ON THE SECOND STIEFEL–WHITNEY CLASS OF SCALED TRACE FORMS OF CENTRAL SIMPLE ALGEBRAS

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Abstract

The second Stiefel–Whitney class of the quadratic form $\mathbf{tr}_{A/k}(ax^2)$ is computed, where A is a central simple algebra over a perfect field k of characteristic different from 2, $a \in A$ is a fixed element, and $\mathbf{tr}_{A/k}$ is the reduced trace. This class is related on the one hand to the class of A in the Brauer group, and on the other hand to corestrictions of quaternion algebras over certain factors arising from $E \otimes_k E$, where E is a commutative étale algebra over k that depends on the semisimple part of a.

1. *Introduction*

Trace forms and their variants arise naturally in the study of finite-dimensional algebras such as commutative étale algebras, central simple algebras and Lie algebras. It is natural to ask for the associated Stiefel–Whitney classes.

In 1984 J.-P. Serre [**18**] expressed the second Stiefel–Whitney invariant of the trace form of a commutative étale algebra in terms of other cohomological invariants. His formula had important applications to embedding problems and to the inverse Galois problem [**23**]. Serre's formula was generalized to all higher Stiefel–Whitney invariants by B. Kahn [**4**].

In the case of central simple algebras, D. Saltman (1987, unpublished as far as we know) and later Serre [**19**] described by different methods the second Stiefel–Whitney class of the form $\mathbf{tr}(x^2)$, where $\mathbf{tr}$ is the reduced trace. (See also Tignol [**22**] and Lewis–Morales [**9**].) More recently, A. Quéguiner [**14**, **15**] computed this invariant for the form $\mathbf{tr}(\sigma(x)\,x)$ of a central simple algebra equipped with an involution σ.

In this paper we shall be interested in *scaled* trace forms of central simple algebras, that is, quadratic forms of the type $Q_{A,a}(x) = \mathbf{tr}_{A/k}(ax^2)$, where $\mathbf{tr}_{A/k}$ is the reduced trace of a central simple algebra A over a perfect field k of characteristic different from 2 and a is a fixed element of A. We shall assume throughout that the form $Q_{A,a}$ is nonsingular, or equivalently, that a and $-a$ have no common eigenvalue.

The form $Q_{A,a}$ was studied by D. Lewis in [**7**], where he established its general properties and gave formulas for its signature and discriminant. The next question was naturally the computation of the Hasse invariant (second Stiefel–Whitney invariant), which was left as an open problem in [**7**].

The motivation for this work was the discussions of the second author with David Lewis a few years ago. Our results are a complement to his results since they provide a full computation of the second Stiefel–Whitney invariant of $Q_{A,a}$. We warmly thank him for having brought the question to our attention and for sharing his ideas with us.

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In Section 2 we introduce the notation and the terminology that will be used throughout the paper.

In Section 3 we show that the forms $Q_{A,a}$ and Q_{A,a_s}, where a_s is the semisimple part of a, are isometric.

Section 4 deals with the case where $A = M_n(k)$. We show that $Q_{M_n(k),a}$ is isometric to the trace of a certain rank-1 hermitian form over $E \otimes_k E$, where E is a commutative étale algebra containing a_s (Proposition 4.1). This description allows us to express $w_2(Q_{M_n(k),a})$ as a sum involving corestrictions of certain quaternion algebras over the factors of the subalgebra of $E \otimes E$ fixed under the canonical involution $x \otimes y \longmapsto y \otimes x$ (Theorem 4.6).

Section 5 differs from the other sections in that it does not deal directly with trace forms or with quadratic forms for that matter (other than the presence of orthogonal and spinor groups). In this section we develop the tools from representation theory that are needed for the general case. These tools are probably known to specialists; we include them here because either we have not been able to find them explicitly in the literature, or they are not in a form or language suited to our purposes. The main result in this section is an explicit description, in terms of weights, of the obstruction to lift a rational orthogonal representation $G \longrightarrow \mathbf{SO}$ of a reductive algebraic group G to a spinor representation $G \longrightarrow \mathbf{Spin}$ (Theorem 5.9).

Finally, Section 6 is devoted to the computation of $w_2(Q_{A,a})$ for a general central simple algebra A over k. This is done by establishing a 'comparison' formula with $w_2(Q_{M_n(k),b})$, where $b \in M_n(k)$ is an element whose similarity class is canonically determined by a according to Lemma 6.1. If n is odd, it is easily seen by Springer's theorem that $Q_{A,a}$ and $Q_{M_n(k),b}$ are actually isometric (Proposition 6.2), so in this case $w_2(Q_{A,a}) = w_2(Q_{M_n(k),b})$. If n is even, we use Springer's spinor interpretation of w_2 and the results on orthogonal representations from Section 5 to show that $w_2(Q_{A,a}) = w_2(Q_{M_n(k),b}) + (n/2)[A]$, where $[A]$ is the class of A in the Brauer group (Theorem 6.8). This result, together with the computation of $w_2(Q_{M_n(k),b})$ of Section 4, yields the general formula (Theorem 6.9).

2. *Notation and definitions*

Let k be a perfect field of characteristic different from 2. We shall denote by k_s a separable closure of k and by Γ the Galois group $\mathrm{Gal}(k_s/k)$.

2.1. *Pointed algebras*

A *pointed algebra* over k is for us a pair (A, a), where A is a central simple algebra over k and a is a fixed element of A. Two pointed algebras (A, a) and (B, b) are said to be *isomorphic* if there exists a k-algebra isomorphism $\varphi : A \longrightarrow B$ with $\varphi(a) = b$.

A *twist* of (A, a) is a pointed algebra (B, b) over k such that $(A \otimes k_s, a) \cong (B \otimes k_s, b)$ as pointed algebras over the separable closure k_s.

It will be shown in Lemma 6.1 that all pointed algebras can be obtained as twists of a pointed algebra of the form $(M_n(k), b)$.

2.2. *Scaled trace forms*

The *scaled trace form* associated to a pointed algebra (A, a) is by definition $Q_{A,a}(x) = \mathbf{tr}_{A/k}(ax^2)$, where $\mathbf{tr}_{A/k}$ is the reduced trace. By invariance of the trace under inner automorphism, it is immediate that the isometry class of $Q_{A,a}$ depends only on the isomorphism class of the pointed algebra (A, a).

2.3. *Some algebraic groups*

Let q be a quadratic form over a vector space V over k. The symbols $\mathbf{O}(q)$, $\mathbf{SO}(q)$, $\mathbf{Spin}(q)$ will denote respectively the orthogonal group, the special orthogonal group, and the spinor group of the quadratic space (V, q). These groups will always be regarded as algebraic groups defined over k. We shall also use the notation $\mathbf{O}(V)$, $\mathbf{SO}(V)$, $\mathbf{Spin}(V)$ when the form q is unambiguously defined by the context.

In Section 6 we shall be interested in the automorphism group G of the pointed algebra $(M_n(k), b)$, regarded as algebraic group over k. This group consists of the algebra automorphisms of $M_n = M_n(k_s)$ that commute with b; we shall study its structure in more detail in Section 6. Notice that since G preserves the scaled trace form $Q = Q_{M_n, b}$, we can regard G as a subgroup of $\mathbf{O}(Q)$.

2.4. *Galois cohomology*

For an algebraic group H defined over k, we shall denote by $H^i(k, H)$ the profinite cohomology set $H^i(\Gamma, H(k_s))$ as defined in [**17**] ($i \leqslant 1$ if H is not abelian).

Recall that $H^1(k, \mathbf{O}(q))$ is in one-to-one canonical correspondence with the set of isometry classes of quadratic forms Q over k of the same rank as q [**20**, Theorem 2.2]. Similarly, $H^1(k, \mathbf{SO}(q))$ is in one-to-one canonical correspondence with the set of isometry classes of quadratic forms Q over k of the same rank and discriminant as q [**20**, Theorem 2.3].

By standard descent theory, the set $H^1(k, G)$, where $G = \mathbf{Aut}(M_n, b)$, is in one-to-one canonical correspondence with the isomorphism classes of twists of the pointed algebra (M_n, b). More explicitly, if (A, a) is a twist of (M_n, b), we choose an isomorphism $\varphi: (A \otimes k_s, a) \longrightarrow (M_n(k_s), b)$ and set $c_{A,a}(\gamma) = \varphi\gamma(\varphi^{-1})$. The map $c_{A,a}: \Gamma \longrightarrow G$ is a 1-cocycle and its class $[c_{A,a}]$ in $H^1(k, G)$ is independent of the choice of φ. The correspondence $(A, a) \longrightarrow [c_{A,a}]$ is a bijection between the set of isomorphism classes of twists of (M_n, b) and the cohomology set $H^1(k, G)$.

Moreover, it is easy to see that if $i: G \longrightarrow \mathbf{O}(Q)$ is the natural inclusion, then the induced map $i_*: H^1(k, G) \longrightarrow H^1(k, \mathbf{O}(Q))$ maps the cohomology class corresponding to a pointed algebra (A, a) to the cohomology class of the corresponding associated scaled trace form $Q_{A,a}$.

2.5. *Stiefel–Whitney classes*

Let q be a nonsingular quadratic form of rank n over k. For $i \geqslant 0$, we denote by $w_i(q) \in H^i(k, \mathbb{Z}/2)$ the *ith Stiefel–Whitney class of* q, in the sense of [**2**]. More explicitly, if $\langle a_1, \ldots, a_n \rangle$ is a diagonalization of q, then

$$w_i(q) = \sum_{r_1 < \ldots < r_i} (a_{r_1}) \ldots (a_{r_i}) \in H^i(k, \mathbb{Z}/2),$$

where (a) denotes the class in $H^1(k, \mathbb{Z}/2)$ corresponding to $a \in k^*/k^{*2}$ via the Kummer isomorphism and the product is the cup product in the ring $H^*(k, \mathbb{Z}/2)$. Delzant [**2**] showed that this definition is independent of the diagonalization chosen for q.

With the canonical identifications $H^1(k, \mathbb{Z}/2) = k^*/k^{*2}$ and $H^2(k, \mathbb{Z}/2) = \mathrm{Br}_2(k)$ (the subgroup of the Brauer group of elements of order dividing 2), $w_1(q)$ is equal to the discriminant of q, and $w_2(q)$ is equal to the Hasse invariant of q with respect to the quaternion symbol, as defined in [**16**].

3. *Reduction to the semisimple case*

In this section we reduce the problem to the case where the scaling factor $a \in A$ is a semisimple element.

PROPOSITION 3.1. *Let a_s be the semisimple part of a in its Jordan decomposition. Then $Q_{A,a} \simeq Q_{A,a_s}$.*

Proof. Let $a = a_s + a_n$ be the Jordan decomposition of a, where a_s is semisimple and a_n is nilpotent. We shall in fact show that $Q_{A,a}$ and Q_{A,a_s} are Witt-equivalent, which will be sufficient since the two forms have the same rank.

Recall that if $U \subset A$ is a totally isotropic subspace, then A is Witt-equivalent to the space $U^\perp/U$ endowed with the form induced by $Q_{A,a}$ (see [**5**, Proposition 3.7.9]). Hence it will be enough to show that there is a subspace $U \subset A$ totally isotropic with respect to $Q_{A,a}$ such that $\mathbf{tr}_{A/k}(a_n x^2) = 0$ for all $x \in U^\perp$.

Let $U \subset A$ be a subspace satisfying the following conditions:

(i) $\mathbf{tr}_{A/k}(x^2) = 0$ for all $x \in U$;

(ii) $aU \subset U$ and $Ua \subset U$;

(iii) U is maximal among the subspaces of A satisfying (i) and (ii) above.

Let $U^\perp$ be the orthogonal complement of U with respect to the form $\mathbf{tr}_{A/k}(x^2)$ (which is, by virtue of (ii), also the orthogonal of U with respect to $\mathbf{tr}_{A/k}(ax^2)$). We shall show that the form $\mathbf{tr}_{A/k}(a_n x^2)$ is identically zero on $U^\perp$ by showing the inclusion $a_n U^\perp \subset U$.

Consider the ring $R = k[a] \otimes_k k[a]$ equipped with the involution given by $\overline{x \otimes y} = y \otimes x$. We define an R-module structure on A by $(x \otimes y) \cdot \alpha = x\alpha y$ for $x, y \in k[a]$ and $\alpha \in A$. Notice that by virtue of (ii), the subspace U is actually an R-submodule of A.

Define $\langle \alpha, \beta \rangle = \mathbf{tr}_{A/k}(\alpha\beta)$ for $\alpha, \beta \in A$. We verify immediately that for $z \in R$ we have $\langle z\alpha, \beta \rangle = \langle \alpha, \bar{z}\beta \rangle$ (it is enough to see this for z of the form $z = x \otimes y$, in which case it is obvious). It follows immediately from this property that $U^\perp$ is an R-submodule as well.

The next step is to show that (iii) implies that $U^\perp/U$ is a semisimple R-module. Indeed, let $\mathfrak{r}$ be the radical of R and let $m \geqslant 1$ be the smallest integer such that $\mathfrak{r}^m U^\perp \subset U$ ($\mathfrak{r}$ is a nilpotent ideal). Suppose that $m \geqslant 2$ and let l be the smallest integer with $l \geqslant m/2$. Then

$$\begin{aligned} \langle \mathfrak{r}^l U^\perp, \mathfrak{r}^l U^\perp \rangle &= \langle U^\perp, \bar{\mathfrak{r}}^l \mathfrak{r}^l U^\perp \rangle \\ &= \langle U^\perp, \mathfrak{r}^{2l} U^\perp \rangle \\ &= \{0\}, \end{aligned}$$

since $\mathfrak{r} = \bar{\mathfrak{r}}$ and $\mathfrak{r}^{2l} U^\perp \subset U$. Hence $\mathfrak{r}^l U^\perp$ is a totally isotropic R-submodule, and therefore so is $U + \mathfrak{r}^l U^\perp$. By (iii) we must have $U + \mathfrak{r}^l U^\perp = U$, that is, $\mathfrak{r}^l U^\perp \subset U$. This is a contradiction with the minimality of m; therefore $m = 1$.

Since $a_n \otimes 1$ is in the radical $\mathfrak{r}$, we have in particular $a_n U^\perp \subset U$. Therefore $\mathbf{tr}_{A/k}(a_n xy) = \mathbf{tr}_{A/k}((a_n x) y) = 0$ for all $x, y \in U^\perp$, as claimed. $\square$

4. *The case where A is a split algebra*

Throughout this section we shall assume that $A = M_n(k)$. Let $b \in M_n(k)$ be a fixed element so that $Q_{M_n(k),b}$ is nonsingular. In view of Proposition 3.1 we can assume, without loss of generality, that $b \in A$ is semisimple. For simplicity, we shall write Q_b for $Q_{M_n(k),b}$.

For the remainder of this section, we fix a commutative étale algebra $E \subset M_n(k)$ of degree n over k containing b. We identify the matrix algebra $M_n(k)$ with the algebra of k-endomorphisms $\mathrm{End}_k(E)$.

Let $L = E \otimes_k E$ and let $\varphi : L \longrightarrow \mathrm{End}_k(E)$ be the k-linear homomorphism given on pure tensors by

$$\varphi(x \otimes y)(z) = \mathbf{tr}_{E/k}(yz)\, x. \tag{1}$$

It is easy to see that φ is an isomorphism of k-vector spaces (since it is essentially the canonical isomorphism $E \otimes_k E^* = \mathrm{End}_k(E)$ as k-vector spaces, where E and its dual E^* have been identified via the trace form of E).

Proposition 4.1. *Let φ be the map of* (1). *Then for all $u, v \in L$ we have*

$$\mathbf{tr}_{L/k}((b \otimes 1)\, u\bar{v}) = \mathbf{tr}(b\varphi(u)\,\varphi(v)), \tag{2}$$

where $^-$ *is the involution on L given by $\overline{x \otimes y} = y \otimes x$. In particular, the map φ is an isometry between the quadratic spaces* $(L, \mathbf{tr}_{L/k}((b \otimes 1)\, u\bar{u}))$ *and* $(\mathrm{End}_k(E), \mathbf{tr}(bz^2))$.

Proof. It is enough to prove (2) for u and v of the form $u = x \otimes y$ and $v = x' \otimes y'$. By direct computation we have

$$\begin{aligned}
[\varphi(u)\,\varphi(v)\, b](w) &= \varphi(u)\,(\mathbf{tr}_{E/k}(y'bw)\, x') \\
&= \mathbf{tr}_{E/k}(y'bw)\,\varphi(u)\,(x') \\
&= \mathbf{tr}_{E/k}(y'bw)\,\mathbf{tr}_{E/k}(yx')\, x \\
&= \mathbf{tr}_{E/k}(yx')\,[\mathbf{tr}_{E/k}(y'bw)\, x] \\
&= \mathbf{tr}_{E/k}(yx') \cdot \varphi(x \otimes y'b)\,(w).
\end{aligned}$$

Therefore $\varphi(u)\,\varphi(v)\, b = \mathbf{tr}_{E/k}(yx')\,\varphi(x \otimes y'b)$. From (1) we see that $\mathbf{tr}(\varphi(x \otimes y'b)) = \mathbf{tr}_{E/k}(y'bx)$; hence

$$\begin{aligned}
\mathbf{tr}(\varphi(u)\,\varphi(v)\, b) &= \mathbf{tr}_{E/k}(bxy')\,\mathbf{tr}_{E/k}(yx') \\
&= \mathbf{tr}_{L/k}(bxy' \otimes yx') \\
&= \mathbf{tr}_{L/k}((b \otimes 1)\,(x \otimes y)\,(\overline{x' \otimes y'})) \\
&= \mathbf{tr}_{L/k}((b \otimes 1)\, u\bar{v}). \qquad \square
\end{aligned}$$

Proposition 4.1 allows us to reduce our problem to the case where the underlying algebra is a commutative étale algebra. This situation is generally better understood. In particular, Proposition 4.1 will allow us to compute invariants for $\mathbf{tr}_{A/k}(bx^2)$.

The following result was proved by D. Lewis in [**8**, Section 4]. This can also be proved easily from Proposition 4.1.

Corollary 4.2. *If $Q_b = \mathbf{tr}(bx^2)$ is nonsingular, then its discriminant is equal to* $(-1)^{n(n-1)/2} \det(b)$.

Let $e_1, e_2, \dots, e_n$ be the system of indecomposable idempotents for the algebra $E_s = E \otimes k_s$. Then $\mathscr{S} := \{e_i \otimes e_j : i, j = 1, \dots, n\}$ is a system of indecomposable idempotents for $E_s \otimes E_s$.

The factors in the decomposition of $L = E \otimes E$ as a product of fields are in one-to-one correspondence with the orbits of the Galois group Γ acting on $\mathscr{S}$. The set $\mathscr{S}$ splits as a disjoint union of Γ-stable subsets

$$\mathscr{S} = \{e_i \otimes e_i : i = 1, \dots, n\} \coprod \{e_i \otimes e_j : i \neq j; i,j = 1, \dots, n\},$$

which corresponds to a splitting of L as a product of algebras

$$L = E \times M. \tag{3}$$

Note that the involution $\overline{x \otimes y} = y \otimes x$ on $E \otimes E$ is the identity on the factor E in (3) and acts nontrivially on all the idempotents corresponding to M. An immediate consequence of the last observation is the following lemma.

Lemma 4.3. *The algebra M of* (3) *splits as an algebra with involution in the form*

$$M = \prod_{i=1}^{r} L_i, \tag{4}$$

where L_i is either a field preserved by the involution on which the involution is not the identity, or a product $L_i = F_i \times F_i$, where F_i is a field and the involution on L interchanges the two factors of L_i.

Given $b \in E$, we denote by T_b the form $\mathbf{tr}_{E/k}(bx^2)$ on E and by S_b the form $\mathbf{tr}_{M/k}(b \otimes 1 z\bar{z})$ on M. Using Proposition 4.1 and the above decomposition (3), we can write Q_b as an orthogonal sum

$$Q_b \simeq T_b \perp S_b. \tag{5}$$

Notice that by Corollary 4.2 we have $\operatorname{disc}(S_b) = (-1)^{n(n-1)/2} d_{E/k}$; in particular, $\operatorname{disc}(S_b)$ is independent of b.

We shall next calculate the invariant w_2 for the form Q_b. It is easier first to calculate w_2 of the form $Q_1 \perp Q_b$ and then use the addition formulae. Recall that if q_1 and q_2 are quadratic forms over k, then

$$w_2(q_1 \perp q_2) = w_2(q_1) + w_2(q_2) + (\operatorname{disc}(q_1), \operatorname{disc}(q_2)). \tag{6}$$

(See, for instance, [**16**, Lemma 12.6].)

From (5) we obtain

$$Q_1 \perp Q_b \simeq (T_1 \perp T_b) \perp (S_1 \perp S_b). \tag{7}$$

By taking w_2 on both sides, we get

$$w_2(Q_1 \perp Q_b) = w_2(T_1 \perp T_b) + w_2(S_1 \perp S_b). \tag{8}$$

(Note that $\operatorname{disc}(S_b \perp S_1) = 1$.)

Let $F_i \subset L_i$ be the field fixed by the involution and let $\beta_i \in F_i$ be the component of $\beta = (1 \otimes b + b \otimes 1)/2$ in L_i. Let $S_b^i(z) = \mathbf{tr}_{L_i/k}(\beta_i z\bar{z})$. With this notation, we have

$$S_1 \perp S_b = \perp_{i=1}^{r} (S_1^i \perp S_b^i). \tag{9}$$

The following result gives an expression for w_2 of each of the terms on the right-hand side of (9).

Proposition 4.4. *Let $\beta_i \in F_i$ be the component of $\beta = (1 \otimes b + b \otimes 1)/2$ in L_i. Write $L_i = F_i[t]/(t^2 - d_i)$, with $d_i \in F_i$. Then*

$$w_2(S_1^i \perp S_b^i) = \operatorname{Cor}_{F_i/k}(d_i, -\beta_i) + [F_i : k](-1, -1), \tag{10}$$

where $\operatorname{Cor}_{F_i/k} : \operatorname{Br}(F_i) \longrightarrow \operatorname{Br}(k)$ is the corestriction map.

Proof. The proposition is essentially proved in [**11**, Proposition 2.1]. The reader should be aware of the differences in the definitions. The 'Hasse–Witt invariant' used in [**11**] is not equal to the second Stiefel–Whitney class, but it is related to it by the formula (in the notation of [**11**])

$$\phi_k(S_1^i \perp S_b^i) = w_2(S_1^i \perp S_b^i) + [F_i : k](-1, -1).$$

(We refer to [**16**, p. 81] for the general conversion formulae.)

An alternative proof of (10) can be given using the general formula of B. Kahn [**4**, Théorème 2] for the corestriction of total Stiefel–Whitney invariants. □

As an immediate application, we have the following.

COROLLARY 4.5. *Keeping the above notation,*

$$w_2(S_1 \perp S_b) = \frac{n(n-1)}{2}(-1, -1) + \sum_{i=1}^{r} \mathrm{Cor}_{F_i/k}(d_i, -\beta_i).$$

Proof. This follows immediately from Proposition 4.4 and equation (6); recall that $\mathrm{disc}(S_1^i \perp S_b^i)$ is a square, so w_2 behaves additively on (9). □

We can now state the corresponding result for Q_b.

THEOREM 4.6. *With the notation above, we have*

$$w_2(Q_b) = w_2(Q_1) + w_2(\tfrac{1}{2}\mathbf{tr}_{E'/k}(x^2)) + \frac{n(n-1)}{2}(-1, \det b) + \sum_{i=1}^{r} \mathrm{Cor}_{F_i/k}(d_i, -\beta_i), \tag{11}$$

where $E' = E[t]/(t^2 - b)$.

Proof. From (8) and Corollary 4.5 we obtain

$$w_2(Q_1 \perp Q_b) = w_2(T_1 \perp T_b) + \sum_{i=1}^{r} \mathrm{Cor}_{F_i/k}(d_i, -\beta_i) + \frac{n(n-1)}{2}(-1, -1).$$

Hence by (6)

$$\begin{aligned} w_2(Q_b) &= w_2(Q_1) + w_2(Q_1 \perp Q_b) + (\mathrm{disc}(Q_1), \mathrm{disc}(Q_b)) \\ &= w_2(Q_1) + w_2(T_1 \perp T_b) + \frac{n(n-1)}{2}(-1, -\det b) \\ &\quad + \frac{n(n-1)}{2}(-1, -1) + \sum_{i=1}^{r} \mathrm{Cor}_{F_i/k}(d_i, -\beta_i) \\ &= w_2(Q_1) + w_2(T_1 \perp T_b) + \frac{n(n-1)}{2}(-1, \det b) \\ &\quad + \sum_{i=1}^{r} \mathrm{Cor}_{F_i/k}(d_i, -\beta_i). \end{aligned}$$

(Recall that $\mathrm{disc}(Q_b) = \det(b)(-1)^{n(n-1)/2}$ and note that $T_1 \perp T_b = \frac{1}{2}\mathbf{tr}_{E'/k}(x^2)$.) □

REMARK 4.7. The term $w_2(Q_1)$ of the right-hand side of (11) is easy to compute directly: $w_2(Q_1) = (m(m-1)/2)(-1,-1)$, where $m = n(n-1)/2$.

The term $w_2(\frac{1}{2}\mathbf{tr}_{E'/k}(x^2))$ of (11) can be described using Serre's formula [**18**], adjusted using [**6**, formula 3.16] to take the factor $\frac{1}{2}$ into account. Let $\mathfrak{S}_{2n}$ be the symmetric group on $2n$ elements; then

$$w_2(\tfrac{1}{2}\mathbf{tr}_{E'/k}(x^2)) = e^*_{E'}(s_{2n}),$$

where $e_{E'}:\Gamma \longrightarrow \mathfrak{S}_{2n}$ is the homomorphism defining E', and s_{2n} is a certain canonical class in $H^2(\mathfrak{S}_{2n}, \mathbb{Z}/2)$ (see [**18**]).

5. *Orthogonal representations and weights*

For an affine algebraic group H over k, we shall denote by $X(H)$ the group of characters of H, that is, the group of rational homomorphisms $H \longrightarrow \mathbf{GL}_1$ defined over k_s. For convenience, we shall write additively the group operation in $X(H)$.

If H is defined over k, then $X(H)$ has a natural structure of Γ-module. If $g: H_1 \longrightarrow H_2$ is a morphism of algebraic groups, then we shall denote by g^* the dual homomorphism $g^*: X(H_2) \longrightarrow X(H_1)$, that is, the homomorphism defined by $g^*(\chi) = \chi \circ g$ for $\chi \in X(H_2)$.

DEFINITION 5.1. Let $\rho: G \longrightarrow \mathrm{GL}(V)$ be a rational representation of the reductive group G and let T be a maximal torus of G. For $\alpha \in X(T)$, let

$$V_\alpha = \{v \in V : \rho(t)v = \alpha(t)v \text{ for all } t \in T\}.$$

If the subspace V_α is not $\{0\}$, we say that $\alpha \in X(T)$ is a *weight of* ρ relative to T. The integer $m_\alpha = \dim(V_\alpha)$ is called the *multiplicity* of α. (See, for instance, [**3**, Chapter XI].)

The following result is well known.

LEMMA 5.2. *Let V be a representation of an algebraic torus T over k_s. Then*

$$V = \bigoplus_{\alpha \in X(T)} V_\alpha. \tag{12}$$

Proof. See, for instance, [**21**, 2.5.2]. □

If $\rho: G \longrightarrow \mathrm{GL}(V)$ is a representation over k_s of a reductive group G and T is a maximal torus of G, then the decomposition (12) is called the *weight decomposition* of ρ relative to T.

The *dual* (also called *contragradient*) representation $\rho^*: G \longrightarrow \mathbf{GL}(V^*)$ is defined by $\rho^*(g)(\lambda) = \lambda \circ \rho(g)^{-1}$ for $g \in G$ and $\lambda \in V^*$. If ρ is isomorphic to ρ^* we say that ρ is *self-dual*. It is easy to see that self-dual representations are exactly the ones that leave invariant some nonsingular bilinear form.

LEMMA 5.3. *Let G be a reductive group and let $\rho: G \longrightarrow \mathbf{GL}(V)$ be a self-dual representation. If $\alpha \in X(T)$ is a weight of ρ, then $-\alpha \in X(T)$ is also a weight of ρ and $m_\alpha = m_{-\alpha}$.*

Proof. Identifying $(V_\alpha)^*$ with the set of linear forms on V that vanish on V_β for all weights $\beta \neq \alpha$, one has by straightforward computation $(V_\alpha)^* = (V^*)_{-\alpha}$. The lemma follows immediately by self-duality. □

PROPOSITION 5.4. *Let T be an algebraic torus and let $\rho: T \longrightarrow \mathbf{GL}(V)$ be a representation preserving a nonsingular quadratic form q on V (that is, $\rho(T) \subset \mathbf{SO}(V,q)$). For each weight $\alpha \neq 0$ of ρ, we choose one element of the set $\{\alpha, -\alpha\}$. Let $\alpha_1, \ldots, \alpha_r$ be the weights chosen in this way. Then the weight spaces $V_{\pm\alpha_i}$ are totally isotropic (with respect to q) and we have an orthogonal decomposition*

$$V \simeq V_0 \perp_{i=1}^{r} (V_{\alpha_i} \oplus V_{-\alpha_i}), \tag{13}$$

where V_0 is the space of fixed points and $(V_{\alpha_i} \oplus V_{-\alpha_i})$ is hyperbolic for $i = 1, \ldots, r$.

Proof. Let $\langle\, , \rangle$ denote the symmetric bilinear form associated with q and let $v \in V_\alpha$ and $w \in V_\beta$. Since $\rho(t)$ preserves q, we have

$$\begin{aligned} \langle v, w \rangle &= \langle \rho(t)v, \rho(t)w \rangle \\ &= \langle \alpha(t)v, \beta(t)w \rangle \\ &= \alpha(t)\beta(t)\langle v, w \rangle. \end{aligned}$$

Hence if $\langle v, w \rangle \neq 0$ then $\beta(t)\alpha(t) = 1$, that is, in additive notation, $\beta = -\alpha$. This shows that V_α is orthogonal to V_β for all $\beta \neq -\alpha$; hence the subspaces $V_{\alpha_i} \oplus V_{-\alpha_i}$ are pairwise orthogonal. The same computation shows also that each V_{α_i} is totally isotropic since $\alpha_i \neq -\alpha_i$. □

Let M be a maximal torus of $\mathbf{SO}(V,q)$ containing $\rho(T)$. Let $s = \operatorname{rank}(M) = \lfloor \dim V/2 \rfloor$ and let $\chi_1, \ldots, \chi_s \in X(M)$ be such that $\pm\chi_1, \ldots, \pm\chi_s$ are all the nonzero weights for the action of M on V. It is easy to see directly that $\chi_1, \ldots, \chi_s$ is a basis for $X(M)$. Since M preserves each factor in the decomposition (13), for each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, \ldots, r\}$ such that $\rho^*(\chi_i) \in \{\alpha_j, -\alpha_j, 0\}$. Replacing χ_i by $-\chi_i$ if necessary, we can assume that $\rho^*(\chi_i) \in \{\alpha_j, 0\}$.

LEMMA 5.5. *Let $\alpha_1, \ldots, \alpha_r$ and $\chi_1, \ldots, \chi_s$ be as above and let m_i be the multiplicity of α_i. Let $d = \chi_1 + \ldots + \chi_s$. Then $\rho^*(d) = m_1\alpha_1 + \ldots + m_r\alpha_r$.*

Proof. On the one hand, each of the subspaces V_{α_j} of (13) has a decomposition

$$V_{\alpha_j} = \bigoplus_{\rho^*(\chi_i) = \alpha_j} V_{\chi_i}.$$

On the other hand, the weight spaces V_{χ_i} have dimension 1. Thus there are exactly m_j weights χ_i with $\rho^*(\chi_i) = \alpha_j$. □

REMARK 5.6. Note that the class of $\rho^*(d)$ in the quotient group $X(T)/2X(T)$ is independent of the choices we have made to define d. We shall see below that this class is exactly the obstruction for ρ to admit a lifting $\tilde{\rho}: T \longrightarrow \mathbf{Spin}(V,q)$.

Let $\pi: \mathbf{Spin}(V,q) \longrightarrow \mathbf{SO}(V,q)$ be the canonical projection and let $\tilde{M} \subset \mathbf{Spin}(V,q)$ be the maximal torus with $\pi(\tilde{M}) = M$. Identifying $X(M)$ with its image $\pi^*(X(M))$, we can regard $X(\tilde{M})$ as a lattice in $X(M) \otimes \mathbb{Q}$ containing $X(M)$.

LEMMA 5.7. *The group $X(\tilde{M})/X(M)$ is generated by the class of $\frac{1}{2}d = \frac{1}{2}\sum_i \chi_i$.*

Proof. It is well known from the theory of algebraic groups (see, for instance, [**13**, Theorem 2.6]) that $X(\tilde{M})$ can be identified with the lattice of fundamental weights of the root system associated with the Lie algebra $\mathfrak{so}(V, q)$, which is known to be D_s if $\dim V = 2s$ and B_s if $\dim V = 2s+1$. The lemma follows immediately from the explicit description of the lattice of fundamental weights for D_s and B_s found in the tables, for example [**12**, p. 294] (note that in the notation of [**12**], $\chi_i = \varepsilon_i$ and $\frac{1}{2}d = \pi_s$). □

The following result is certainly well known, but we have not been able to find it explicitly in the literature.

PROPOSITION 5.8. *Let H be a connected reductive group over k and let $T \subset H$ be a maximal torus. Let $\eta : H \longrightarrow \mathbf{SO}(V)$ be a rational representation. Then η lifts to $\mathbf{Spin}(V)$ if and only if $\eta|_T : T \longrightarrow \mathbf{SO}(V)$ lifts to $\mathbf{Spin}(V)$.*

Proof. The 'only if' part being trivial, we shall only prove the 'if' part of the statement.

Let us assume first that H is a semisimple algebraic group. Let $p : \tilde{H} \longrightarrow H$ be its universal covering. Let $\tilde{T} \subset \tilde{H}$ be the preimage of T by p and let $\tilde{\eta} : \tilde{H} \longrightarrow \mathbf{Spin}(V)$ be the lifting of η to $\tilde{H}$.

The subgroup $\ker p$ is central; hence it is contained in $\tilde{T}$. By hypothesis, $\tilde{\eta}$ vanishes on $\tilde{T} \cap \ker p = \ker p$. Thus $\tilde{\eta}$ factors through H.

Now suppose that H is reductive. Then the derived group H' of H is semisimple and $H = Z(H) \cdot H'$ with $Z(H) \cap H'$ finite [**3**, 27.5].

Write $T = S \cdot Z(H)$, where S is a maximal torus in H'. Let $\eta_1 : T \longrightarrow \mathbf{Spin}(V)$ be the lifting of $\eta|_T$ given by our hypothesis and let $\eta_2 : H' \longrightarrow \mathbf{Spin}(V)$ be the lifting of $\eta|_{H'}$ given by the semisimple case considered above. Notice that η_1 and η_2 coincide on $Z(H) \cap H'$; hence the map $\tilde{\eta}(zh') = \eta_1(z)\eta_2(h')$ $(z \in Z(H), h' \in H')$ is well defined and is a homomorphism on $H = Z(H) \cdot H'$. □

We are now ready to prove the main theorem of this section.

THEOREM 5.9. *Let G/k be a connected reductive group and let $\rho : G \longrightarrow \mathbf{SO}(V, q)$ be a representation rational over k. Let $\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_r$ be the set of nonzero weights of ρ relative to some maximal torus $T \subset G$ and let $m_1, m_2, \ldots, m_r$ be the corresponding multiplicities.*

Then there exists a k-homomorphism $\tilde{\rho} : G \longrightarrow \mathbf{Spin}(V)$ such that $\pi \circ \tilde{\rho} = \rho$ if and only if $\sum_i^r m_i \alpha_i \in 2X(T)$.

Proof. It is easy to see that since G is connected, the homomorphism $\tilde{\rho}$, if it exists, is unique. It follows immediately from uniqueness that if ρ is defined over k, then so is $\tilde{\rho}$, for it must be equal to all its Galois conjugates. Hence it is enough to prove the theorem over k_s.

Let M be the maximal torus of $\mathbf{SO}(V)$ such that $\rho(T) \subset M$. Let $\rho^* : X(M) \longrightarrow X(T)$ be the map induced by ρ. By Proposition 5.8, it is enough to show that the restriction of ρ to T can be lifted to $\tilde{\rho} : T \longrightarrow \tilde{M}$ if and only if $\rho^*(d) \in 2X(T)$.

By duality, the map $\rho : T \longrightarrow M \subset \mathbf{SO}(V)$ can be lifted to $\mathbf{Spin}(V)$ if and only if $\rho^* : X(M) \longrightarrow X(T)$ can be extended to $X(\tilde{M})$. Since $X(\tilde{M}) = X(M) + \frac{1}{2}d\mathbb{Z}$, the homomorphism ρ^* can be extended to $X(\tilde{M})$ if and only if $\rho^*(d) \in 2X(T)$. □

6. *The general case*

Let (A, a) be a general pointed algebra. By Proposition 3.1, we can assume that $a \in A$ is semisimple. The following lemma shows that we can also assume without loss of generality that (A, a) is a twist of pointed algebra of the form $(M_n(k), b)$.

LEMMA 6.1. *Let A be a central simple algebra over k and let $a \in A$. Then there exists $b \in M_n(k)$, unique up to conjugacy, such that (A, a) is a twist of $(M_n(k), b)$.*

Proof. Since A is a central simple algebra, there exists an isomorphism $\varphi : A \otimes k_s \longrightarrow M_n(k_s)$, where n is the degree of A over k. By the Skolem–Noether theorem, $\gamma(\varphi)\varphi^{-1}$ is an inner automorphism for all $\gamma \in \Gamma$. In particular, $\varphi(a)$ and $\gamma(\varphi(a))$ are similar in $M_n(k_s)$ for all $\gamma \in \Gamma$. It follows that the elementary divisors associated with $\varphi(a)$ have coefficients in k. Thus $\varphi(a)$ is similar to a matrix in $M_n(k)$. Uniqueness follows from the fact that a similarity class is uniquely determined by its elementary divisors. □

In this section we shall compare the quadratic forms $Q_{A,a}$ and $Q_{M_n(k),b}$, where (A, a) and $(M_n(k), b)$ are as in Lemma 6.1. For simplicity, we shall write Q_b in place of $Q_{M_n(k),b}$, as we did in Section 4.

When n is odd, the situation is particularly simple.

PROPOSITION 6.2. *If n is odd then $Q_{A,a} \simeq Q_b$.*

Proof. Let D be the division algebra in the same class of A in $\mathrm{Br}(k)$. It is well known that any maximal subfield L of D is a splitting field for A. Such a field L has odd degree over k, since D does. The forms $Q_{A,a}$ and Q_b become isometric over L, so by a theorem of Springer [**16**, Chapter 2, 5.4], $Q_{A,a} \simeq Q_b$ over k. □

For the case where n is even, we shall establish relations between the lower Stiefel–Whitney invariants of $Q_{A,a}$ and Q_b using Galois cohomology.

Let G be the automorphism group of the pointed algebra (M_n, b), regarded as an algebraic group over k. We shall first investigate the structure of G.

By the Skolem–Noether theorem, the elements of G are the inner automorphisms of M_n, and they must fix b. Hence

$$G \simeq Z_{\mathbf{GL}_n}(b)/\mathbf{GL}_1,$$

where $Z_{\mathbf{GL}_n}(b)$ is the centralizer of b in $\mathbf{GL}_n$ and $\mathbf{GL}_1$ is the subgroup of scalar matrices.

Since b is semisimple, the vector space $V = k_s^n$ is equal to the direct sum of its eigenspaces $V_1, V_2, \ldots, V_r$. Hence, over the separable closure k_s, the group $Z_{\mathbf{GL}_n}(b)$ admits the decomposition

$$Z_{\mathbf{GL}_n}(b) = \prod_{i=1}^{r} \mathbf{GL}(V_i). \tag{14}$$

This shows that G is a connected reductive algebraic group.

Clearly the group G acts on M_n by automorphisms of Q_b, that is, we have $G \subset \mathbf{O}(Q_b)$. Since G is connected, we must actually have $G \subset \mathbf{SO}(Q_b)$. This simple observation has a nontrivial consequence.

PROPOSITION 6.3. $w_1(Q_{A,a}) = w_1(Q_b)$.

Proof. Let $\det_*: H^1(k, \mathbf{O}(Q_b)) \longrightarrow H^1(k, \mathbb{Z}/2)$ be the map induced by the determinant map $\mathbf{O}(Q_b) \longrightarrow \mathbb{Z}/2$. It is easy to see directly on the cocycles that if c_q is the class in $H^1(k, \mathbf{O}(Q_b))$ corresponding to a quadratic form q, then $w_1(q) = w_1(Q_b) + \det_*(c_q)$. In particular, if c_q is represented by a cocycle with values in $\mathbf{SO}(Q_b)$, as is the case for $q = Q_{A,a}$, then $w_1(q) = w_1(Q_b)$. □

REMARK 6.4. D. Lewis proved the equivalent of Proposition 6.3 using generic splitting fields [**7**, Proposition 3.1].

Let $G^1 = Z_{\mathbf{GL}_n}(b) \cap \mathbf{SL}_n$. Observe that the restriction to G^1 of the canonical projection $Z_{\mathbf{GL}_n}(b) \longrightarrow G$ is an isogeny. Its kernel is $\boldsymbol{\mu}_n$, the group of nth roots of unity.

Let $S \subset Z_{\mathbf{GL}_n}(b)$ be a maximal torus defined over k. Then $T = S/\mathbf{GL}_1$ and $T^1 = S \cap \mathbf{SL}_n$ are maximal tori in G and G^1 respectively.

By (14), the rank of S is n, so S is conjugated in $\mathbf{GL}_n$ to the group of diagonal matrices $\mathbf{D}_n$, say $S = g\mathbf{D}_n g^{-1}$. Let $\pi_1, \ldots, \pi_n$ be the canonical projections $\mathbf{D}_n \longrightarrow \mathbf{GL}_1$ and define $\chi_i: S \longrightarrow \mathbf{GL}_1$ by $\chi_i(t) = \pi_i(g^{-1}tg)$ for $1 \leqslant i \leqslant n$ (notice that the χ_i are the weights for S acting on k_s^n via the inclusion $S \hookrightarrow \mathbf{GL}_n$).

The following result is well known (and easy to prove).

LEMMA 6.5. *Let $\rho: Z_{\mathbf{GL}_n}(b) \longrightarrow \mathbf{GL}(M_n)$ be the restriction of the adjoint representation $\mathbf{GL}_n \longrightarrow \mathbf{GL}(M_n)$. Then the weights of ρ relative to S are $\chi_i - \chi_j$ for $i, j = 1, \ldots, n$.*

PROPOSITION 6.6. *Let $\pi: \mathbf{Spin}(Q_b) \longrightarrow \mathbf{SO}(Q_b)$ be the canonical projection. Let $\rho: G^1 \longrightarrow \mathbf{SO}(Q_b)$ be the adjoint representation. Then there exists a homomorphism $\tilde{\rho}: G^1 \longrightarrow \mathbf{Spin}(Q_b)$ such that $\pi \circ \tilde{\rho} = \rho$.*

Proof. We shall verify the condition of Theorem 5.9. According to Lemma 6.5, the nonzero weights of ρ are $\chi_i - \chi_j$ $(i \neq j)$. Since the multiplicities of these weights are 1, by Theorem 5.9, the obstruction for the existence of $\tilde{\rho}$ is given by the class of $\rho^*(d) = \sum_{i<j} (\chi_i - \chi_j)$ in $X(T^1)/2X(T^1)$. This class is shown below to be trivial:

$$\begin{aligned}\rho^*(d) &\equiv \sum_{i<j} (\chi_i + \chi_j) \pmod{2X(T^1)} \\ &\equiv (n-1)(\chi_1 + \ldots + \chi_n) \pmod{2X(T^1)} \\ &\equiv 0 \pmod{2X(T^1)}. \qquad (15)\end{aligned}$$

(Note that $\chi_1 + \ldots + \chi_n = 0$ in $X(T^1)$.) □

Let $\rho: G \longrightarrow \mathbf{SO}(M_n, Q_b)$ be the representation given by $\rho(g)(x) = gxg^{-1}$. The following diagram of algebraic groups over k is commutative and the rows are exact sequences.

$$\begin{tikzcd} 1 \arrow[r] & \boldsymbol{\mu}_n \arrow[r] \arrow[d, "\tilde{\rho}"'] & G^1 \arrow[r] \arrow[d, "\tilde{\rho}"'] & G \arrow[r] \arrow[d, "\rho"'] & 1 \\ 1 \arrow[r] & \boldsymbol{\mu}_2 \arrow[r] & \mathbf{Spin}\,(Q_b) \arrow[r] & \mathbf{SO}\,(Q_b) \arrow[r] & 1 \end{tikzcd} \qquad (16)$$

LEMMA 6.7. *For n even, the induced map* $\tilde{\rho}: \boldsymbol{\mu}_n \longrightarrow \boldsymbol{\mu}_2$ *in* (16) *is nontrivial.*

Proof. By the same computation as in (15), we have $\rho^*(d) \equiv (n-1)(\chi_1 + \ldots + \chi_n)$ $(\mathrm{mod}\, 2X(T))$. Since $\chi_1 + \ldots + \chi_n \neq 0$ in $X(T)$ and $(n-1)$ is odd, $\rho^*(d) \notin 2X(T)$. Thus, by Theorem 5.9, the homomorphism $\rho: G \longrightarrow \mathbf{SO}(Q_b)$ cannot be lifted to $\mathbf{Spin}(Q_b)$. Hence $\tilde{\rho}: \boldsymbol{\mu}_n \longrightarrow \boldsymbol{\mu}_2$ is nontrivial, for otherwise $\tilde{\rho}: G^1 \longrightarrow \mathbf{Spin}(Q_b)$ would factor through G. □

We are now ready to prove the main result of this section.

THEOREM 6.8. *Let* Q_b *and* $Q_{A,a}$ *be as above. Then*

$$w_2(Q_{A,a}) = w_2(Q_b) + \frac{n(n-1)}{2}[A],$$

where $[A]$ *is the class of A in the Brauer group* $\mathrm{Br}(k) = H^2(k, \mathbf{GL}_1)$.

Proof. If n is odd, we have $w_2(Q_{A,a}) = w_2(Q_b)$ by Proposition 6.2. Note that in this case $(n(n-1)/2)[A] = 0$, since the order of $[A]$ is a divisor of n.

Let us now assume that n is even. The set $H^1(k, G)$ classifies all the pointed algebras that are twists of $(M_n(k), b)$. Let $[A, a]$ denote the cohomology class in $H^1(k, G)$ associated with (A, a) and let $[Q_{A,a}] \in H^1(k, \mathbf{SO}(Q_b))$ be the class associated with the form $Q_{A,a}$. Taking cohomology in (16), we obtain the commutative diagram

$$\begin{array}{ccc} H^1(k, G) & \xrightarrow{\partial} & H^2(k, \boldsymbol{\mu}_n) \\ {\scriptstyle \rho_*}\downarrow & & \downarrow{\scriptstyle \tilde{\rho}_*} \\ H^1(k, \mathbf{SO}(Q_b)) & \xrightarrow{\partial'} & H^2(k, \boldsymbol{\mu}_2) \end{array} \tag{17}$$

where ∂ and ∂' are the coboundary maps.

It is easy to see that $\rho_*[A, a] = [Q_{A,a}]$ and that $\partial[A, a] = [A]$, the class of A in $H^2(k, \boldsymbol{\mu}_n) = \mathrm{Br}_n(k)$.

By Springer's theorem [**20**, Formula 4.7] (in additive notation), we have

$$\begin{aligned} w_2(Q_{A,a}) &= w_2(Q_b) + \partial' \rho_*[A, a] \\ &= w_2(Q_b) + \tilde{\rho}_*[A], \end{aligned} \tag{18}$$

where the second equality uses the commutativity of diagram (17).

On the other hand, by Lemma 6.7, we have

$$\tilde{\rho}_*[A] = \frac{n}{2}[A].$$

Combining this equality with (18) we get

$$w_2(Q_{A,a}) = w_2(Q_b) + \frac{n}{2}[A]. \qquad \square$$

Finally, putting together Theorem 6.8 and Theorem 4.6 we obtain the most general formula.

THEOREM 6.9. *With the notation of Proposition* 4.4 *and Theorem* 4.6, *we have*

$$w_2(Q_{A,a}) = w_2(Q_1)+\frac{n(n-1)}{2}[A]+w_2(\tfrac{1}{2}\mathbf{tr}_{E'/k}(x^2))$$
$$+\frac{n(n-1)}{2}(-1,\det b)+\sum_{i=1}^{r}\mathrm{Cor}_{F_i/k}(d_i,-\beta_i), \qquad (19)$$

where $E' = E[t]/(t^2-b)$.

See Remark 4.7 for the computation of the terms $w_2(Q_1)$ and $w_2(\frac{1}{2}\mathbf{tr}_{E'/k}(x^2))$ in (19).

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