Henriksen and Isbell on f-rings

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Abstract. This paper gives an account of the contributions of Melvin Henriksen and John Isbell to the abstract theory of f-rings and formally real f-rings, with particular attention to the manner in which their work was framed by universal algebra. I describe the origins of the Pierce-Birkhoff Conjecture and present some other unsolved problems suggested by their work.

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A conjecture or hypothesis may become as significant in a mathematician's legacy as a finished piece of work, as the manner in which we refer to great mathematical questions by the names of great mathematicians attests. In his work on f-rings, Mel Henriksen contributed to the base of existing knowledge and also raised questions that are driving some of the most challenging and intriguing research I know.

An *f-ring*—the name is short for "function ring"—is a subring of a product of totally-ordered rings that is also closed under the natural lattice operations. These objects were first named and studied systematically by Birkhoff and Pierce in their paper [BP]. In [HI], Henriksen and Isbell picked up where Birkhoff and Pierce left off, proving several deep results about the equational theory of f-rings and adding many important results on the structure of f-rings, as well. In the present essay, I will concentrate on the former theme and the unanswered questions it leads to. For a presentation of the structure theory, one may consult [BKW], section 9.4.

The notorious problem now known as the "Pierce-Birkhoff Conjecture" was first formulated by Henriksen and Isbell during their collaboration on [HI]. I heard the story directly from Mel, who with characteristic animation and good humor told how his numerous "proofs" were shot down, one after another, by Isbell. It was like listening to a fisherman talk of an encounter with a legendary fish, too big and too sly to be caught. The conjecture is that every continuous piecewise polynomial function on $\mathbb{R}^n$ can be expressed as a finite lattice-combination of polynomials, i.e., as a sup of infs of finitely many polynomials. Here, of course, we are concerned with piecewise polynomials that are defined by giving a *finite* cover of $\mathbb{R}^n$ by closed semialgebraic sets and and stipulating a polynomial on each. Functions that are piecewise polynomial in a more general sense, e.g., requiring infinitely many pieces, are not generally finite lattice-combinations of polynomials. The conjecture was publicized in the early 1980s by Isbell, who believed that the methods of real-algebraic geometry then being introduced might be capable of capturing it. Using Thom's Lemma, Mahé reeled in the $n = 2$ case soon after; see [ML1]. After this the conjecture became widely known. Since Mahé's work, some new techniques have been explored but there has been no definitive progress on the cases with $n \geq 3$.

Many people ask why the names of Birkhoff and Pierce appear in reverse alphabetical order in the name of the conjecture. My guess is that it just sounds better to have the one-syllable name first. As a matter of fact, it is arguable whether Birkhoff and Pierce should really be regarded as the authors of this problem. The likely inspiration is an unsolved problem stated at the end of [BP], but a careful examination suggests that Birkhoff and Pierce may actually have meant to ask something different. To be fair (and alphabetical), perhaps the name should be the "Birkhoff-Henriksen-Isbell-Pierce Conjecture," but I don't expect this to catch on. Whatever it's called, it seems that everyone who has taken it up has experienced it in much the same way as Mel did, believing at first that all the pieces of a proof are at hand only to discover that the crux of the problem has not been touched, and finally marveling at the mysterious depths.

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1. Universal algebra. I shall review some basic ideas of universal algebra in order to provide a conceptual and terminological frame of reference. Readers may skim the present section and refer back to it as needed, in case questions about meanings should arise. An algebra, in the sense of universal algebra, is a set equipped with distinguished elements and operations. A collection of symbols acting as names for these elements and operations is called the signature of the algebra. For example, a group (if written additively) is an algebra with signature $(0, -, +)$, with the understanding that 0 names a fixed element of the algebra, $-$ names a unary operation on it, and $+$ names a binary operation on it. A ring with identity may be given signature $(0, -, +, \cdot)$. If a signature Ω is given then an algebra with that signature is called an Ω-algebra. Though when speaking of a particular Ω-algebra I may identify the symbols in Ω with the corresponding elements or operations of the algebra, in general one distinguishes symbols from the elements or operations they name, so that one may speak of corresponding elements or operations in different algebras, and thus clarify notions like homomorphism and isomorphism. For a rigorous discussion of the concept of 'signature' (under different names), the reader may refer to the definition of "*language* (or *type*) of algebras in [BS], page 23, or to the definition of "*operator domain* ... Ω" and "Ω-*algebra*" in [C], page 48.

Not every $(0, -, +)$-algebra is a group. A group satisfies the additional requirements that $+$ be associative, that 0 be a left and right identity for $+$ and that $-x$ be a left and right inverse for x. These requirements can be stated as *equational laws*, i.e., as universally quantified sentences in which the quantifier-free part is an equation in the language with the constant and function symbols from the signature. Equational laws are also called *identities*.

Fix a signature Ω, and let X be an arbitrary class of Ω-algebras. It is easy to see that any equational law satisfied by every element of X is also satisfied by the subalgebras, products and homomorphic images that may be formed from the algebras in X. The smallest class containing X and closed under these formations can be built in three steps: first take all products of elements of X, then adjoin all sub-Ω-algebras of these and finally adjoin all homomorphic images of these. This class is denoted $\mathbf{HSP}(X)$. A famous theorem of Birkhoff asserts the converse: if $X = \mathbf{HSP}(X)$, then X is defined by a set of equational laws (or, as one says, X is an *equational class*). We write $\mathbf{HSP}(X, \Omega)$ if it is necessary to make the signature clear. We write $\mathbf{HSP}(A)$ if $X = \{A\}$, in which case we say that A is a generator of the class.

Universal algebra concerns the equational laws that may be satisfied by the algebras of a given signature, the implications among them and the classes of algebras defined by them. For example, when A is an Ω-algebra one may want to know whether finitely many equational laws define $\mathbf{HSP}(A)$; or, if an equational class X has been given, one may seek an Ω-algebra B of some particularly simple kind such that $X = \mathbf{HSP}(B)$.

2. ℓ-groups, ℓ-rings and f-rings. An *abelian ℓ-group* is a $(0, -, +, \vee)$-algebra that is an abelian group with respect to 0, $-$ and $+$ and in which $\vee$ is a binary operation that is associative, commutative and idempotent and that satisfies the following distributive law:

$$\forall\, x, y, z : \; x + (y \vee z) = (x + y) \vee (x + z).$$

Any totally-ordered group may be viewed as an ℓ-group by defining $x \vee y$ to be the maximum of x and y. In an ℓ-group, one defines $x \wedge y := -(-x \vee -y)$, $x^+ := x \vee 0$ and $x^- := (-x) \vee 0$. If x is an element of an ℓ-group, we call it *positive* if $x \wedge 0 = 0$ and we call it *strictly positive* if it is positive and non-zero.

It is a fact that every abelian ℓ-group is a sub-ℓ-group of a product of totally-ordered groups; see [BKW], 4.2. It follows that under $\vee$ and $\wedge$ any abelian ℓ-group is is a distributive lattice. It is a consequence of Theorem 3.10 of [HI] that the class of all abelian ℓ -groups is $\mathbf{HSP}(\mathbb{Z}, 0, -, +, \vee)$; this result is also discussed in the Appendix of [BKW], where further references are given.

An *ℓ-ring* is a $(0, -, +, \cdot, \vee)$-algebra that: *i*) is a ring with respect to 0, $-$, $+$ and $\cdot$, *ii*) is an abelian ℓ-group with respect to 0, $-$, $+$ and $\vee$ and *iii*) satisfies:

$$\forall\, x, y :\ (x^+ y^+) \wedge 0 = 0.$$

This law assures that any product of positive elements is positive. An *f-ring* is an ℓ-ring that satisfies the stronger laws:

$$\forall\, x, y :\ (x^+ y^+) \wedge (x^-) = 0 \quad \text{and} \quad \forall\, x, y :\ (y^+ x^+) \wedge (x^-) = 0.$$

Any totally-ordered ℓ-ring is an f-ring. Birkhoff and Pierce showed that any f-ring is a sub-f-ring of a product of totally-ordered rings; this is one of the main results of [BP]. Because of this, any equational class of f-rings is completely determined by the totally-ordered rings that are in it.

When ℓ-rings and f-rings were first investigated, they were not assumed to have multiplicative identity. We will stick to tradition. When used without qualification, the word "ring" refers to a possibly non-commutative ring, possibly without identity. The equational class of *rings with identity* differs from the class of rings in having a signature that includes the constant 1 and in satisfying the equational laws stating that 1 is a left and right multiplicative identity. Similarly, *f-rings with identity* differ from f-rings in the presence or absence of 1 in the signature together with the laws of identity in the defining equations. In the following, **FR** will denote the equational category of f-rings and **FR1** will denote the equational category of f-rings-with-identity. **FR1** is a subcategory of **FR**, but it is not a full subcategory. For example, if A is an **FR1**-object, then $a \mapsto (a, 0) : A \to A \times A$ is an **FR**-morphism but not an **FR1**-morphism.

3. Equational classes of f-rings. We call an element of the free ring in variables $x_1, x_2, \ldots$ a *polynomial*. In general, we don't assume that the variables commute. Henriksen and Isbell showed that every f-ring identity is equivalent to a conjunction of identities of a particularly simple form; see their Corollary 3.6, quoted below. This important result is treated almost as a passing observation in [HI]. The proof rests on the fact that, using the defining equations for f-rings, any f-ring word w can be rewritten as a supremum of infima of finitely many polynomials:

$$w = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell_i} f_{ij}.$$

This is Corollary 3.5 of [HI]. Henriksen and Isbell sketched a proof of 3.5, which (though convincing) is not as complete as many who have read it have wished. A treatment that makes all details fully explicit now appears in [HJ], section 2.

Corollary [HI] 3.6. *Any equational class of f-rings is defined by the identities defining f-rings together with laws of the form*

$$\forall\, x_1, \ldots, x_m :\ \big(g_1 \wedge g_2 \wedge \cdots \wedge g_\ell\big)^+ = 0,$$

where each g_j is a polynomial.

Proof. Observe that $w = 0$ is equivalent to $w^+ = 0$ & $(-w)^+ = 0$. By [HI] 3.5, both w and $-w$ may be written as suprema of infima of polynomials. Finally, $\big(\bigvee_{i=1}^k F_i\big)^+ = 0$ is equivalent to the conjunction of the equations $F_i^+ = 0$, $i = 1, \ldots, k$. ∎

Interpreted in a totally-ordered ring A, the identity in [HI] 3.6 simply says that whenever the polynomials $g_1, \ldots, g_\ell$ are evaluated at elements $a_1, \ldots, a_m \in A$, not all of them are strictly positive.

4. Example – unitable f-rings. Call an f-ring *unitable* if it can be embedded in an f-ring that possesses a multiplicative identity. Henriksen and Isbell showed that the class of unitable f-rings—which we call **uFR**—is defined by the following equational laws:

$$\forall\, x, y :\ \big(x \wedge y \wedge (x^2 - x) \wedge (y - xy)\big)^+ = 0$$
$$\forall\, x, y :\ \big(x \wedge y \wedge (x^2 - x) \wedge (y - yx)\big)^+ = 0.$$

Their proof runs as follows. By the result of Birkhoff and Pierce referred to above, an f-ring is unitable if and only if it may be embedded in a product of totally-ordered f-rings with multiplicative identity. Now, the equations above are satisfied by any totally-ordered f-ring that has a multiplicative identity, for in any such f-ring, if $x > 0$ and $x^2 - x > 0$ then $x > 1$, and thus if $y > 0$ then $y - xy \le 0$ and $y - yx \le 0$. On the other hand, suppose A is a totally-ordered f-ring that satisfies the identities. Then either $x^2 \le x$ for all $x > 0$ in A or there is some $x > 0$ such that $xy \ge y$ and $yx \ge y$ for all $y > 0$. In either case, as Henriksen and Isbell show, A may be embedded in a totally-ordered f-ring with identity. For the sake of brevity, I will not reproduce the proof; a nice exposition can be found in [BKW], 9.6. For examples of non-unitable f-rings, see [BKW], 9.4. [HI] also contains the interesting result that every unitable f-ring is contained in a unique smallest f-ring with identity (5.11.i). Because **FR1** is not full in **FR**, this "unital hull" is not a monoreflection. I don't know if it has any interesting functorial properties.

5. Formally real f-rings. Henriksen and Isbell call an f-ring *formally real* if it satisfies all the f-ring identities that are true in the totally-ordered field of rational numbers. An f-ring, in other words, is formally real if and only if it belongs to $\mathbf{HSP}(\mathbb{Q})$. The most remarkable parts of [HI] concern this equational class.

By the result in §4 above, because $\mathbb{Q}$ has a multiplicative identity, the members of $\mathbf{HSP}(\mathbb{Q})$ are all unitable. Clearly, they are also commutative. It is not at all obvious that there are any commutative f-rings in **uFR** that are not in $\mathbf{HSP}(\mathbb{Q})$. [HI] includes a remarkable example. Observe that in $\mathbb{Q}$, if $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$ are all strictly positive and $x_1 z_1 > y_1 z_2$, $x_2 z_2 > y_2 z_3$, and $x_3 z_3 > y_3 z_1$, then $x_1 x_2 x_3 > y_1 y_2 y_3$. Using the observations in section 3, this easily translates into an equational law of the kind exhibited in Corollary 3.6. In [HI], the authors present a totally-ordered algebra over the reals that violates this law. It is a semigroup-algebra over a semigroup with 79 elements. The semigroup is an initial segment of a particular 9-generator numerical semigroup, but it is modified in a peculiar way: the penultimate element is declared to be absorbing, and the order is changed so that the last element comes before the penultimate one.

In correspondence with Isbell in the 1990s, I found a much simpler example with only three generators that nicely illustrates the idea of the construction. Consider the following equational law:

$$\forall\, x, y, z : \; \big(x \wedge y \wedge z \wedge (yz - x^3) \wedge (xz - y^2) \wedge (x^2 y - z^2)\big)^+ = 0. \tag{$*$}$$

This is an identity in $\mathbb{Q}$, for suppose x, y and z are strictly positive rational numbers and $yz - x^3 > 0$ and $xz - y^2 > 0$. Then $xyz^2 > x^3 y^2$, and therefore $x^2 y - z^2 < 0$. We now display an f-ring in which this law fails. Let $S := \{9, 12, 14, 18, 21, 23, 24, 26, 27, 30, \infty\}$. Define an addition $\oplus$ in S:

$$a \oplus b := \begin{cases} a + b, & \text{if } a + b \leq 30 \text{ and } a + b \neq 28; \\ \infty, & \text{otherwise.} \end{cases}$$

In effect, 28 has been renamed ∞, placed after 30 and made into an absorbing element. Let A denote the semigroup ring over S, with the absorbing element of S identified with 0. An arbitrary element of A may be written in the form $a_9 t^9 + a_{12} t^{12} \cdots + a_{27} t^{27} + a_{30} t^{30}$, where the a_i are integers and t is an indeterminate. Order A by declaring such an expression to be positive if the first non-zero coefficient is positive. Then, as one may check, any product of nonnegative elements is nonnegative. Thus, A is a totally-ordered f-ring. Now, the elements $x = t^9, y = t^{12}, z = t^{14} \in A$ violate the law above, for they are all positive and

$$\begin{aligned} yz - x^3 &= t^{26} - t^{27} > 0, \\ xz - y^2 &= t^{23} - t^{24} > 0, \\ x^2 y - z^2 &= t^{30} - 0 > 0. \end{aligned}$$

By way of an explanation, the semigroup S is a modification of the numerical semigroup $G := \langle 3, 4, 5 \rangle$. S is obtained by replacing 5 by $5 - \epsilon$, identifying large elements with ∞ and re-ordering. The reasons why the example works can be traced back to the relations that hold between the generators 3, 4 and 5:

$$3 + 3 + 3 = 4 + 5, \quad 4 + 4 > 3 + 5, \quad 5 + 5 = 3 + 3 + 4.$$

It is clear that these parallel the polynomials that appear in line $(*)$. The theme of this example is elaborated in [MJ2] and [MJ3].

The example of [HI] is based on the same kind of construction, taking advantage of more elaborate relations in a larger numerical semigroup. It accomplishes more than just illustrating that $\mathbf{HSP}(\mathbb{Q}) < \mathbf{FR}$, however. It is an f-algebra over $\mathbb{R}$ with the property that every sub-f-algebra with 8 or fewer generators is in $\mathbf{HSP}(\mathbb{Q})$, and it demonstrates, therefore, that any set of equational laws defining $\mathbf{HSP}(\mathbb{Q})$ must use at least 9 variables. In a later paper, Isbell generalized the example to show that no set of equational laws in finitely many variables can define $\mathbf{HSP}(\mathbb{Q})$; see [I].

We cannot end our discussion of $\mathbf{HSP}(\mathbb{Q})$ without including an important observation that Henriksen and Isbell made about formally real f-rings. In paragraph 3.8 they demonstrate that every totally-ordered field is formally real. Thus, any equational law of f-rings violated in some totally-ordered field is already violated in $\mathbb{Q}$, and if k is any totally-ordered field, then $\mathbf{HSP}(k) = \mathbf{HSP}(\mathbb{Q})$.

6. Free f-rings. The *absolutely free* Ω*-algebra* on the generators $x_1, \ldots, x_n$ is the set of all expressions that can be formed from the constants in Ω and the symbols $x_1, \ldots, x_n$ using the function symbols in Ω. Each element in the absolutely free algebra is called a *word* and is nothing but a well-formed string of symbols. If $\mathbf{C}$ is an equational class of Ω-algebras, then the free $\mathbf{C}$-algebra on $x_1, \ldots, x_n$ is defined to be the quotient of the absolutely free algebra by the least equivalence relation that respects the operations and identifies all words that are equal by virtue of the equational laws defining $\mathbf{C}$. The *word problem* for the free $\mathbf{C}$-algebra is to provide an algorithm that decides if two words denote the same element in the free $\mathbf{C}$-algebra.

The definition of the free $\mathbf{C}$-algebra that we have just given is syntactic. A well-known proposition of universal algebra—see [C], 3.13—says that if $\mathbf{C} = \mathbf{HSP}(A)$, then each free $\mathbf{C}$-algebra has a nice representation within a product of copes of A:

Proposition. *Fix a signature Ω, and let A be an Ω-algebra. Then the free $\mathbf{HSP}(A)$-algebra on n generators is isomorphic to the sub-algebra of the Ω-algebra of all A-valued functions on A^n that is generated by the projections $\pi_i : A^n \to A$, $i = 1, \ldots, n$.*

If we put this together with results 3.5 and 3.8 of [HI], we get immediately that the free formally-real unital f-ring is the sub-f-ring of k^{k^n} consisting of all finite sups of infs of polynomials with integer coefficients; this is essentially the content of [HI], 4.4, except that the assertion in [HI] is phrased to apply to the non-unital category.

7. Origin of the Pierce-Birkhoff Conjecture. The final section of [BP] is a list of unsolved problems. The third asks for solutions of the word problems for several varieties of lattice-ordered algebraic structure. It also includes an ambiguous parenthetical remark. I quote it in full:

> *Solve the word problem for the free, commutative, real ℓ-algebra (ℓ-group) with n generators. (We conjecture that it is isomorphic with the ℓ-group of real functions which are continuous and piecewise polynomial of degree at most n over a finite number of pieces.) Same problem for free (commutative) ℓ-rings, for free f-rings. (The former is probably very difficult.)*

What do we know today about these problems? For commutative ℓ-groups, vector-lattices over a totally-ordered field, formally real f-rings and formally real f-algebras over a totally-ordered field we have an answer. Each of these equational classes is $\mathbf{HSP}(k)$ (the appropriate signature being understood) for some totally-ordered field k. The free algebra in each of these classes, therefore, is an algebra of k-valued functions on some k^n. These functions are defined by finitely many algebraic inequalities, and therefore by Tarski's Theorem, the question of whether two are equal is decidable. (Note that by [HR] 3.8, there is no essential loss of generality in assuming that k is real-closed.) Thus, we have solutions for the word problem for the free algebras in each of these classes. For free f-rings and for free f-algebras (not assumed formally real) the word problem is not yet solved, nor is the word problem for ℓ-rings or ℓ-algebras. The theory presented in section 3 above shows that the word problem for free f-rings would be solved by giving an algorithm that could decide, given a finite set of polynomials $g_i \in \mathbb{Z}[x_1, \ldots, x_n]$, if there is a totally-ordered ring A (not necessarily reduced) and elements $a_1, \ldots, a_n \in A$ such that $g_i(a_1, \ldots, a_n) > 0$ for all i.

Returning to the quotation, the statement in parentheses is the likely origin of the Pierce-Birkhoff Conjecture since it is the only reference to piecewise polynomials in [BP]. It appears that the authors are referring to the free ℓ-group, but then the restriction on degree does not make sense. (The equational class of ℓ-groups is $\mathbf{HSP}(\mathbb{Z})$, and so the free ℓ-group should consist of piecewise *linear* functions.) It follows from [HI], 3.10 and the proposition above that the free real vector-lattice on n generators is isomorphic to a vector-lattice of piecewise-homogeneous linear functions on $\mathbb{R}^n$. It is not obvious that it includes all of them, but this was proved by Beynon in [B1] and [B2], thus completing the proof of what we might call the "linear Pierce-Birkhoff Conjecture."

8. Unsolved Problems. I have spent a good deal of this essay viewing f-rings through the lens of universal algebra, for universal algebra was one of the main instruments that Henriksen and Isbell used to investigate them. There are numerous interesting unsolved problems. Here are two that I think are interesting enough and difficult enough to be worth anyone's efforts:

Problem. *The example in §5 shows that there is an f-ring generated by three elements that is not formally real. Is there an f-ring generated by two elements that is not formally real? (I think not.)*

Problem. *Suppose X is the class of all formally real f-rings on 3 generators. Is there a finite set of f-ring identities that defines $\mathbf{HSP}(X)$?*

These questions have interesting connections to classical work in commutative algebra. The structure of valuations, which is much simpler in dimension two than in higher dimensions, suggests that the kind of pathological order that occurs in rings that require more than two generators may be absent when there are only two. The interested reader may find some leads in [AJM]. An important reference for anyone interested in the second question is [H].

The Pierce-Birkhoff Conjecture itself arose in the setting of universal algebra. The free formally real f-algebra over $\mathbb{R}$ is the sub-f-ring of the f-ring of all real-valued functions in $\mathbb{R}^n$ generated by the coordinate projections. Therefore, its elements are piecewise-polynomial functions on $\mathbb{R}^n$. As in the case of vector-lattices, one wonders whether it contains *all* piecewise polynomial functions. This is the Pierce-Birkhoff Conjecture. What makes it so difficult? Experience suggests that the real crux of the matter is in the analysis of singularities of algebraic sets. Let me say a few words about this. The frontier of any "piece" of a piecewise polynomial function on $\mathbb{R}^n$ is a codimension-one algebraic subset of $\mathbb{R}^n$. It is a certain lack of understanding about the the behavior of polynomials near the singularities that may occur in such a set that has proven to be the main obstacle to solving the conjecture. The reason that the 2-dimensional case has been resolved but that higher dimensional cases have so far resisted all attempts is simply that singularities of plane curves are easier to analyze than singularities of surfaces or higher-dimensional algebraic sets.

In the last few years, the conjecture has received some renewed attention. The most important recent references are [LMSS], [ML2] and [W]. The Henriksen Festschrift [DM] contains other relevant articles.

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