

Lecture 3. Order of Operations¹

Some arithmetic expressions appear to allow different interpretations. For example, most people will simplify $6 - 3 + 2$ as follows:

$$6 - 3 + 2 = (6 - 3) + 2 = 3 + 2 = 5,$$

but a few may read $6 - 3 + 2$ as $6 - (3 + 2)$, which equals 1. Most people will simplify $6 + 3 \times 2$ this way:

$$6 + 3 \times 2 = 6 + (3 \times 2) = 6 + 6 = 12,$$

but a few will read $6 + 3 \times 2$ as $(6 + 3) \times 2$, which is 18. To assure consistency, mathematics users have agreed on a set of rules to govern the order in which operations are to be performed. These rules explicitly sanction specific readings of expressions that might be read in more than one way.

The rules for the order of operations

- I. A string of additions and/or subtractions is to be evaluated from left to right. Each minus-sign operates on the entire expression to its left, but only on the number that is immediately to its right. Thus,

$$6 - 1 + 5 - 2 - 3$$

is to be interpreted in the following manner:

$$\boxed{\boxed{6 - 1} + 5} - 2 - 3.$$

The steps in simplifying the expression are:

$$\begin{aligned} 6 - 1 + 5 - 2 - 3 &= \boxed{\boxed{6 - 1} + 5} - 2 - 3 \\ &= \boxed{5 + 5} - 2 - 3 \\ &= \boxed{10 - 2} - 3 \\ &= 8 - 3 \\ &= 5. \end{aligned}$$

- II. A string of multiplications and/or divisions is also to be evaluated from left to right. Thus,

$$8 \div 4 \times 5 \div 2 \times 3$$

is to be interpreted as:

$$\boxed{\boxed{8 \div 4} \times 5} \div 2 \times 3.$$

¹ *Acknowledgement.* These notes were written in 2008 in cooperation with Eric Hsu, San Francisco State University. Dr. Hsu's web site is <http://math.sfsu.edu/hsu>

- III. In a string involving additions and/or subtractions *as well as* multiplications and/or divisions, the multiplications and/or divisions are to be done before the additions and/or subtractions. Thus,

$$7 + 2 \times 6 \div 3 \times 5 - 9 + 5 \times 4$$

is to be interpreted as:

$$7 + \boxed{2 \times 6 \div 3 \times 5} - 9 + \boxed{5 \times 4} = 7 + 20 - 9 + 20 = 38.$$

- IV. Parentheses may be used to override any of the rules above. Expressions inside parentheses are evaluated first, no matter what operations they involve.

$$\begin{aligned} (1 + 2) \times 3 &= 3 \times 3 = 9, \\ 5 - (3 - 2) &= 5 - 1 = 4, \\ 5 - (5 - (3 - 2)) &= 5 - (5 - 1) = 5 - 4 = 1. \end{aligned}$$

When division is represented using a horizontal bar (as in $\frac{3+1}{2}$), the bar effectively creates parentheses around the numerator and denominator:

$$\frac{4+5}{1+2} = (4+5) \div (1+2) = 9/3 = 3.$$

- V. Exponents act on the numbers immediately preceding them. Thus, 3×5^2 is interpreted as $3 \times \boxed{5^2}$. When a minus sign is involved, as in -5^2 , it is not interpreted as part of the base; -5^2 means $-(5^2)$.

Simplifying arithmetic expressions

An arithmetic expression can be simplified by attending to the parentheses first. Within them, one performs the multiplications and divisions and then additions and subtractions. If there are several layers of parentheses, the process is iterated, beginning with the innermost parentheses. When an operation is performed in an arithmetic expression, a number is written in place of a complex expression involving an operation symbol, reducing the total number of operations. The following example illustrates this. There are 5 operations in the first expression, 4 in the next, then 3, 2, 1, and finally none.

$$\begin{aligned} (1 + 3 \cdot (\underline{3 + 2 \cdot 5})) / 2 &= (1 + 3 \cdot (\underline{3 + 10})) / 2 \\ &= (\underline{1 + 3 \cdot 13}) / 2 \\ &= (\underline{1 + 39}) / 2 \\ &= \underline{40} / 2 \\ &= 20. \end{aligned}$$

When all the operations have been performed, an arithmetic expression is fully simplified. What remains at the end of the process is a number.

Arithmetic expressions versus algebraic expressions

So far, we have discussed only arithmetic expressions, *i.e.*, expressions made from numbers and the arithmetic operations of addition, subtraction, multiplication and division. *Algebraic expressions* are obtained if, in addition, we also allow variables.

- A *polynomial* is an expression that can be built from variables and constants by using addition, subtraction and multiplication (but not division). There are some important special cases. A *monomial* is a constant times a product of variables. A *linear term* is a constant times a variable. A *linear form* is a sum of linear terms.
- A *rational expression* is an expression that can be built from variables and constants by using addition, subtraction, multiplication and division.
- More general algebraic expressions are obtained by using the four arithmetic operations as well as additional operations, such as n -th roots $\sqrt[n]{\quad}$.

Algebraic expressions do not always allow simplification in sense that arithmetic ones do, since every arithmetic expression simplifies to a number, but most algebraic expressions do not.

Equal expressions, equivalent expressions and normal form

In algebra, one of the most common and important tasks is determining if two expressions are equal. To tell whether two arithmetic expressions are equal is easy: one simplifies each expression (using the rules for the order of operations), and if the resulting numbers are equal then the expressions were equal. Determining if two polynomials are equal is more involved, but the problem can be solved mechanically. By following an algorithmic procedure, one can rewrite any polynomial as a sum of monomials. Two polynomials are equal if and only if when thus expanded as sums of monomials, the same products of variables occur in each with the same coefficients. (Rational expressions are more challenging, but there is an algorithm that reduces the task to determining if two polynomials are equal.)

In execution, the algorithm for rewriting a polynomial is different from the algorithm for simplifying an arithmetic expression. To see where the difference lies, consider the following examples:

$$\begin{aligned}
 4 \cdot (1 + 2) &= 4 \cdot \boxed{1 + 2} & x \cdot (y + z) &= x \cdot \boxed{y + z} \\
 &= 4 \cdot \boxed{3} & &= x \cdot \boxed{y} + x \cdot \boxed{z} \\
 &= 12 & &
 \end{aligned}$$

The boxes on the second line are there only to show what has happened to the contents of the boxes on the first line. In the arithmetic example on the left, the expression “1 + 2” inside the parentheses is treated as a complete expression, which is evaluated to yield the number 3. Then this 3 is multiplied by 4 to yield 12. In the algebraic example on the right, the expression inside the parentheses cannot be further simplified. Instead, the distributive law is applied, creating an expression with two multiplications and an addition. There are *more* operations than in the original expression, but as we will see, an advantage has been gained.

In an algebraic expression, it may be impossible to make a modification that eliminates an operation: $x + y$ cannot be written in a simpler form. If we measure the complexity of an algebraic expression by the number of operation symbols in it, then many expressions just do not simplify. But there are modifications that reduce a different kind of complexity. Consider the following example:

$$\begin{aligned}
 x + x(x + x(x + x^2)) &= x + x(x + x^2 + x^3) \\
 &= x + x^2 + x^3 + x^4
 \end{aligned}$$

Including the multiplications implicit in the exponents, there are 6 operations in the first expression but 9 in the last. In terms of *alternations* between additions and multiplications, however, the last expression is much simpler. To explain what is meant, we need to analyze the structure of expressions more carefully.

Because addition is both commutative and associative, we may view it as an operation on finite collections of inputs, independent of grouping and order. Let us represent the sum of several inputs x_1, x_2, \dots, x_n by $A[x_1, x_2, \dots, x_n]$. The inputs may be constants, variables, or the outputs of other operations. We will assume that none of the inputs to A are outputs of addition, since any A s that occur as heads of inputs to A are redundant (e.g., $A[a, A[y, z]] = A[x, y, z]$). The analogue is true of multiplication. Let us represent the product of several inputs x_1, x_2, \dots, x_n , which again may be constants, variables, or the outputs of any operations *other than multiplication*, by $M[x_1, x_2, \dots, x_n]$.

Any polynomial expression can be written using A and M in conformity with the rules we have just stated, together with the minus-sign to denote the operation of taking the additive inverse. For example:

$$\begin{aligned} 1 + x + x^2 &= A[1, x, M[x, x]]; \\ (x + 1)(y - 2) &= M[A[x, 1], A[y, -2]]; \\ xy + (1 + x)(xy + (x + 1)^2) &= A[M[x, y], M[A[1, x], A[M[x, y], M[A[x, 1], A[x, 1]]]]]. \end{aligned}$$

We can reveal the structure of a polynomial expression by writing it with M s and A s and then deleting all the constant and variable symbols and minus signs and retaining only such commas and brackets as are needed to group occurrences of A and M . For example, the three polynomials above yield $A[M]$, $M[A, A]$ and $A[M, M[A, A[M, M[A, A]]]]$. Any M (respectively, A) other than the head of the entire expression appears as an argument of some A (respectively, M), which we say is *immediately above* it. We call an A or an M a *terminus* if it does not have an M or an A below it. From any terminus, we can read upwards to the head of the whole expression, or from the head we can read downward to a terminus. (Going *up* means going outside of brackets, going down means going inside.) If we read from the head to a terminus, we get a *branch*. For example, the branches in $A[M, M[A, A[M, M[A, A]]]]$ are AM , AMA , and $AMAM$ and $AMAMA$. Clearly, every branch is an alternating sequence of A s and M s. The *MA-complexity* of an expression is the number of MA s in the longest branch.

The distributive law transforms any expression of the form $M[A[\dots], \dots, A[\dots]]$ into an expression of the form $A[M[\dots], \dots, M[\dots]]$. For example:

$$(u + v)(w + x)(y + z) = uwy + uwz + uxy + uxz + vwy + vwz + vxy + vxz.$$

Thus, the distributive law reduces the the MA -complexity of any expression with head M . Repeated applications of the distributive law, therefore, ultimately result in an expression of the form $A[M, \dots, M]$ (or M or A). The MA -complexity 0 of such an expression is zero.