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Grothendieck Groups of Sesquilinear Forms over a Ring with Involution

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Introduction

For any ring with unit R equipped with an involution, we consider the sets $FP(R)$ and $F(R)$ of isomorphism classes of unimodular sesquilinear forms defined on finitely generated projective, respectively free, R-modules. We do not require the forms to satisfy any symmetry condition. The orthogonal sum of sesquilinear forms puts a monoid structure on these sets.

We also define a natural notion of exactness for triples of elements of $FP(R)$ and $F(R)$ and consider the corresponding Grothendieck groups $KFP(R)$ and $KF(R)$. Each can be viewed as the quotient of the Grothendieck group associated to the monoid by the subgroup generated by all "exactness relations". This fits into the formalism developed in [11, Sects. 1, 2].

The aim of this article is to determine the groups $KFP(R)$ and $KF(R)$ in terms of known algebraic objects.

Their study was motivated by the fact that the first author tried to use $KF(\mathbb{Z})$ as an obstruction group for a question that arose in the theory of high-dimensional knots [15]. The question however turned out to have a positive answer and this provided a computation of $KF(\mathbb{Z})$ and indeed with some modification of $KF(R)$ for R any euclidian ring. The method used failed for principal ideal domains and this led us to a study of these groups for rings with an involution in general.

Here is an outline of the contents and main results of this article.

Section 1 contains the basic definitions and shows that $KF(R)$ can be described in terms of matrices. The related notion of stable equivalence of matrices is introduced.

In Sect. 2 we give an exact sequence connecting $KF(R)$, $KFP(R)$ and a subgroup of the projective class group $\tilde{K}_0(R)$ (Theorem 2.2).

In Sect. 3 we show that the block sum operation puts an abelian group structure on the set $\Sigma(R)$ of stable equivalence classes of matrices (Proposition 3.4). The groups $KF(R)$ and $\Sigma(R)$ are essentially quotients of the K-theory group $K_1(R)$ (Theorems 3.2 and 3.6).

These quotients depend on the way the transpose-conjugation acts on $K_1(R)$. As a consequence, we show in Sect. 4 that $\Sigma(R)$ is trivial for instance in the case of a field, a euclidian ring and the ring of algebraic integers in a number field (Theorem 4.2), but can fail to be finitely generated even for a principal ideal domain (Example 4.3). Finally, using topological K-theory, we give examples of different ways in which the transposition acts on $K_1(R)$ and compute the corresponding $\Sigma(R)$.

1. Definitions

Let R be a ring with unit equipped with an involution, that is a map $x \to \bar{x}$ on R such that $\overline{x+y} = \bar{x} + \bar{y}$, $\overline{xy} = \bar{y}\bar{x}$, and $\bar{\bar{x}} = x$ for x and y in R. We denote by $U(R)$ the group of units of R.

We shall always assume that the rank of free finitely generated modules over R is well-defined, that is: if R^n is isomorphic to R^m then $n = m$; (for a discussion of this condition, see [17, Chap. II]).

All the R-modules considered in this article are finitely generated projective left R-modules. For an R-module P, we denote by P^* the dual $\mathrm{Hom}_R(P, R)$ of P with R operating on the left by $(a\varphi)(x) = \varphi(x)\bar{a}$ for a in R and φ in P^*. For a homomorphism $f: P_1 \to P_2$, we denote by f^* the dual homomorphism $P_2^* \to P_1^*$.

By a *sesquilinear form* (or simply a *form*) B on P we mean a R-homomorphism $B: P \to P^*$. Following the usual convention we say that B is *unimodular* if B is an isomorphism.

Two sesquilinear forms $B_i: P_i \to P_i^*$ $(i = 1, 2)$ are *isomorphic* if there exists a R-isomorphism $f: P_1 \to P_2$ such that $f^* B_2 f = B_1$.

Let $B_i: P_i \to P_i^*$ $(i = 1, 2)$ be sesquilinear forms; we denote by $B_1 \oplus B_2$ the *orthogonal sum* of B_1 and B_2.

A triple (B_1, B_2, B_3) of unimodular forms $B_i: P_i \to P_i^*$ $(i = 1, 2, 3)$ is *exact* if there exists a R-homomorphism $C: P_1 \to P_3^*$ such that the forms $\begin{bmatrix} B_1 & 0 \\ C & B_3 \end{bmatrix}$ and B_2 are isomorphic. For instance $(B_1, B_1 \oplus B_3, B_3)$ is an exact triple.

Let $FP(R)$ be the set of isomorphism classes of unimodular forms defined on finitely generated projective R-modules and denote by $\langle B \rangle$ the class of the form B.

Let $F(R)$ be the subset of $FP(R)$ consisting of isomorphism classes of forms defined on free R-modules.

Both $F(R)$ and $FP(R)$ are commutative monoids with respect to the orthogonal sum of forms $\oplus$, the zero element being represented by the unique form on the zero module.

The *Grothendieck groups* $KFP(R)$ and $KF(R)$ are defined as follows:

$KFP(R)$ [respectively $KF(R)$] is the quotient of the free abelian group on $FP(R)$ [respectively $F(R)$] by the subgroup generated by all expressions of the form $\langle B_2 \rangle - \langle B_2 \rangle - \langle B_3 \rangle$ where (B_1, B_2, B_3) is an exact triple of forms on projective (respectively free) R-modules.

Alternatively, $KFP(R)$ [respectively $KF(R)$] can be viewed as the quotient of the Grothendieck group associated to the monoid $(FP(R), \oplus)$ [respectively $(F(R), \oplus)$] by the exactness relations.

The map $F(R) \to \mathbb{Z}$ which associates to a form the rank of its underlying free module gives a surjective homomorphism $\varrho: KF(R) \to \mathbb{Z}$ and we set $\widetilde{KF}(R) = \ker \varrho$.

The homomorphism which sends 1 to the class of the rank one form $\langle 1\rangle$ on R is a section for ϱ and gives a canonical splitting $KF(R) \simeq \widetilde{KF}(R) \oplus \mathbb{Z}$.

The group $KF(R)$ can also be described in terms of matrices with coefficients in R as follows:

By convention the empty matrix ϕ is the unique invertible matrix of rank 0. Two matrices A and B are *congruent* if there exists an invertible matrix U such that $U^*AU = B$, where U^* stands for the transpose-conjugate of U.

$F(R)$ can be identified with the set of congruence classes of invertible matrices. The orthogonal sum of forms corresponds to the block sum $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ of the matrices A_1 and A_2.

Let (A_1, A_2, A_3) be a triple of invertible matrices. We say that (A_1, A_2, A_3) is *exact* if there exists a matrix X with coefficients in R such that A_2 is congruent to $\begin{pmatrix} A_1 & 0 \\ X & A_3 \end{pmatrix}$.

The group $KF(R)$ can therefore be viewed using the identification above as the quotient of the free abelian group on $F(R)$ by the subgroup generated by all elements of the form $\langle A_2\rangle - \langle A_1\rangle - \langle A_3\rangle$ where (A_1, A_2, A_3) is an exact triple of matrices.

Remark. For $R = \mathbb{Z}$, the exactness condition on triples of matrices has a geometric interpretation in knot theory; it corresponds to the plumbing operation on two fibre-surfaces of fibred knots (see [14, Sect. 2]).

Closely related to the study of $KF(R)$ is the following notion of stable equivalence of matrices. We borrow our notation from simple-homotopy theory (see [5, Sect. 4]).

Definitions. Let A_1 and A_2 be two invertible matrices with coefficients in R. We say that A_2 is an *elementary expansion* of A_1 (denoted by $A_1 \nearrow^{e} A_2$) if there exist u in $U(R)$, $x_1, \ldots, x_n$ in R such that A_2 is congruent to

$$\left[\begin{array}{c|c} A_1 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline x_1, \ldots, x_n & u \end{array}\right]$$

where $n = \operatorname{rank} A_1$. The matrix A_1 *expands* to A_2 $(A_1 \nearrow A_2)$ if there is a sequence of elementary expansions connecting A_1 and A_2; A_2 *collapses* to A_1 $(A_2 \searrow A_1)$ if A_1 expands to A_2. The matrix A_1 is *stably equivalent* to A_2 $(A_1 \wedge\!\!\searrow A_2)$ if there is a sequence of expansions and collapses connecting A_1 and A_2. This is clearly an equivalence relation.

Remarks. i) If A_1 is congruent to A_2, A_1 is stably equivalent to A_2.

ii) If $A_1 \wedge\!\!\searrow A_2$, there is a matrix B such that $A_1 \nearrow B$ and $B \searrow A_2$.

Let us denote by $\Sigma(R)$ the set of stable equivalence classes of invertible matrices over R. We shall see in Sect. 3 that the block sum operation puts an abelian group structure on $\Sigma(R)$.

When $\Sigma(R)$ is trivial, the ring R has the following property:

For any invertible matrix A there exist invertible triangular matrices T_1 and T_2 and a matrix X such that $\begin{pmatrix} A & 0 \\ X & T_1 \end{pmatrix}$ is congruent to T_2.

This will be the case in particular when R is a field or the ring of integers (see Theorem 4.2).

2. An Exact Sequence Connecting *KF*(*R*) and *KFP*(*R*)

The map $P \to P^*$ which sends a projective module over R to its dual determines an action of the cyclic group of order 2 on the projective class group $\tilde{K}_0(R)$. We denote by $\tilde{K}_0^+(R)$ the subgroup of elements of $\tilde{K}_0(R)$ that are fixed under this action.

We recall that a projective module is *self-dual* if it admits a unimodular form.

Lemma 2.1. i) *Each class of* $\tilde{K}_0^+(R)$ *contains a self-dual projective module.*

ii) *If* P *is a self-dual projective module, there is a self-dual projective module* Q *such that* $P \oplus Q$ *is free.*

Proof. i) Let $[P]$ be in $\tilde{K}_0^+(R)$, then $[P]=[P^*]$ so that $P \oplus R^s \simeq P^* \oplus R^t$ for some integers $s, t \geqq 0$. Since projective modules are canonically isomorphic to their biduals [4, Chap. II, 2.7], the dualization of this isomorphism gives $P^* \oplus R^s \simeq P \oplus R^t$. Combining these two isomorphisms one sees that $P \oplus R^{2s} \simeq P \oplus R^{2t}$. Since P is projective, there is a module Q such that $P \oplus Q \simeq R^m$ for some m. Thus R^{m+2s} is isomorphic to R^{m+2t} and therefore $s=t$. Set $P'=P \oplus R^s$, we have:

$$(P')^* \simeq P^* \oplus (R^s)^* \simeq P^* \oplus R^s \simeq P \oplus R^s = P'$$

so that P' is self-dual and $[P']=[P]$.

ii) Let P be a self-dual projective module and set $x=-[P]$; then $x^*=x$ so that by i), x is represented by a self-dual module Q and there exist integers s and $t \geqq 0$ such that $P \oplus Q \oplus R^s \simeq R^t$. The module $Q'=Q \oplus R^s$ is clearly self-dual and $P \oplus Q'$ is free.

The inclusion of $F(R)$ in $FP(R)$ determines a homomorphism $i: KF(R) \to KFP(R)$; the map $FP(R) \to \tilde{K}_0^+(R)$ which associates to a form the class of its underlying projective module induces a homomorphism $\pi: KFP(R) \to \tilde{K}_0^+(R)$ and we have:

Theorem 2.2. *The sequence*

$$0 \to KF(R) \xrightarrow{i} KFP(R) \xrightarrow{\pi} \tilde{K}_0^+(R) \to 0$$

is exact.

Proof. The map π is surjective by Lemma 2.1 and clearly $\pi \circ i = 0$. Let y be in $KFP(R)$ such that $\pi(y)=0$; we can represent y as $y=[B_1]-[B_2]$ where $B_i: P_i \to P_i^*$ is unimodular and P_i is projective. By Lemma 2.1, there is a self-dual module Q_2 equipped with a form B_2' such that $P_2 \oplus Q_2$ is free. We have $y=[B_1 \oplus B_2'] - [B_2 \oplus B_2']$. As $[P_1 \oplus Q_2]=[P_1]-[P_2]=\pi(y)=0$, there are integers s and t such that $P_1 \oplus Q_2 \oplus R^s \simeq R^t$. Let C_3 be a unimodular form on R^s and denote by C_4 the form $B_1 \oplus B_2' \oplus C_3$. The equality $y=[C_4]-[C_3]-[B_2 \oplus B_2']$ shows that y is in the image of i.

Let $x=[B_1]-[B_2]$ be an element of $KF(R)$, where $B_i: L \to L_i^*$ is a unimodular form defined on a free module and suppose that $i(x)=0$. There exist unimodular forms $C_k^\alpha: P_k^\alpha \to (P_k^\alpha)^*$, $\alpha=1,2,3$ where P_k^α is a projective module, such that

(C_k^1, C_k^2, C_k^3) is an exact triple and such that the following equality holds in the free abelian group on $FP(R)$:

$$\langle B_1\rangle - \langle B_2\rangle = \sum_k \beta_k(\langle C_k^1\rangle + \langle C_k^3\rangle - \langle C_k^2\rangle)$$

with β_k in $\mathbb{Z}$. By Lemma 2.1, there exist forms $D_k^\alpha : Q_k^\alpha \to (Q_k^\alpha)^*$, $\alpha = 1, 3$ such that $P_k^\alpha \oplus Q_k^\alpha$ is free. For each k,

$$(C_k^1 \oplus D_k^1, C_k^2 \oplus D_k^1 \oplus D_k^3, C_k^3 \oplus D_k^3)$$

is an exact triple of forms defined on free modules, so that

$$x = \sum_k \beta_k([C_k^1 \oplus D_k^1] + [C_k^3 \oplus D_k^3] - [C_k^2 \oplus D_k^1 \oplus D_k^3]) = 0 \quad \text{in } KF(R).$$

Example. For a Dedekind ring D with trivial involution, $KFP(D)$ is an extension of $KF(D)$ by the subgroup of elements of order $\leqq 2$ of the ideal class group of D.

3. Determination of $KF(R)$ and $\Sigma(R)$

Let G be an abelian group written multiplicatively and suppose that the cyclic group of order two C_2 acts on G by $g \to \bar{g}$. We denote by NG the norm subgroup

$$NG = \{y \in G \mid y = \bar{x}x \text{ for some } x \text{ in } G\}$$

Let $U(R)^{ab}$ denote the abelianization of $U(R)$. The involution on R gives a C_2-action on $U(R)^{ab}$.

Recall that $K_1(R)$ is the abelian group defined as the quotient of the infinite general linear group $GL(R)$ by the subgroup generated by the elementary matrices over R (see [19, Chap. 13] for the basic facts about $K_1(R)$). We shall write the group operation multiplicatively.

The map

$$GL(R) \to GL(R),$$
$$A \mapsto A^*$$

which sends a matrix A to its transpose-conjugate yields a C_2-action on $K_1(R)$.

The canonical homomorphism $U(R) = GL_1(R) \to K_1(R)$ induces a homomorphism $j: U(R)^{ab} \to K_1(R)$ and we set $\bar{K}_1(R) = \operatorname{coker} j$.

The homomorphism j is compatible with the actions on $U(R)^{ab}$ and $K_1(R)$, so that there is an induced C_2-action on $\bar{K}_1(R)$.

We can therefore consider the norm subgroups $NU(R)^{ab}$, $NK_1(R)$, and $N\bar{K}_1(R)$.

Let $\mathbb{1}$ denote the unit matrix in $GL_n(R)$.

Lemma 3.1. i) *If A and B are in $GL_n(R)$ the equality $[A] + [B] = [AB] + [\mathbb{1}]$ holds in $KF(R)$.*

ii) *For every element x in $KF(R)$ there is an integer n and a matrix C in $GL_n(R)$ such that $x = [C] - [\mathbb{1}]$.*

Proof. i) The matrix

$$U = \begin{pmatrix} A^* - \mathbb{1} & \mathbb{1} \\ A^* & \mathbb{1} \end{pmatrix} = \begin{pmatrix} -\mathbb{1} & \mathbb{1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ A^* & \mathbb{1} \end{pmatrix}$$

is invertible and we have the equality:

$$(*) \qquad U^*\begin{pmatrix} A & 0 \\ \mathbb{1}-A-B & B \end{pmatrix} U = \begin{pmatrix} AB & 0 \\ A^*+B-\mathbb{1} & \mathbb{1} \end{pmatrix}$$

so that $[A]+[B]=[AB]+[\mathbb{1}]$ in $KF(R)$.

ii) Any element x of $\widetilde{KF}(R)$ can be written as $x=[A]-[B]$ with A and B in $GL_n(R)$ for some n. Set $C=AB^{-1}$, then $[C]-[\mathbb{1}]=[A]-[B]$ using i).

The map

$$F(R) \to K_1(R)/NK_1(R),$$
$$\langle A \rangle \mapsto [A]$$

is clearly well-defined and induces a homomorphism $KF(R) \to K_1(R)/NK_1(R)$. To show this, suppose that (A_1, A_2, A_3) is an exact triple of matrices so there exist an invertible matrix U and a matrix X such that

$$A_2 = U^*\begin{pmatrix} A_1 & 0 \\ X & A_3 \end{pmatrix} U = U^*\begin{pmatrix} A_1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ X & \mathbb{1} \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ 0 & A_3 \end{pmatrix} U.$$

Since $\begin{pmatrix} \mathbb{1} & 0 \\ X & \mathbb{1} \end{pmatrix}$ is a product of elementary matrices we have

$$[A_2][A_1^{-1}][A_3^{-1}]=[U^*][U]=1$$

in $K_1(R)/NK_1(R)$. Let Φ denote the restriction of this homomorphism to $\widetilde{KF}(R)$.

Conversely, Lemma 3.1 i) shows that the maps

$$GL_n(R) \to \widetilde{KF}(R),$$
$$A \mapsto [A]-[\mathbb{1}]$$

are homomorphisms. They induce a homomorphism $K_1(R) \to \widetilde{KF}(R)$ which vanishes on $NK_1(R)$ since $[U^*U]-[\mathbb{1}]=0$. Let $\Psi: K_1(R)/NK_1(R) \to \widetilde{KF}(R)$ denote the induced homomorphism.

Clearly $\Phi \circ \Psi$ is the identity on $K_1(R)/NK_1(R)$. Let x in $KF(R)$ be represented as $x=[C]-[\mathbb{1}]$ with C in $GL_n(R)$ using Lemma 3.1 ii); $\Psi \circ \Phi(x) = \Psi([C]) = x$. We therefore deduce the following theorem which characterizes $\widetilde{KF}(R)$:

Theorem 3.2. *The homomorphism $\Phi: \widetilde{KF}(R) \to K_1(R)/NK_1(R)$ is an isomorphism.*

We now turn to the determination of $\Sigma(R)$.

Lemma 3.3. i) *Let A be an $m \times m$ invertible matrix, B be an $n \times n$ invertible matrix, and X be any $n \times m$ matrix, then:*

$$\begin{pmatrix} A & 0 \\ X & B \end{pmatrix} \textit{ is stably equivalent to } \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

ii) *Let A and B be two invertible matrices of the same rank, then:*

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \textit{ is stably equivalent to } AB.$$

Proof. We denote by $\mathbb{1}_k$ the unit matrix in $GL_k(R)$.

i) $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ expands to $\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ V & -B & \mathbb{1}_n \end{pmatrix}$

where V will be determined below. The matrix

$$U=\begin{pmatrix}\mathbb{1}_m & 0 & 0\\ 0 & \mathbb{1}_n & 0\\ 0 & \mathbb{1}_n & \mathbb{1}_n\end{pmatrix}\begin{pmatrix}\mathbb{1}_m & 0 & 0\\ 0 & \mathbb{1}_n & B^*-\mathbb{1}_n\\ 0 & 0 & \mathbb{1}_n\end{pmatrix}$$

is invertible and

$$U^*\begin{pmatrix}A & 0 & 0\\ 0 & B & 0\\ V & -B & \mathbb{1}_n\end{pmatrix}U=\begin{pmatrix}A & 0 & 0\\ V & \mathbb{1}_n & B^*\\ BV & 0 & B\end{pmatrix}.$$

This last matrix is congruent to

$$\begin{pmatrix}A & 0 & 0\\ BV & B & 0\\ V & B^* & \mathbb{1}_n\end{pmatrix}$$

which collapses to $\begin{bmatrix}A & 0\\ BV & B\end{bmatrix}$. Setting $V=B^{-1}X$ proves i).

ii) By i), $\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}$ is stably equivalent to $\begin{bmatrix}A & 0\\ \mathbb{1}-A-B & B\end{bmatrix}$. The equality (*) in the proof of Lemma 3.1 shows that the latter is congruent to $\begin{bmatrix}AB & 0\\ A^*+B-\mathbb{1} & \mathbb{1}\end{bmatrix}$ which collapses to AB.

Proposition 3.4. $\Sigma(R)$ *forms an abelian group for the operation induced by the block sum of matrices.*

Proof. To see that the addition is well-defined, it clearly suffices to prove that if $A_1 \nearrow_e A_2$ then $A_1 \oplus B \nearrow_e A_2 \oplus B$ for any invertible matrix B. Suppose that A_2 is congruent to

$$\left(\begin{array}{c|c} A_1 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ \hline x_1,\dots,x_n & u\end{array}\right)$$

where the x_i are in R and u is in $U(R)$; $A_2\oplus B$ is congruent to

$$\left(\begin{array}{c|c|c} A_1 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & 0\\ \hline x_1\dots x_n & u & 0\dots 0\\ \hline 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & B\end{array}\right) \text{ and therefore to } \left(\begin{array}{cc|c} A_1 & 0 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\\ 0 & B & \\ \hline x_1\dots x_n & 0\dots 0 & u\end{array}\right)$$

which collapses to $A_1\oplus B$.

The zero element is represented by the class of the empty matrix. If A is an invertible matrix, $[A]$ admits $[A^T]=[A^{-1}]$ as an inverse since by Lemma 3.3, $\begin{bmatrix} A & 0 \\ 0 & A^* \end{bmatrix} \nwarrow\!\!\searrow AA^*$ which is congruent to $\mathbb{1}$, $\begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \nwarrow\!\!\searrow AA^{-1}=\mathbb{1}$ and $\mathbb{1}$ collapses to ϕ.

The map

$$U(R)\to \widetilde{KF}(R),$$

$$u \mapsto [u]-[1]$$

is a homomorphism by Lemma 3.1 and induces a homomorphism $U(R)^{ab}\to\widetilde{KF}(R)$ which vanishes on $NU(R)^{ab}$. Denote by $j' : U(R)^{ab}/NU(R)^{ab}\to\widetilde{KF}(R)$ the induced homomorphism.

The map

$$F(R)\to\Sigma(R),$$

$$\langle A\rangle\to[A]$$

induces a surjective homomorphism $KF(R)\to\Sigma(R)$ since $\begin{bmatrix} A_1 & 0 \\ X & A_3 \end{bmatrix}$ is stably equivalent to $\begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix}$ for any invertible matrices A_1 and A_3 and any matrix X. We denote by μ its restriction to $\widetilde{KF}(R)$.

Proposition 3.5. *There is an exact sequence:*

$$U(R)^{ab}/NU(R)^{ab} \xrightarrow{j'} \widetilde{KF}(R) \xrightarrow{\mu} \Sigma(R)\to 0$$

Proof. The homomorphism μ is clearly surjective and $\mu\circ j'=0$. Let x in $\widetilde{KF}(R)$ be represented as $x=[C]-[\mathbb{1}]$ with C in $GL_n(R)$. If $\mu(x)=1$, C is stably equivalent to the empty matrix, so there is a sequence $\phi=A_0, A_1, \ldots, A_k=C$ such that $A_i \nearrow_e A_{i+1}$ or $A_{i+1} \nearrow_e A_i$. This shows that there exist elements u_i in $U(R)$ such that $[A_{i+1}]=[A_i]+\varepsilon_i[u_i]$ in $KF(R)$ where $\varepsilon_i=+1$ if $A_i \nearrow_e A_{i+1}$, $\varepsilon_i=-1$ if $A_{i+1} \nearrow_e A_i$. Moreover we have $\sum_{i=1}^k \varepsilon_i=n$. Thus the equality $[C]=\sum_{i=1}^k \varepsilon_i[u_i]$ holds in $KF(R)$ and $[C]-[\mathbb{1}]=\sum_{i=1}^k \varepsilon_i([u_i]-[1])$ is in the image of j'.

Theorem 3.6. *The group $\Sigma(R)$ is isomorphic to $\overline{K_1}(R)/N\overline{K_1}(R)$.*

Proof. The homomorphism $j : U(R)^{ab}\to K_1(R)$ induces

$$\bar{j}: U(R)^{ab}/NU(R)^{ab}\to K_1(R)/NK_1(R)$$

and the diagram

$$\begin{array}{ccc} U(R)^{ab}/NU(R)^{ab} & \xrightarrow{\quad j' \quad} & \widetilde{KF}(R) \\ & {\scriptstyle \bar{j}}\searrow \quad \swarrow{\scriptstyle \Phi} & \\ & K_1(R)/NK_1(R) & \end{array}$$

clearly commutes. Proposition 3.5 shows that $\Sigma(R)$ is isomorphic to cokerj' and it is easy to see that $\overline{K_1}(R)/N\overline{K_1}(R)$ is isomorphic to coker$\bar{j}$. The result follows from the fact that Φ is an isomorphism.

Remark. Neither j nor $\bar{j}$ are injective in general. For instance, let R be the ring of 2×2 matrices over $\mathbb{Z}$ together with the transposition of matrices as an involution. The group $K_1(R)$ is isomorphic to C_2 while $U(R)^{ab}$ is isomorphic to $C_2\times C_2$. Moreover the C_2-actions on $U(R)^{ab}$ and $K_1(R)$ induced by the transposition are trivial; this shows that $\bar{j}$ is not injective.

When R is a commutative ring the determinant induces a split epimorphism $\det: K_1(R)\to U(R)$ so that $SK_1(R)=\ker\det$ can be identified with $\bar{K}_1(R)$. This identification commutes with the C_2-actions induced on $SK_1(R)$ and $\bar{K}_1(R)$ by the transpose-conjugation of matrices and we get:

Corollary 3.7. *For a commutative ring R,*

$$\Sigma(R) \text{ is isomorphic to } SK_1(R)/NSK_1(R),$$

$$KF(R) \text{ is isomorphic to } \mathbb{Z}\oplus U(R)/NU(R)\oplus SK_1(R)/NSK_1(R).$$

Remark. This corollary shows that for a commutative ring R the sequence of Proposition 3.5 can be extended to a short exact sequence.

The following corollary gives a "stable range" condition for $\Sigma(R)$.

Corollary 3.8. *Let R be a commutative ring which is a finite algebra over a ring of Krull dimension d, then every element of $\Sigma(R)$ can be represented by an invertible matrix of* rank $d+1$.

Proof. A theorem of Bass (see [19], Theorem 12.3 and Theorem 13.5) shows that in this situation the natural map $GL_{d+1}(R)\to K_1(R)$ is surjective.

In particular we obtain the following:

Corollary 3.9. *Let R be a commutative ring which is a finite algebra over a ring of Krull dimension* 1 *and suppose that the involution on R is trivial, then $x^*=x^{-1}$ in $SK_1(R)$ and $\Sigma(R)$ is isomorphic to $SK_1(R)$.*

Proof. The maps $GL_2(R)\to K_1(R)$ and therefore $SL_2(R)\to SK_1(R)$ are surjective. Since any matrix C in $SL_2(R)$ satisfies

$$C^*\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$x^*=x^{-1}$ holds in $SK_1(R)$.

4. Examples

Example 4.1. Let G be a torsion abelian group. By a theorem of Bak ([2]), the involution $g\mapsto g^{-1}$ for g in G induces the trivial C_2-action on $SK_1(\mathbb{Z}G)$. This shows that for this involution $\Sigma(\mathbb{Z}G)$ is isomorphic to $SK_1(\mathbb{Z}G)/SK_1(\mathbb{Z}G)^2$.

From now on we restrict ourselves to the case where R is a commutative ring with *trivial involution* and compute the corresponding group $\Sigma(R)$.

The condition in Corollary 3.9 is fulfilled for instance in the following cases:

- $R=\mathbb{Z}G$, where G is a finite abelian group;
- R is a Dedekind ring;
- R is a field.

Let C_n denote the cyclic group of order n, we deduce:

Theorem 4.2. i) *$\Sigma(R)$ is trivial in the following cases:*
– *R is a euclidian ring (in particular $\mathbb{Z}$, the p-adic integers $\mathbb{Z}_p$ or a field)*
– *R is the ring of algebraic integers in a number field.*
ii) *If G is a finite abelian group, $\Sigma(\mathbb{Z}G)$ is trivial if and only if*
– *G is either an elementary abelian 2-group or*
– *every p-Sylow subgroup of G is either cyclic or of the form $C_p \times C_p^n$.*

Proof. It is well known that if R is euclidian (in particular a field or a discrete valuation ring), $SK_1(R)$ is trivial. A theorem of Bass, Serre, and Milnor (see [16], Sect. 16) shows that $SK_1(R)=0$ in the case of the ring of algebraic integers in a number field. For the result mentioned about group rings, see [1, Theorem 4.9].

Remark. The fact that $\Sigma(\mathbb{Z})$ is trivial has a geometric interpretation in knot theory: it shows that every high-dimensional fibred knot is stably obtained by Hopf plumbing and gives another proof of [15], Theorem 1.

We now give examples of rings for which $\Sigma(R)$ is non trivial.

Example 4.3. Bass [3, Sect. 9.2] gives a method for constructing examples of principal ideal domains B such that $SK_1(B)$ and therefore $\Sigma(B)$, although generated by rank 2 matrices, are not finitely generated. It can be shown that the ring $B=\mathbb{Q}(t)[X, Y]/(Y^2-X^3-7)$ is an instance of such a ring.

Example 4.4. Let R be the coordinate ring of an affine algebraic variety X defined over the reals such that the set of real points $X_{\mathbb{R}}$ of X is a non-empty compact connected topological space. Topological K-theory can be used to show that $\Sigma(R)$ is non trivial.

The group $\widetilde{KO}^{-1}(X_{\mathbb{R}})$ is isomorphic to the group of homotopy classes $[X_{\mathbb{R}}; SL(\mathbb{R})]$ and the inclusion of SO in $SL(\mathbb{R})$ induces an isomorphism $\Psi: [X_{\mathbb{R}}; SO] \to [X_{\mathbb{R}}; SL(\mathbb{R})]$ (see [6, Sect. 3] which clearly preserves transposition. The natural map $SL(R) \to [X_{\mathbb{R}}; SL(\mathbb{R})]$ induces a homomorphism $\Phi: SK_1(R) \to [X_{\mathbb{R}}; SL(\mathbb{R})]$ and the composite $\Psi^{-1} \circ \Phi$ vanishes on $NSK_1(R)$. This gives a well-defined homomorphism $\Sigma(R) \to \widetilde{KO}^{-1}(X_{\mathbb{R}})$.

Consider for instance $R_m = \mathbb{R}[X_0, X_1, \ldots, X_m]/(X_0^2 + \ldots + X_m^2 - 1)$, the coordinate ring of the m-sphere S^m.

For $m=1,3$ the matrices

$$A_1=\begin{pmatrix} X_0 & -X_1 \\ X_1 & X_0 \end{pmatrix}, \quad A_3=\begin{pmatrix} X_0 & -X_1 & -X_2 & -X_3 \\ X_1 & X_0 & -X_3 & X_2 \\ X_2 & X_3 & X_0 & -X_1 \\ X_3 & -X_2 & X_1 & X_0 \end{pmatrix}$$

represent elements in $\Sigma(R_m)$.

The maps

$$S^m \to SO(m+1),$$

$$x \mapsto A_m(x)$$

correspond to the multiplication of complex and quaternionic numbers of unit norm respectively and give generators for the groups $\Pi_m(SO)$, $m=1,3$. ($\Pi_m(SO)$ is cyclic of order 2 for $m=1$ and infinite cyclic for $m=3$; see [12, Chap. V, Sect. 3] and [20].)

The matrices above are therefore specific examples of matrices that are not stably trivial. For $m=7$, a similar example can be constructed using Cayley numbers.

Even when R is a commutative ring with trivial involution, we shall show that the equation $x^*=x^{-1}$ does not necessarily hold in $SK_1(R)$.

Let $C(Y)$ denote the ring of continuous real valued functions on the topological space Y. Recall that if Y is compact and connected, $\tilde{K}_0(C(Y))$ is isomorphic to $\widetilde{KO}(Y)$ [18, Theorem 2] and $SK_1(C(Y))$ is isomorphic to $\widetilde{KO}^{-1}(Y)$ [6, Lemma 3.1].

Let R_m be the coordinate ring of S^m, $m \geqq 1$, and let

$$S=\{r\in R_m | r(x)\neq 0 \text{ for all } x \text{ in } S^m\}$$

The set S is multiplicative and we consider the ring of fractions $A_m=S^{-1}R_m$. Since R_m is a regular integral domain, so is A_m.

It is well known that R_m and therefore A_m can be viewed as dense subalgebras of $C(S^m)$. Using [6, Theorem 2.7], [7, Theorem 1] and [8] it can be shown that $\tilde{K}_0(A_m)$ is isomorphic to $\tilde{K}_0(C(S^m))$ and $SK_1(A_m)$ is isomorphic to $SK_1(C(S^m))$.

Set $\Lambda_m=A_m[X,X^{-1}]$. Since A_m is a regular integral domain,

$$U(\Lambda_m)\simeq \mathbb{Z}\times U(A_m)$$

and $SK_1(\Lambda_m)\simeq \tilde{K}_0(A_m)\oplus SK_1(A_m)\simeq \widetilde{KO}(S^m)\oplus\widetilde{KO}^{-1}(S^m)$ [19, Corollary 16.5].

The transposition in $SK_1(\Lambda_m)$ corresponds to the dualization of modules over A_m and hence of bundles over S^m [18, Sect. 2]. Since every bundle is isomorphic to its dual, the transposition acts trivially on the first summand. On the second summand it corresponds to the transposition in $[S^m; SO]$ and therefore $x^*=x^{-1}$ holds in $\widetilde{KO}^{-1}(S^m)$. We deduce that

$$\Sigma(\Lambda_m)\simeq \widetilde{KO}(S^m)/2\widetilde{KO}(S^m)\oplus\widetilde{KO}^{-1}(S^m).$$

Example 4.5 [where $x^*=x$ holds in $SK_1(\Lambda)$]:

For $m\equiv 4\,(8)$, $\widetilde{KO}(S^m)\simeq\mathbb{Z}$, and $\widetilde{KO}^{-1}(S^m)=0$ [13, Chap. 9, Sect. 5] so that $x^*=x$ holds in $SK_1(\Lambda_m)$ and $\Sigma(\Lambda_m)\simeq\mathbb{Z}/2$.

The ring $C(\mathbb{R}P^m)$ of continuous real valued functions on the projective space $\mathbb{R}P^m$ can be identified with the subring of even functions of $C(S^m)$. Let $\bar{R}_m$ denote the subring of R_m whose elements are represented by even polynomials. Set $\bar{S}=S\cap\bar{R}_m$ and consider the ring of fractions $\bar{A}_m=\bar{S}^{-1}\bar{R}_m$. It can be shown that $\bar{R}_m$ and therefore $\bar{A}_m$ are regular integral domains which inject as dense subalgebras into $C(\mathbb{R}P^m)$. Using [7, Theorem 1], [10, Sect. 6] and [6, Theorem 2.7], we see that $\tilde{K}_0(\bar{A}_m)$ is isomorphic to $\tilde{K}_0(C(\mathbb{R}P^m))$ and $SK_1(\bar{A}_m)$ is isomorphic to $SK_1(C(\mathbb{R}P^m))$.

Set $\bar{\Lambda}_m=\bar{A}_m[X,X^{-1}]$. The same arguments as above show that

$$SK_1(\bar{\Lambda}_m)\simeq\tilde{K}_0(\bar{A}_m)\oplus SK_1(\bar{A}_m)\simeq\widetilde{KO}(\mathbb{R}P^m)\oplus\widetilde{KO}^{-1}(\mathbb{R}P^m)$$

and

$$\Sigma(\bar{\Lambda}_m)\simeq\widetilde{KO}(\mathbb{R}P^m)/2\widetilde{KO}(\mathbb{R}P^m)\oplus\widetilde{KO}^{-1}(\mathbb{R}P^m).$$

Example 4.6 [where neither $x^*=x$ nor $x^*=x^{-1}$ holds in $SK(\Lambda)$]:

For $m=8r+3$ (respectively $8r+7$), $\widetilde{KO}(\mathbb{R}P^m)\simeq\mathbb{Z}/2^{4r+2}$ (respectively $\mathbb{Z}/2^{4r+3}$) and $\widetilde{KO}^{-1}(\mathbb{R}P^m)\simeq\mathbb{Z}\oplus\mathbb{Z}/2$ (see [9, Theorem 1]); therefore $\Sigma(\bar{\Lambda}_m)\simeq\mathbb{Z}/2\oplus(\mathbb{Z}\oplus\mathbb{Z}/2)$ so that neither $x^*=x$ nor $x^*=x^{-1}$ holds in $SK_1(\bar{\Lambda}_m)$.

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References

1. Alperin, R.C., Dennis, R.K., Oliver, R., Stein, M.R.: SK_1 of finite abelian groups II. Invent. Math. **87**, 253–302 (1987)
2. Bak, A.: The involution on Whitehead torsion. Gen. Topology Appl. **7**, 201–206 (1977)
3. Bass, H.: Some problems in classical algebraic K-theory. Algebraic K-theory II. Batelle Institute Conf. 1972 (Lect. Notes Mathematics, Vol. 342, pp. 3–73.) Berlin Heidelberg New York: Springer 1973
4. Bourbaki, N.: Algèbre. Paris: Hermann 1970
5. Cohen, M.M.: A course in simple-homotopy theory. (Graduate Texts Mathematics, Vol. 10) Berlin Heidelberg New York: Springer 1973
6. Dayton, B.H.: SK_1 of commutative normed algebras. Algebraic K-theory. Evanston 1976 (Lect. Notes Mathematics, Vol. 551, pp. 30–43) Berlin Heidelberg New York: Springer 1976
7. Evans, E.G., Jr.: Projective modules as fibre bundles. Proc. Am. Math. Soc. **27**, 623–626 (1971)
8. Fossum, R.: Vector bundles over spheres are algebraic. Invent. Math. **8**, 222–225 (1969)
9. Fujii, M.: K_0-groups of projective spaces. Osaka J. Math. **4**, 141–149 (1967)
10. Geramita, A.V., Roberts, L.C.: Algebraic vector bundles on projective space. Invent. Math. **10**, 298–304 (1970)
11. Heller, A.: Some exact sequences in algebraic K-theory. Topology **4**, 389–408 (1965)
12. Hilton, P.J.: An introduction to homotopy theory. Cambridge Tracts 43. Cambridge University Press 1961
13. Husemoller, D.: Fibre bundles (Graduate Texts Mathematics, Vol. 20) Berlin Heidelberg New York: Springer 1966
14. Lines, D.: On odd-dimensional fibred knots obtained by plumbing and twisting. J. London Math. Soc. (2) **32**, 557–571 (1985)
15. Lines, D.: Stable plumbing for high odd-dimensional fibred knots. Canad. Math. Bull. **30**, 429–435 (1987)
16. Milnor, J.: Introduction to algebraic K-theory. Ann. Math. Stud., Vol. 72. Princeton University Press 1971
17. Silvester, J.R.: Introduction to algebraic K-theory. London New York: Chapman and Hall 1981
18. Swan, R.: Vector bundles and projective modules. Trans. Am. Math. Soc. **105**, 264–277 (1962)
19. Swan, R.: Algebraic K-theory (Lect. Notes Mathematics, Vol. 76). Berlin Heidelberg New York: Springer 1968
20. Toda, H., Saito, Y., Yokota, I.: Note on the generator of $\pi_7 SO(n)$. Mem. Coll. Sci. Univ. Kyoto Ser. A, Math. **30**, 227–230 (1957)

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