W}$, monomorphisms are injective but epimorphisms need not be surjective; see [BH1]. Every $\mathbf{W}$-object A is torsion-free, and hence has a divisible hull $\mathbb{Q} \otimes A$. For any set X, $F(X)$ denotes the sub-ℓ-group of the ℓ-group of all functions from $\mathbb{R}^X$ to $\mathbb{R}$ that is generated by the coordinate projections and the constant function 1. It is known—see [H1]—that $F(X)$, with the constant function 1 as weak unit, is the free $\mathbf{W}$-object on X. In other words, any set map from X to a $\mathbf{W}$-object has a unique extension to a $\mathbf{W}$-morphism from $F(X)$ to that $\mathbf{W}$-object.

The Yosida Representation Theorem states that every $\mathbf{W}$-object A is isomorphic to an ℓ-group of almost-everywhere-defined continuous $\mathbb{R}$-valued functions on a compact Hausdorff space $Y(A)$, and every $\mathbf{W}$-morphism $\phi : A \to B$ is induced by a continuous map $Y(\phi) : Y(B) \to Y(A)$. The localic version of the Yosida Theorem (see [HM]) states that every A is isomorphic to an ℓ-group $\widehat{A}$ of continuous $\mathbb{R}$-valued functions on a regular Lindelöf locale $\mathcal{Y}(A)$, and every $\mathbf{W}$-morphism $\phi : A \to B$ is induced by a continuous map $\mathcal{Y}(\phi) : \mathcal{Y}(B) \to \mathcal{Y}(A)$ such that

$$\widehat{\phi(a)} = \widehat{a} \circ \mathcal{Y}(\phi). \tag{1.1}$$

In the cases of interest in the present paper, the Yosida locales have dense point sets and the locale morphisms are completely determined by their behavior on these sets. Specifically, let n be a positive integer or ω, and let $\mathcal{Y}^n$ denote $\mathcal{Y}\big(F(n)\big)$, the Yosida locale of the free $\mathbf{W}$-object on n generators. Then $\mathcal{Y}^n$ is homeomorphic to $\mathbb{R}^n$, where the latter has the usual (Tychonoff) topology; indeed, if $x_i \in F(n)$ is the ith generator, then

$$(\widehat{x_1}, \widehat{x_2}, \ldots, \widehat{x_n}) : \mathcal{Y}^n \to \mathbb{R}^n$$

is a homeomorphism; see [HM]. (This includes the case $n = \omega$, where we should write $(\widehat{x_1}, \widehat{x_2}, \ldots)$.) Since $F(n)$ is by definition an ℓ-group of functions on $\mathbb{R}^n$, the distinction between $F(n)$ and its representation $\widehat{F(n)}$ is usually inconsequential, but we will use the hat when the context makes it natural to do so.

Let μ be a monoreflection of $\mathbf{W}$ and, maintaining the assumption that $n \in \{1, 2, \ldots, \omega\}$, let $\mathcal{Y}^n_\mu$ denote the Yosida locale of $\mu F(n)$. The reflection morphism $\mu_{F(n)}$ induces a locale morphism $f^n_\mu : \mathcal{Y}^n_\mu \to \mathcal{Y}^n$. As discussed in [HM], the point set $\mathrm{pt}\,\mathcal{Y}^n_\mu \subseteq \mathcal{Y}^n_\mu$ is dense, and f^n_μ maps $\mathrm{pt}\,\mathcal{Y}^n_\mu$ bijectively onto $\mathrm{pt}\,\mathcal{Y}^n = \mathbb{R}^n$. Thus, we may identify $\mathrm{pt}\,\mathcal{Y}^n_\mu$ with $\mathbb{R}^n$. Note that the topology that

$\mathbb{R}^n$ inherits from $\mathcal{Y}_\mu^n$ may be stronger than the Tychonoff. In fact, μ is essential (see [H1]) if and only if $\mathcal{Y}(\mu_A)$ is a homeomorphism for all A in **W**.

The H-closed epireflections of **W** have a very nice description, which we now present; more detail may be found in [H2], [HM] and the references of these papers. Every **W** object is the image of some $F(X)$ under a surjective **W**-morphism, so each H-closed epireflection of **W** is completely determined by its action on the free objects $F(X)$. Hager has shown that it suffices to consider countably generated free objects; see [H1]; the essential argument for this is also presented in Lemma 4.2 of [HM]. Because each $F(n)$ is a retract of $F(\omega)$, any H-closed epireflection ρ of **W** is completely determined by $\rho(F(\omega))$.

Another useful description of the H-closed epireflections of **W** was presented by Hager in [H2]. Suppose $e : A \to A'$ is a morphism in **W**. A **W**-object B is said to be e-injective if it satisfies the following condition: for any morphism $g : A \to B$, there is a morphism $g' : A' \to B$ such that $g = g'e$. If E is a class of morphisms, then B is said to be E-injective—or to belong to the *injectivity class* $\mathbf{Inj}(E)$—if it is injective for each morphism in E. The main theorem in [H2] applies to **W**, yielding the following:

Proposition. *A class* **R** *of* **W**-*objects forms an H-closed epireflective subcategory if and only if* $\mathbf{R} = \mathbf{Inj}(E)$ *for some class E of* **W** *epimorphisms, each with domain some $F(n)$, n a positive integer or ω.*

"Arity". Assume that $(\mathbf{R}, \rho)$ is an H-closed epireflection. We say that **R** has *arity* n, (with n finite or ω) if: $\mathbf{R} = \mathbf{Inj}(E)$ for some E in which all the domains are free of rank n or less and $R \neq \mathbf{Inj}(E')$ if all the domains in E' have rank strictly less that n. In principle, **R** could fail to have finite arity, and yet not have arity ω, in which case, we say the arity is unbounded but not ω. Since $F(m)$ is a retract of $F(n)$ when $m < n$, it follows that if **R** has arity n, then $\mathbf{R} = \mathbf{Inj}(\rho_n)$, where $\rho_n : F(n) \to \rho F(n)$ is the reflection morphism.

2. Injectivity classes and closure under composition

Throughout this section, we assume $n \in \{1, 2, \ldots, \omega\}$. The strongest monoreflection of **W** (that is to say, the monoreflective subcategory with smallest object class) consists of the epiclosed **W**-objects. It is known that the epireflection $\epsilon F(n)$ of $F(n)$ is isomorphic to the ℓ-group $B(\mathbb{R}^n)$ of Baire functions on $\mathbb{R}^n$. This follows from a theorem of Ball, Comfort, Garcia-Ferreira, Hager, van Mill and Robertson (see [B]) as described in [HM]. The discussion in section 1 of the present paper shows how the Yosida representation makes this isomorphism explicit.

The reflection $\mu F(n)$ of $F(n)$ under *any* monoreflection μ is isomorphic to a sub-ℓ-group of $B(\mathbb{R}^n)$; this follows from Lemma 1.2 in [HM] or 8.1 of [MS]. Suppose A is an epic extension of $F(n)$ within $B(\mathbb{R}^n)$. We will show that there is an epireflection ρ such that $A = \rho F(n)$ if and only if A has the property in the following definition:

Definition. A set of functions $A \subseteq \mathbb{R}^{\mathbb{R}^I}$ is said to be *closed under composition* if for any $f \in A$ and any indexed set $(g_i | i \in I)$ of elements of A, the composition $f(\ldots, g_i, \ldots)$ belongs to A.

Lemma 1. *If μ is a monoreflection in* **W**, *then* $\widehat{\mu F(n)} \subseteq B(\mathbb{R}^n)$ *is closed under composition.*

Proof. Suppose $a, b_1, b_2, \ldots \in \mu F(n)$. In this proof, we let $\widehat{a}$, $\widehat{b}_i$, which properly are real-valued functions on $\mathcal{Y}_\mu^n$, stand for elements of $B(\mathbb{R}^n)$ by restricting their domains to pt $\mathcal{Y}_\mu^n = \mathbb{R}^n$. Let $\beta : F(n) \to \mu F(n)$ be the **W**-morphism determined by sending the generator x_i of $F(n)$ to b_i. Then by (1.1), $\widehat{b}_i = \widehat{x}_i \circ \mathcal{Y}(\beta)$. Since the x_i are the coordinate projections on $\mathbb{R}^n$, we have $\mathcal{Y}(\beta) = (\ldots, \widehat{b}_i, \ldots)$. Now let $\overline{\beta} : \mu F(n) \to \mu F(n)$ be the canonical extension. Then

$$\widehat{a}(\ldots, \widehat{b}_i, \ldots) = \widehat{a} \circ \mathcal{Y}(\overline{\beta}) = \widehat{\overline{\beta}(a)} \in \widehat{\mu F(n)}. \qquad /////$$

3

Lemma 2. *Suppose A is an epic extension of $F(n)$ within $B(\mathbb{R}^n)$. Let $e : F(n) \subseteq A$ be the containment. If A is closed under composition, then A is e-injective, and A is the reflection of $F(n)$ in the epireflection $\mathbf{Inj}(e)$.*

Proof. Let $g : F(n) \to A$ be any $\mathbf{W}$-morphism, and set $b_i := g(x_i) \in A$. Since A is closed under composition, if $a \in A$, then $a(\ldots, b_i, \ldots) \in A$. Let $\overline{g} : A \to A$ be defined by $\overline{g}(a) := a(\ldots, b_i, \ldots)$. Because composition respects the operations of $\mathbf{W}$ (e.g., $(a + a') \circ b = (a \circ b) + (a' \circ b)$), $\overline{g}$ is a $\mathbf{W}$-morphism, and $\overline{g}e(x_i) = x_i(\ldots, b_i, \ldots) = b_i$, so $\overline{g}e = g$. Thus, A is e-injective. Let $\rho_n : F(n) \to \rho F(n)$ be the reflection morphism for $\mathbf{Inj}(e)$. Since $A \in \mathbf{Inj}(e)$, there is $\overline{e} : \rho F(n) \to A$ such that $\overline{e}\rho_n = e$, and since $\rho F(n) \in \mathbf{Inj}(e)$, there is $\overline{\rho_n} : A \to \rho F(n)$ such that $\overline{\rho_n}\rho_n = e$. Now, $\overline{\rho_n} \circ \overline{e} = \mathrm{id}_{\rho F(n)}$ by the universal mapping property of $\rho F(n)$, and $\overline{e} \circ \overline{\rho_n} = \mathrm{id}_A$ because e is an epimorphism. Thus $\rho F(n) \cong A$. /////

The arity of the epireflection $\mathbf{Inj}(e)$ appearing in the lemma is at most n, and it may be strictly less than n. The hypotheses of the lemma assure that $\mathbf{Inj}(e)$ is a *mono*reflective subcategory. To prove this, it suffices to show that every W-object is contained in an $\mathbf{Inj}(e)$-object. Every $\mathbf{W}$-object is contained in its epicompletion. We will show that every *epicomplete* W-object is e-injective. Suppose that E is epicomplete and $g : F(n) \to E$ is any W-morphism. Then g factors as $g = \overline{g}\epsilon_{F(n)}$, where $\epsilon_{F(n)} : F(n) \to \epsilon F(n)$ is the epicompletion morphism and $\overline{g} : \epsilon F(n) \to E$. As noted earlier, $\epsilon F(n) = B(\mathbb{R}^n)$. Let g' be the restriction of $\overline{g}$ to A. This shows that E is e-injective. (I thank A. W. Hager for pointing this argument out to me.) It is possible to find injective epimorphisms $f : F(n) \to A$, where A is not isomorphic to a sub-ℓ-group of $B(\mathbb{R}^n)$. We give another example at the end of this paper. For such A, $\mathbf{Inj}(f)$ is not a monoreflection.

3. Piecewise-linear and almost-piecewise-linear functions

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *piecewise-$\mathbb{Q}$-linear* if there are finitely many rational numbers $q_1 < q_2 < \ldots < q_s$ and $m_i, b_i \in \mathbb{Q}$, $i = 0, \ldots, s$ such that

$$f(x) = \begin{cases} m_0\, x + b_0, & \text{if } x \in (-\infty, q_1]; \\ m_i\, x + b_i, & \text{if } x \in [q_i, q_{i+1}] \text{ for } i = 1, \ldots s - 1; \\ m_s\, x + b_s, & \text{if } x \in [q_s, \infty). \end{cases}$$

Note that a piecewise-$\mathbb{Q}$-linear function is continuous. (The shared endpoints of the intervals in the definition guarantee this.) The set of piecewise-$\mathbb{Q}$-linear functions from $\mathbb{R}$ to $\mathbb{R}$ is the divisible hull $\mathbb{Q} \otimes F(1)$ of $F(1)$. If $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$, we say that f is piecewise-$\mathbb{Q}$-linear if f is the restriction to X of some piecewise-$\mathbb{Q}$-linear function on $\mathbb{R}$.

In the present section, we will construct an ℓ-group $M \subseteq B(\mathbb{R})$ that is closed under composition and contains discontinuous functions that are *almost-piecewise-$\mathbb{Q}$-linear*, in the sense that for each $f \in M$, there is a finite set S of rational points such that f is piecewise-$\mathbb{Q}$-linear on the complement of any neighborhood of S. Essentially, our strategy is to add a single function, m (which we will define momentarily), to $\mathbb{Q} \otimes F(1)$ and then close under the $\mathbb{Q}$-vector-lattice operations and composition. The function m is chosen to have properties that assure that this closure has a manageable description. For each $q \in \mathbb{Q}$, M will also contain functions f and g such that

$$\lim_{x \to q^-} f(x) = +\infty \quad \text{and} \quad \lim_{x \to q^+} g(x) = +\infty.$$

As we explain more fully below, this implies that M is a $\mathbf{W}$-epic extension of $F(1)$. Thus, it produces a 1-ary H-closed epireflection of $\mathbf{W}$.

Definition. Let $m : \mathbb{R} \to \mathbb{R}$ be the function whose graph consists of the sets:

- $\{(x, 2) \mid x \geq 1\}$,
- the line segments joining $(\frac{1}{n}, 2^n)$ and $(\frac{1}{n+1}, 2^{n+1})$ for $n = 1, 2, 3, \ldots$, and
- $\{(x, 0) \mid x \leq 0\}$.

For any constant $k > 0$, let $m_k(x) := m(kx)$.

Remark. We have the following formula for m:

$$m(x) = \begin{cases} 2, & \text{if } x \geq 1; \\ -n(n+1)2^n\, x + (n+2)2^n, & \text{if } \frac{1}{n+1} \leq x \leq \frac{1}{n}; \\ 0, & \text{if } x \leq 0. \end{cases}$$

Remark. Note that

$$\lim_{x \to 0^+} m_k(x) = +\infty.$$

The function m_k is piecewise-$\mathbb{Q}$-linear on $[q, \infty)$ for any $q > 0$, but it is not piecewise-$\mathbb{Q}$-linear on any interval $(0, q]$, $q > 0$ because it has "infinitely many pieces."

Lemma 1. *For all $x \in (0, 1/k)$,*

$$2^{\frac{1}{kx}} \leq m_k(x) \leq 2 \cdot 2^{\frac{1}{kx}}.$$

Proof. It is enough to prove this for $k = 1$, since the general result follows from this case by substituting kx for x. Now, for $x \in (0, 1)$

$$2^{\frac{1}{x}} \leq m(x)$$

because the graph of m over $(0, 1)$ consists of secants to the graph of the convex function $f(x) = 2^{\frac{1}{x}}$. The inequality

$$m(x) \leq 2 \cdot 2^{\frac{1}{x}}$$

can be seen as follows. Assume $\frac{1}{n+1} \leq x \leq \frac{1}{n}$. Then $n + 1 \geq \frac{1}{x} \geq n$, so $1 + \frac{1}{x} \geq n + 1$, so $2^{1+\frac{1}{x}} \geq 2^{n+1} \geq m(x)$. /////

Lemma 2. *Suppose $k, \ell \in \mathbb{R}$, $k > \ell > 0$. Then, for any $N \in \mathbb{N}$, there is $q > 0$ such that*

$$m_\ell(x) > N\, m_k(x), \text{ for all } x \in (0, q].$$

Proof. Pick $s \in \mathbb{N}$ such that $2^{s-1} > N$. Then, if $x > 0$,

$$2^{\frac{1}{\ell x}} \geq 2^s \cdot 2^{\frac{1}{kx}} \iff \frac{k - \ell}{k \ell s} \geq x.$$

Thus, if $0 < x < \frac{k-\ell}{k \ell s}$, then by Lemma 1,

$$m_\ell(x) \geq 2^{\frac{1}{\ell x}} \geq 2^s \cdot 2^{\frac{1}{kx}} > N \cdot 2 \cdot 2^{\frac{1}{kx}} \geq N\, m_k(x). \qquad /////$$

Corollary. *The functions x, 1 and m_k, $0 < k \in \mathbb{Q}$, are linearly independent over $\mathbb{R}$.*

Lemma 3. *Suppose $f : (0, \infty) \to \mathbb{R}$ is of the form*

$$f(x) = c_0\, x + c_1 + c_2\, m_{k_2}(x) + \cdots + c_n\, m_{k_n}(x),$$

5

where $c_i, k_i \in \mathbb{Q}$ and $k_2 > k_3 > \cdots > k_n > 0$ and $c_n > 0$. Then there is $q > 0$ such that $f(x) > 0$ for all $x \in (0, q]$.

Proof. This is immediate from Lemma 2, which says that the order of the pole of m_{k_n} at 0^+ exceeds the orders of the poles of all the other terms of f. $/////$

Let G_{0+} be the $\mathbb{Q}$-vector-space of functions on $\mathbb{R}$ spanned by the functions x, 1 and m_k, $0 < k \in \mathbb{Q}$. Let M_{0+} be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ for which there is a rational number $0 < q$ such that:
- $f = g$ on $(0, q]$ for some $g \in G_{0+}$, and
- f is piecewise-$\mathbb{Q}$-linear on $(-\infty, 0]$ and on $[q, \infty)$.

Remarks:
 i) M_{0+} includes all the piecewise-$\mathbb{Q}$-linear functions on $\mathbb{R}$.
 ii) If $f \in M_{0+}$, then outside every neighborhood of 0, f is piecewise-$\mathbb{Q}$-linear.
 iii) f is continuous from the left at 0 but $\lim_{x \to 0^+} f(x)$ may be infinite or may be finite and different from $f(0)$.
 iv) If $\lim_{x \to 0^+} f(x)$ is finite, then f is piecewise-$\mathbb{Q}$-linear on $[0, \infty)$. (*Proof.* Lemma 2 implies that the only elements of G_{0+} with finite limit from the right at 0 are the linear functions $f(x) = ax + b$.)

Lemma 4. M_{0+} *is a $\mathbb{Q}$-vector-lattice under the pointwise operations. It is generated as a $\mathbb{Q}$-vector-lattice by 1, x and the functions m_k, $0 < k \in \mathbb{Q}$.*

Proof. Clearly, M_{0+} is a $\mathbb{Q}$-vector-space, so for the first assertion it suffices to prove that if $f \in M_{0+}$, then $f \vee 0 \in M_{0+}$. But this follows from Lemma 3 and Remarks *i*) and *ii*) just above. Now suppose $f \in M_{0+}$. We seek to show that f is in the $\mathbb{Q}$-vector-lattice generated by 1, x and the functions m_k, $0 < k \in \mathbb{Q}$. It suffices to treat the case where $f = 0$ on $(-\infty, 0]$, since we may add or subtract from f a piecewise-linear function that agrees with f on $(-\infty, 0]$ without affecting membership in M. If $\lim_{x \to 0^+} f(x) = 0$, then f is piecewise-$\mathbb{Q}$-linear, so it remains to consider the case where there is a discontinuity at 0, and since we can multiply by -1, it suffices to treat the following cases:

- Case 1. $0 < \lim_{x \to 0^+} f(x) = a \in \mathbb{Q}$. Then f is piecewise-linear on $(0, \infty)$. Let h be the piecewise-$\mathbb{Q}$-linear function that coincides with f on $(0, \infty)$ and is equal to the constant a on $(-\infty, 0]$. Pick $0 < b \in \mathbb{Q}$ so large that $m(x) \vee b\,x > f(x)$ on $(0, \infty)$. Then $f(x) = (m(x) \vee bx) \wedge h(x)$.
- Case 2. $\lim_{x \to 0^+} f(x) = +\infty$. By Case 1, we can assume $f = g$ on $(0, q]$, where $g = c_2 m_{k_2} + \cdots + c_n m_{k_n} \in G_{0+}$. Note that $f(x) = g(x) = 0$ for all $x \in (-\infty, 0]$. Pick $0 < b \in \mathbb{Q}$ so large that $g(x) \vee b\,x \geq f(x)$ for all $x > 0$. Let q_0 be the smallest positive solution to $b\,x = f(x)$. Let h be the piecewise-$\mathbb{Q}$-linear function that is identically 0 on $(-\infty, q_0]$ and that is equal to $f(x) - b\,x$ for $x > q_0$. Then $f(x) = (g(x) \vee b\,x) + h(x)$ for all $x \in \mathbb{R}$. $/////$

Definition. For any $k, q \in \mathbb{Q}$, $k \neq 0$, let

$$m_{k,q}(x) := m(k(x - q)).$$

Let M be the sub-$\mathbb{Q}$-vector-lattice of $\mathbb{R}^{\mathbb{R}}$ generated by 1, x and the functions $m_{k,q}(x)$.

Note that $m_{k,q}$ is the result of applying m after an affine transformation of $\mathbb{R}$. We will show in Lemma 5 that we can recognize the elements of M directly. Let $M_{q+} := \{\, g(x - q) \mid g \in M_{0+} \,\}$ and $M_{q-} := \{\, g(q - x) \mid g \in M_{0+} \,\}$. Note that M_{q+} is the translation of M_{0+} by q, and M_{q-} is the reflection of M_{q+} in the line $x = q$. By Lemma 4, M_{q+} is generated as a $\mathbb{Q}$-vector-lattice by the functions 1, $(x - q)$, and $m_{k,q}$, $0 < k \in \mathbb{Q}$, and M_{q-} is generated as a $\mathbb{Q}$-vector-lattice by the functions 1, $(q - x)$, and $m_{k,q}$, $0 > k \in \mathbb{Q}$.

Lemma 5. $f \in M \Leftrightarrow$ there is a finite set $q_1 < q_2 < \cdots < q_s$ of rational numbers such that on each interval $[q_i, q_{i+1}]$, $i = 1, \ldots s-1$, f coincides with an element of $M_{q_i^+}$ or of $M_{q_{i+1}^-}$ and on $(-\infty, q_1]$ $([q_s, +\infty))$ f coincides with an element of $M_{q_1^-}$ $(M_{q_s^+})$.

Proof. The direction $\Rightarrow$ follows from the fact that each element of M is defined using only finitely many of the functions $m_{k,q}$. The other direction follows from the fact that we can represent a function of the kind described as a sum of elements of M_{q^+} and M_{q^-}. For each $i = 1, \ldots, s-1$, we choose q_i', q_i'' such that $q_i < q_i' < q_i'' < q_{i+1}$ and we choose an element of $M_{q_i^+}$ that agrees with f on (q_i, q_i') and is 0 on $(-\infty, q_i]$ and on $[q_i', \infty)$; similarly, we choose an element of $M_{q_i^-}$ that agrees with f on (q_i'', q_{i+1}) and is 0 on $(-\infty, q_i'']$ and on $[q_{i+1}, \infty)$. Also, to the left of q_1 and to the right of q_s, we choose functions in $M_{q_1^-}$ and $M_{q_s^+}$ that agree with f on $(-\infty, q_1)$ and (q_s, ∞) and are 0 elsewhere. If we subtract all these functions from f, we get a piecewise-linear function, showing that f is indeed in M. $/////$

Lemma 6. If $f, g \in M$, then $f \circ g \in M$.

Proof. Since M is a $\mathbb{Q}$-vector-lattice, it suffices to show that if $g \in M$, then $m_{k,q} \circ g \in M$ for all functions $m_{k,q}$. We can further reduce our task to showing that $m \circ g \in M$, since M is closed under composition on the right or left by affine transformations of $\mathbb{R}$. Now, the zero-set of g consists of finitely many points and rational intervals (which may be open, closed, half-open or half-infinite in the standard topology on $\mathbb{R}$). Let $\{ w_i \mid i = 1, \ldots s \}$ be the set of all rational numbers such that

$$g(w_i) = 0, \quad \lim_{x \to w_i^-} g(x) = 0 \quad \text{and } g(x) > 0 \text{ on some interval } (v_i, w_i).$$

Similarly, let $\{ y_i \mid i = 1, \ldots s \}$ be the set of all rational numbers such that

$$g(y_i) = 0, \quad \lim_{x \to y_i^+} g(x) = 0 \quad \text{and } g(x) > 0 \text{ on some interval } (y_i, z_i).$$

(An isolated point in the zero set of g may give rise to a w_i and/or a y_j.) We may choose the v_i and z_i so that g is linear on $[v_i, w_i]$ and on $[y_i, z_i]$. On the complement of the intervals (v_i, w_i) and (y_i, z_i), $m \circ g$ is piecewise-$\mathbb{Q}$-linear, while on each interval $[v_i, w_i]$ or $[y_i, z_i]$, $m \circ g = m_{k,q}$ for some rationals k and q. $/////$

Summarizing the lemmas, we have the following theorem:

Theorem. M is a $\mathbb{Q}$-vector-lattice of almost-piecewise-$\mathbb{Q}$-linear functions on $\mathbb{R}$ that is closed under composition and that lies between $\mathbb{Q} \otimes F(1)$ (the $\mathbb{Q}$-vector-lattice of all piecewise-$\mathbb{Q}$-linear functions on $\mathbb{R}$) and $B(\mathbb{R})$, the ℓ-group of all real-valued Baire functions on $\mathbb{R}$. The elements of M have at most finitely many discontinuities, which occur only at rational numbers. Every rational number is a point of discontinuity for some element of M. The elements of M have left and right limits (possibly infinite) at all points of discontinuity, and for each rational number q there is an element M that tends to infinity when approaching q from the left and an element M that tends to infinity when approaching q from the right.

4. $F(1) \subseteq M$ **is epic**

In order to show that M is an epic extension of $F(1)$, we will use the criterion that Ball and Hager proved in [BH1]. In order to apply it, we will need to know the Yosida spaces of $F(1)$ and of M. To determine these spaces, we appeal to the following lemma, which is also described in [BH].

Lemma 1. *Let X be a compact Hausdorff space, let $D(X)$ denote the set of all continuous $\mathbb{R} \cup \{\pm\infty\}$-valued functions on X that are finite on a dense set, and let A be a sub-$\mathbf{W}$-object of $D(X)$ that contains 1 as weak unit and separates points. Then $Y(A) \cong X$.* $/////$

From the lemma, it is immediate that $Y(F(1)) = Y(\mathbb{Q} \otimes F(1)) = \mathbb{R} \cup \{\pm\infty\}$. We simply note that if f is piecewise-linear on $\mathbb{R}$, then there is a positive constant K such that f is affine linear on $(-\infty, -K)$ and on (K, ∞). Thus the limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$ both exist as elements of $\mathbb{R} \cup \{\pm\infty\}$. Therefore, we can extend f to the compact domain $\mathbb{R} \cup \{\pm\infty\}$ by defining the values at the endpoints according to the limits, and we have the conditions of the lemma. Thus, $Y(F(1)) \cong \mathbb{R} \cup \{\pm\infty\}$.

To describe the Yosida space of M requires more work. The points of $\mathbb{R}$ at which elements of M have discontinuities have multiple pre-images in $Y(M)$. However, all functions in M have right and left limits at all points of $\mathbb{R}$, and this suggests that there ought to be at most three pre-images. We confirm this shortly, but first we introduce a general construction that always yields a compact Hausdorff space.

Let X be an arbitrary subset of $\mathbb{R}$. For each point $x \in X$ we add to $\mathbb{R}$ two additional points, denoted x^- and x^+. Let $X^- := \{\, x^- \mid x \in X \,\}$, $X^+ := \{\, x^+ \mid x \in X \,\}$, and let

$$\mathbb{R}_X := \mathbb{R} \cup X^- \cup X^+ \cup \{-\infty, +\infty\}.$$

We order $\mathbb{R}_X$ by making $-\infty$ the smallest element, $+\infty$ the largest, giving the reals the usual order and, for each $x \in X$, placing x^- less than x but larger than all $y \in \mathbb{R}$—and if they exist, y^- and y^+—if $y < x$. Similarly, x^+ is greater than x but less than y, y^- and y^+ if $x < y \in \mathbb{R}$. We give $\mathbb{R}_X$ the weakest topology in which the following sets are open:

i) singletons $\{x\}$, for all $x \in X$;
ii) intervals of the form $(a, b) := \{\, z \in \mathbb{R}_X \mid a < z < b \,\}$, with $a, b \in \mathbb{R}_X$.
iii) intervals of the form $(a, +\infty]$ and of the form $[-\infty, a)$, with $a \in \mathbb{R}_X$.

Note that if $x \in X$, intervals of the form $(a, x^-] = (a, x)$ form an open neighborbood base at x^-. Similarly, intervals of the form $[x^+, b) = (x, b)$ form an open neighborbood base at x^+.

Lemma 2. *For any $X \subseteq \mathbb{R}$, $\mathbb{R}_X$ is a compact Hausdorff space.*

Proof. The referee suggests the following argument: since a chain is compact in its interval topology if and only if it is complete, and since $\mathbb{R}_X$ is evidently complete, it is compact. The lemma can also be proved by imitating the usual proof of the compactness of $[0, 1]$, but there are numerous details and special cases to attend to, and it is laborious. $/////$

Proposition 1. $Y(M) = \mathbb{R}_\mathbb{Q}$.

Proof. It suffices to show that M is is isomorphic to a point-separating sub-ℓ-group of $D(\mathbb{R}_\mathbb{Q})$, with weak unit 1. By construction, every element of $f \in M$ extends to a $\mathbb{R} \cup \{\pm\infty\}$-valued function on $\mathbb{R}_\mathbb{Q}$ by defining $f(q^-)$ to be the limit of $f(x)$ as x approached q from below and defining $f(q^+)$ to be the limit of $f(x)$ as x approached q from above. $/////$

Proposition 2. $F(1) \subseteq M$ *is epic in* $\mathbf{W}$.

Proof. This is immediate from the criterion for $\mathbf{W}$-epimorphisms presented in [BH1]. $/////$

5. Final Remarks.

The same kind of construction that we have used to make M enables us to make numerous variants. For example, we can construct a **W**-epic extension of $F(1)$ that is closed under composition and is not a $\mathbb{Q}$-vector-lattice by restricting the $m_{k,q}$ to be of the form $k \in \mathbb{Z}$ and $kq \in \mathbb{Z}$. The function m itself is actually piecewise $\mathbb{Z}$-linear on the complement of any neighborhood of the origin. If we adjoin m to $F(1)$ and close under the ℓ-group operations and composition, we get an ℓ-group of almost-piecewise-$\mathbb{Z}$-linear functions, which is contained in M. Every rational point is a point of discontinuity for some element of this ℓ-group, so in terms of Yosida spaces, we don't get an example that is different from M. On the other hand, we can make examples that are larger than M by adjoining m to $k \otimes F(1)$, where k is a subfield of $\mathbb{R}$, and then taking the least k-vector-lattice that contains M and is closed under composition. In this case, we generate functions that have discontinuities at points of k. In particular, if $k = \mathbb{R}$, we get have discontinuities at all points of $\mathbb{R}$.

Other modifications of the basic construction are possible by using functions that have yet higher-order poles at 0 than $2^{1/x}$. It appears that if these are chosen with care, ℓ-groups with similar properties to M may be constructed. For example, in place of m (or in addition to m), use a function whose graph contains the line segments joining $(\frac{1}{n}, 2^{2^n})$ and $(\frac{1}{n+1}, 2^{2^{n+1}})$ for $n = 1, 2, 3, \ldots$.

We close by providing the example promised at the end of Section 2. *Example.* Let J be the ℓ-ideal of M consisting of the functions that are non-zero at only finitely many points. Then $F(1) \subseteq M/J$. The injectivity class of this embedding is obviously not a monoreflection, since the associated reflection morphism $\rho_M : M \to \rho M$ is not injective.

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