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EQUIVARIANT WITT GROUPS

BY

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ABSTRACT. This paper studies for a number field K and a finite group Γ the cokernel of the residue homomorphism $W(K\Gamma) \longrightarrow \bigoplus_{\mathfrak{p}} W(k(\mathfrak{p})\Gamma)$.

Introduction. Let K be a number field and R the ring of integers of K. It is well known that the Witt groups of the ring R, the field K and the residue fields $k(\mathfrak{p})$ are related by an exact sequence

$$(1) \qquad 0 \longrightarrow W(R) \longrightarrow W(K) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} W(k(\mathfrak{p})) \longrightarrow C(K)/C(K)^2 \longrightarrow 0$$

where $C(K)$ is the ideal class group of K and $\mathfrak{p}$ runs over all nonzero prime ideals of R (see [12] Chapter 6 and [5] Chapter 4). The nontrivial step in the proof of the exactness of this sequence is the identification of coker(∂) with $C(K)/C(K)^2$, which involves some version of quadratic reciprocity.

In this paper we shall consider, for a fixed finite group Γ, the equivariant analogue of (1), namely the sequence of equivariant Witt groups

$$(2) \qquad 0 \longrightarrow W(R\Gamma) \longrightarrow W(K\Gamma) \xrightarrow{\partial} \bigoplus_{\mathfrak{p}} W(k(\mathfrak{p})\Gamma)$$

which is seen easily to be exact (see [1] Chap I, Theorem 4.1). Alexander-Conner-Hamrick ([1] Chapter I and IV) and Dress ([4] Section 4 Theorem 5) asked for a description of coker(∂) in order to have a complete equivariant analogue of (1). To the author's knowledge, no general computation of coker(∂) is known so far. This paper gives an answer in some important cases. Here is an outline of its contents .

Section 1 contains the basic definitions.

In Section 2 we consider the case where Γ is a finite abelian group and K is the field of the rational numbers. With these hypotheses the homomorphism ∂ turns out to be surjective.

In Section 3 we consider the case where Γ is a p-group (p odd). We show that in this case coker(∂) is isomorphic to $C(K)/C(K)^2$, as in the non-equivariant case.

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It is in particular independent of Γ. As a subsidiary result we calculate the structure of $W(R\Gamma)$. In Section 4 we propose a conceptual setting for describing coker(∂) in general.

1. **Definitions.** Here we recall the definitions and notations that will be used in the next sections. Let Λ be a ring equipped with an involution J and let L be a finitely generated left Λ-module. A *hermitian form* h on L is a function

$$h : L \times L \longrightarrow \Lambda$$

which is Λ-linear in the first variable and satisfies $h(x,y)^J = h(y,x)$. If in addition the adjoint map $y \longmapsto h(\ ,y)$ is an isomorphism from L to $\mathrm{Hom}_\Lambda(L,\Lambda)$ then h is called *regular*.

A hermitian form (L,h) is said to be *metabolic* if there is a sub-Λ-module $N \subset L$ for which $N = N^\perp$.

The Witt group $W(\Lambda, J)$ is an additive group defined by generators and relations as follows. It is generated by the isomorphism classes $[L,h]$ of regular hermitian forms over Γ, with the relations

$$(R1) \qquad [L_1,h_1] + [L_2,h_2] = [L_1 \boxplus L_2, h_1 \boxplus h_2]$$

$$(R2) \qquad [L,h] = 0 \quad \text{if } (L,h) \text{ is metabolic,}$$

where $\boxplus$ denotes the orthogonal sum. Let R be a Dedekind domain with field of fractions K. Let A be a semi-simple K-algebra with K-involution J, and Λ an R-order of A preserved by J. A *torsion hermitian form* over Λ is a pair (T,h), where T is a finitely generated R-torsion Λ-module, and h is a function

$$h : T \times T \longrightarrow A/\Lambda$$

which is Λ-linear in the first variable and satisfies $h(x,\ y)^J = h(y,x)$. If the adjoint map $y \longmapsto h(\ ,y)$ is a bijection $T \longrightarrow \mathrm{Hom}_\Lambda(T, A/\Lambda)$ then h is called *regular*.

A torsion hermitian form (T,h) is *metabolic* if T admits a sub-Λ-module S which is self-orthogonal with respect to h.

Similarly, the *torsion Witt group* $WT(\Lambda, J)$ is the additive group generated by the isomorphism classes $[T,h]$ of regular torsion hermitian forms with the relations (R1) and (R2).

The torsion Witt group $WT(\Lambda,J)$ is (non-canonically) isomorphic to the direct sum

$$\bigoplus_{\mathfrak{p}} W(\Lambda/\mathfrak{p}\Lambda, J)$$

where $\mathfrak{p}$ runs over all nonzero prime ideals of R (see [1] Chapter I Section 3). There is a canonical homomorphism $\partial : W(A,J) \longrightarrow WT(\Lambda,J)$ defined as follows. Let (V,h) be a regular hermitian form over A. Choose a Λ-lattice $L \subset V$ such that $h(L,L) \subseteq \Lambda$.

Let $L^\sharp$ be the dual lattice $\{x \in V : h(x,L) \subseteq \Lambda\}$. The torsion module $L^\sharp/L$ carries the regular hermitian form $h_L(\bar{x},\bar{y}) = h(x,y) \pmod{\Lambda}$. It is easy to see that the Witt class of $(L^\sharp/L, h_L)$ does not depend on the particular choice of the lattice $L \subset V$ and that $\partial(V,h) = (L^\sharp/L, h_L)$ is a homomorphism (see [1] Chapter I Section 4).

The Witt groups $W(\Lambda,J)$, $W(A,J)$ and $WT(\Lambda,J)$ are connected by a canonical exact sequence

$$0 \longrightarrow W(\Lambda,J) \xrightarrow{\iota} W(A,J) \xrightarrow{\partial} WT(\Lambda,J)$$

where ι is induced by tensor product with K over R and ∂ is the homomorphism defined above (see [1] Chapter I).

We shall mainly deal with the case where Λ is the group ring $R\Gamma$ of a finite group Γ and J is the canonical involution $\gamma \longmapsto \gamma^{-1}$ for $\gamma \in \Gamma$. In this case, using the canonical isomorphism

$$\mathrm{Hom}_R(L,R) \longrightarrow \mathrm{Hom}_{R\Gamma}(L,R\Lambda)$$

$$f \longmapsto \Big(x \longmapsto \sum_{\gamma\in\Gamma} f(\gamma^{-1}x)\gamma\Big),$$

we shall identify the set of hermitian forms on L with the set of Γ-equivariant symmetric bilinear forms over L.

By abuse of language, we shall use the abbreviated notation $W(\Lambda)$ for the Witt group of Λ with respect to some fixed involution, whenever there is no danger of confusion. This abbreviated notation will be applied in particular to group rings, where no involution different from the canonical one will be considered.

For the elementary properties of equivariant Witt groups we refer to [1]. For a general categorical setting we refer to [9] and [12].

2. **The abelian case.** Throughout this section Γ denotes a finite abelian group. Our aim is to prove that the canonical homomorphism $\partial : W(\mathbb{Q}\Gamma) \longrightarrow WT(\mathbb{Z}\Gamma)$ is surjective.

LEMMA 2.1. *Let p be a prime number and $d = p^\nu > 2$. Let ζ be a primitive d^{th} root of unity. Let $\mathfrak{p}$ be the ramified prime ideal of $\mathbb{Z}[\zeta+\zeta^{-1}]$ (which lies above p). Then the canonical map*

$$\mathbb{Z}[\zeta+\zeta^{-1}]^\times \longrightarrow (\mathbb{Z}[\zeta+\zeta^{-1}]/\mathfrak{p})^\times = \mathbb{F}_p^\times$$

is surjective.

PROOF. Let m be a positive integer not divisible by p. We define

$$u = \frac{(\zeta^m - \zeta^{-m})}{(\zeta-\zeta^{-1})}$$

$$= \zeta^{-m+1}\sum_{i=0}^{m-1} \zeta^{2i}. \tag{3}$$

It is not difficult to check that u is a unit (see [14] Lemma 8.1). Let $\mathfrak{P}$ be the prime ideal of $\mathbb{Z}[\zeta]$ lying above $\mathfrak{p}$. Since $\zeta \equiv 1 \pmod{\mathfrak{P}}$, it follows from (3) that $u \equiv m (\text{mod } \mathfrak{p})$. $\square$

Proposition 2.2. *Let ζ be a primitive d^{th} root of unity and $\mathfrak{q}$ a prime ideal of $\mathbb{Z}[\zeta]$ not dividing d and preserved by complex conjugation.*

a) if d is composite (i.e. d is not of the form p^ν or $2p^\nu$ with p prime) then there exists a rank one hermitian form (L, h) over $\mathbb{Z}[\zeta]$ such that $L^\sharp/L \simeq \mathbb{Z}[\zeta]/\mathfrak{q}$.

b) If d is not composite and $d \neq 2$ then there is a rank two skew-hermitian form (L, h) over $\mathbb{Z}[\zeta]$ such that $L^\sharp/L \simeq \mathbb{Z}[\zeta]/\mathfrak{q}$.

Proof. Since $\mathfrak{q}$ is unramified and preserved by complex conjugation, it may be regarded as an ideal of $\mathbb{Z}[\zeta+\zeta^{-1}]$. By class field theory, the norm map $C(\mathbb{Q}(\zeta)) \longrightarrow C(\mathbb{Q}(\zeta+\zeta^{-1})$ is surjective (see [14] Theorem 10.1): there exists an ideal $\mathfrak{a}$ of $\mathbb{Q}(\zeta)$ and $\lambda \in \mathbb{Q}(\zeta+\zeta^{-1})$ such that $\mathfrak{q} = \lambda\mathfrak{a}\bar{\mathfrak{a}}$.

a) Define $L = \mathfrak{a}$ and $h(x, y) = \lambda x\bar{y}$. By construction we have

$$\begin{aligned} L^\sharp/L &= (\lambda\bar{\mathfrak{a}})^{-1}/\mathfrak{a} \\ &\simeq \mathbb{Z}[\zeta]/\lambda\mathfrak{a}\bar{\mathfrak{a}} \\ &= \mathbb{Z}[\zeta]/\mathfrak{q}. \end{aligned}$$

b) Suppose $d = p^\nu$. Let $\mathfrak{p}$ be the prime ideal of $\mathbb{Z}[\zeta]$ above p. It is well known (see [14] Lemma 1.4) that $\mathfrak{p}$ is principal and generated by $\pi = \zeta - \zeta^{-1}$. Without loss of generality we can assume that $\mathfrak{a}$ is contained in $\mathbb{Z}[\zeta]$ and is relatively prime to $\mathfrak{p}$. It follows from the equality $\mathfrak{q} = \lambda\mathfrak{a}\bar{\mathfrak{a}}$ that λ is a $\mathfrak{p}$-unit. Moreover, by Lemma (2.1), we can assume $\lambda \equiv 1 \pmod{\mathfrak{p}}$. Let h be the skew-hermitian form defined on $V = \mathbb{Q}(\zeta) \oplus \mathbb{Q}(\zeta)$ whose matrix with respect to the natural basis is

$$H = \begin{pmatrix} (\lambda-1)\pi^{-1} & 1 \\ -1 & \pi \end{pmatrix}.$$

Let $L = \mathfrak{a} \oplus \mathbb{Z}[\zeta]$; we claim that (L, h) satisfies the conclusion of part b) of the proposition. We can see from the shape of the matrix H that h takes integral values on L if and only if $(\lambda-1)\pi^{-1}\mathfrak{a}\bar{\mathfrak{a}} \subseteq \mathbb{Z}[\zeta]$. Since $\lambda \equiv 1 \pmod{\mathfrak{p}}$ and $\mathfrak{a} \subseteq \mathbb{Z}[\zeta]$, we have $\text{ord}_{\mathfrak{p}}((\lambda-1)\pi^{-1}\mathfrak{a}\bar{\mathfrak{a}}) \geq 0$. For a prime $\mathfrak{r} \neq \mathfrak{p}$ we have

$$\begin{aligned} \text{ord}_{\mathfrak{r}}((\lambda-1)\pi^{-1}\mathfrak{a}\bar{\mathfrak{a}}) &= \text{ord}_{\mathfrak{r}}((\lambda-1)\mathfrak{a}\bar{\mathfrak{a}}) \\ &\geq \text{ord}_{\mathfrak{r}}(\lambda\mathfrak{a}\bar{\mathfrak{a}} + \mathfrak{a}\bar{\mathfrak{a}}) \\ &= \text{ord}_{\mathfrak{r}}(\mathfrak{q} + \mathfrak{a}\bar{\mathfrak{a}}) \\ &\geq 0. \end{aligned}$$

Hence $h(L, L) \subseteq \mathbb{Z}[\zeta]$. By the construction of (L, h) we have

$$\begin{aligned} \text{ord}_{\mathfrak{r}}(\det(L_{\mathfrak{r}}, h)) &= \text{ord}_{\mathfrak{r}}(\mathfrak{a}\bar{\mathfrak{a}}\det(H)) \\ &= \text{ord}_{\mathfrak{r}}(\mathfrak{a}\bar{\mathfrak{a}}\lambda) \\ &= \text{ord}_{\mathfrak{r}}(\mathfrak{q}) = \begin{cases} 1, & \text{if} \quad \mathfrak{r} = \mathfrak{q} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Thus $L^\sharp/L \simeq \mathbb{Z}[\zeta]/\mathfrak{q}$ as claimed. $\square$

THEOREM 2.3. *Let Γ be a finite abelian group. Then the canonical homomorphism*

$$W(\mathbb{Q}\Gamma) \xrightarrow{\partial} WT(\mathbb{Z}\Gamma)$$

is surjective.

PROOF. Let (T,β) be a torsion form over $\mathbb{Z}\Gamma$. We will show that the Witt class of (T,β) is in the image of ∂. The form (T,β) is Witt-equivalent to an orthogonal sum of torsion forms whose underlying $\mathbb{Z}\Gamma$-modules are simple. Thus we may assume that T is simple. Let $k = \mathrm{End}_{\mathbb{Z}\Gamma}(T)$ and $q = \mathrm{char}(k)$. The field k is a finite extension of $\mathbb{F}_q$ and T can be regarded as an absolutely simple $k\Gamma$-module. Let $\chi : \Gamma \longrightarrow k^\times$ be the character of T and $d = |\mathrm{im}(\chi)|$. If $d \leq 2$, i.e. $\chi(\gamma) = \pm 1$, then the group Γ plays no role, and using the surjectivity of $W(\mathbb{Q}) \longrightarrow WT(\mathbb{Z})$ (see [5] Chap. IV Theorem 2.1) we see that (T,β) is in the image of ∂. Thus we may assume that $d > 2$. In this case since $\chi \neq \chi^{-1} = \chi^*$, the adjoint involution on k is not trivial. Let ζ be a primitive d^{th} root of unity. The field k can be viewed as the quotient of $\mathbb{Z}[\zeta]$ by a prime ideal $\mathfrak{q}$ stable by complex conjugation, and the character $\chi : \Gamma \longrightarrow k^\times$ can be lifted to a character $\Gamma \longrightarrow \mathbb{Q}(\zeta)^\times$, which will be denoted again χ. Every $\mathbb{Z}[\zeta]$-module M becomes a $\mathbb{Z}\Gamma$-module by defining

$$\gamma \cdot x = \chi(\gamma)x \quad \text{for} \quad \gamma \in \Gamma \quad \text{and} \quad x \in M.$$

In particular, T is isomorphic to its endomorphism ring k viewed as a $\mathbb{Z}\Gamma$-module.

Let $\mathfrak{D}$ be the different of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$. It is known that $\mathfrak{D}$ is a principal ideal (it is in fact generated by $\Phi_d'(\zeta)$) (see for instance [6] Chap. III, Section 2, Proposition 8). Moreover, a generator α of $\mathfrak{D}$ can be chosen such that

$$\bar{\alpha} = \begin{cases} \alpha, & \text{if } d \text{ is composite} \\ -\alpha, & \text{otherwise} \end{cases}$$

(see also [2] Lemma 1.6).

Let (L,h) be the hermitian (skew-hermitian if d not composite) form over $\mathbb{Z}[\zeta]$ provided by Proposition (2.2). Let

$$\begin{aligned} b : L \times L &\longrightarrow \mathbb{Z} \\ (x,y) &\longmapsto \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\alpha^{-1}h(x,y)). \end{aligned}$$

By construction, b is a symmetric bilinear Γ-equivariant form, and by Proposition (2.2) we have

$$L^\sharp/L \simeq T$$

as $\mathbb{Z}\Gamma-$ modules. Let b_L be the form induced on T by b. It is left to show that $b_L \simeq \beta$. Since T is simple, we have $b_L(x,y) = \beta(ux,y)$, where $u \in k^\times$ and $\bar{u} = u$. Since k is a finite field and the adjoint involution on k is not trivial, there exists $v \in k^\times$ such that $u = v\bar{v}$. Thus $b_L(x,y) = \beta(vx, vy)$, that is b_L is isomorphic to β as required. □

3. **The case of a p-group.** We fix the following notation for this section.

p an odd prime
Γ a finite p-group
K a number field
R the ring of integers of K
$k(\mathfrak{p})$ the residue field $R/\mathfrak{p}$ ($\mathfrak{p}$ is a nonzero prime ideal of R)
A the group algebra $K\Gamma$
Λ the integral group ring $R\Gamma$
S the set of primes of K which divide p
$W_0(\Lambda)$ the kernel of the restriction map $W(\Lambda) \longrightarrow W(R)$
$W_0(\Lambda_S)$, $W_0(A)$, $W_0T(\Lambda)$ and $W_0T(\Lambda_S)$ are defined in an analogous manner as the kernels of the appropriate restriction maps.

THEOREM 3.1. *The localization map $\Lambda \longrightarrow \Lambda_S$ induces isomorphisms*

$$W_0(\Lambda) \xrightarrow{\simeq} W_0(\Lambda_S) \tag{4}$$
$$W_0T(\Lambda) \xrightarrow{\simeq} W_0T(\Lambda_S) \tag{5}$$

PROOF. We first prove that (5) is an isomorphism. Let (T, β) be a torsion form in the kernel of (5). On the one hand, we know that every Witt class has a representative whose underlying module is semisimple (see [8] Lemma 1.2). Thus we may assume that T is semisimple. On the other hand, since (T, β) is in the kernel of (5), T can be written

$$T = \bigoplus_{\mathfrak{p} \in S} T_{\mathfrak{p}}$$

where $T_{\mathfrak{p}}$ is a semisimple $k(\mathfrak{p})\Gamma$-module. Since Γ is a p-group and $\mathfrak{p}$ divides p, we deduce that Γ acts trivially on $T_{\mathfrak{p}}$ (see [3] Theorem 5.24). By hypothesis, (T, β) is also in the kernel of the restriction map $WT(\Lambda) \longrightarrow WT(R)$; thus $T = 0$. The surjectivity of (5) is obvious. The fact that (4) is an isomorphism follows now easily from (5) and the exact sequence (2).

LEMMA 3.2. *Let V be a simple $\mathbb{C}\Gamma$-module. Then V is self-dual if and only if Γ acts trivially on V.*

PROOF. We proceed by induction on the cardinality of Γ. Let $\Gamma \longrightarrow C_p$ be a surjective homomorphism and let Γ_0 denote its kernel. Let $U \subseteq V$ be a simple sub-$\mathbb{C}\Gamma_0$-module of V.

a) If $U = V$, by the induction hypothesis Γ_0 acts trivially on V. Thus V can be regarded as a self-dual $\mathbb{C}C_p$-module. Its character χ is a homomorphism $C_p \longrightarrow \mathbb{C}^\times$ which satisfies $\chi = \chi^* = \chi^{-1}$. Since p is odd, we must have $\chi = 1$.

b) If $U \neq V$ we have $V = \mathrm{Ind}_{\Gamma_0}^{\Gamma}(U)$. Using again the fact that p is odd, we conclude that U must also be self-dual. By the induction hypothesis Γ_0 acts trivially on U and therefore also on V. The same argument as in a) shows that Γ acts trivially on V. □

PROPOSITION 3.3. *Suppose that the group algebra A has a simple component $B \neq K$ which is preserved by the canonical involution. Let E denote the center of B. Then E is a CM-field and the restriction to E of the involution on B coincides with complex conjugation. In particular K is a totally real field.*

PROOF. Let V be the simple A-module corresponding to B. Since B is preserved by the involution, V is self-dual. By a theorem of Schilling (see [10] Theorem 41.9), B is a matrix algebra over its center E. Thus $E = \mathrm{End}_A(V)$ and V can be regarded as an absolutely simple $E\Gamma$-module. For every K-embedding $E \hookrightarrow \mathbb{C}$, the module $V \otimes_E \mathbb{C}$ *is not* self-dual by Lemma (3.2), i.e. the involution on $\mathbb{C}$ is not trivial. This shows both that E is a CM-field and that the involution on E is complex conjugation. Since K is fixed by the involution, K must be totally real. □

THEOREM 3.4. *Let V be a simple self-dual A-module. Then there exists a symmetric Γ-equivariant form $b : V \times V \longrightarrow K$ and a Λ_S-lattice L such that $L = L_b^\sharp$.*

PROOF. If Γ acts trivially on V the theorem is obvious. Thus we may assume that $V^\Gamma \neq V$. Let c be any symmetric nondegenerate Γ-equivariant form on V (since K is totally real such forms exist). Let $M \subset V$ be a Λ_S-lattice. Let $E = \mathrm{End}_A(V)$ and $\mathfrak{a} = \mathrm{Hom}_{\Lambda_S}(M, M_c^\sharp) \subset E$. We leave to the reader to check that $\mathfrak{a}M = M_c^\sharp$ (see [8] Theorem 2.7). Let F be the maximal real subfield of E. The S-ideal $\mathfrak{a}$ of E is by construction stable by complex conjugation and contains no ramification. Thus it can be regarded as an S-ideal of F. Using the fact that the norm map $N_{E/F} : C(E) \longrightarrow C(F)$ is surjective (see [14] Theorem 10.1), we conclude that there is an S-ideal $\mathfrak{b}$ of E such that $\mathfrak{a} = \lambda\mathfrak{b}\bar{\mathfrak{b}}$. Put $L = \mathfrak{b}M$. We have

$$\begin{aligned} L_b^\sharp = (\bar{\mathfrak{b}})^{-1} M_b^\sharp &= (\lambda\bar{\mathfrak{b}})^{-1} M_c^\sharp \\ &= (\lambda\bar{\mathfrak{b}})^{-1}\mathfrak{a}M \\ &= \mathfrak{b}M \\ &= L. \quad \square \end{aligned}$$

THEOREM 3.5. *Let A_i $(i = 1 \ldots r)$ be simple components of A which are different from the trivial component K and are preserved by the canonical involution J. Let $\Lambda_i \subset A_i$ be the image of Γ_S in A_i. Let E_i be the center of A_i and O_i the center of Λ_i. Then there is a commutative diagram*

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_0(\Lambda_S) & \longrightarrow & W_0(A) & \longrightarrow & W_0T(\Lambda_S) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_i W(O_i, {}^-) & \longrightarrow & \bigoplus_i W(E_i, {}^-) & \longrightarrow & \bigoplus_i WT(O_i, {}^-) \end{array}$$

where the bottom row consists of hermitian Witt groups with respect to complex conjugation $^-$ and the vertical arrows are isomorphisms.

PROOF. Let $e_0, \ldots, e_r$ be the indecomposable central idempotents of A fixed by the involution, where e_0 corresponds to the trivial component K. Let $e_{r+1}, e^J_{r+1}, \ldots, e_n, e^J_n$ be the remaining central indecomposable idempotents. Since p is invertible in R_S, the R_S-order Λ_S is maximal and we have

$$\Lambda_S = e_o\Lambda_S \oplus \cdots \oplus e_r\Lambda_S \oplus (e_{r+1}\Lambda_S \oplus e^J_{r+1}\Lambda_S) \oplus \cdots \oplus (e_n\Lambda_S \oplus e^J_n\Lambda_S).$$

Thus, every form (M, b) over Λ_S decomposes canonically as an orthogonal sum

$$M = e_0M \boxplus \cdots \boxplus e_rM \boxplus (e_{r+1}M \oplus e^J_{r+1}M) \boxplus \cdots \boxplus (e_nM \oplus e^J_nM)$$

where the factors $e_iM \oplus e^J_iM$ are clearly hyperbolic. Thus we have canonical isomorphisms

$$W_0(\Lambda_S) \xrightarrow{\simeq} \bigoplus_{i=1}^{r} W(\Lambda_i)$$

and

$$W_0T(\Lambda_S) \xrightarrow{\simeq} \bigoplus_{i=1}^{r} WT(\Lambda_i).$$

We use now the technique of *transfer to the endomorphism ring* (see [12] Chap. VII Section 4 and [9] for a general setting) to show that there is a commutative diagram

$$\text{(6)}\qquad \begin{array}{ccccccc} 0 & \longrightarrow & W(\Lambda_i) & \longrightarrow & W(A_i) & \longrightarrow & WT(\Lambda_i) \\ & & \big\downarrow \Phi_i & & \big\downarrow \Phi'_i & & \big\downarrow \Phi''_i \\ 0 & \longrightarrow & W(O_i, ^-) & \longrightarrow & W(E_i, ^-) & \longrightarrow & WT(O_i, ^-) \end{array}$$

where the vertical arrows are the (non-canonical) isomorphisms defined below. Let V_i the simple self-dual A-module corresponding to A_i. Let b_i be a form on V_i and $L_i \subset V_i$ a Λ_S-lattice unimodular with respect to b_i (such a pair exists by Theorem 3.4). We define

$$\begin{aligned} \Phi_i : W(\Lambda_i) &\longrightarrow W(O_i, ^-) \\ (M, b) &\longmapsto (\Phi_i(M), \Phi_i(b)). \end{aligned}$$

where $\Phi_i(M) = \mathrm{Hom}_{\Lambda_S}(L_i, M)$ and $\Phi_i(b)(f, g)$ is the composite homomorphism

$$L_i \xrightarrow{g} M \xrightarrow{b} M^* \xrightarrow{f^*} L_i^* \xrightarrow{b_i^{-1}} L_i$$

Note that $\Phi_i(M)$ carries a natural structure of right O_i-module. We define Φ'_i : $W(A_i) \longrightarrow W(E_i, ^-)$ in exactly the same manner. Finally, Φ''_i on torsion forms is defined as follows:

$$\begin{aligned} \Phi''_i : WT(\Lambda_i) &\longrightarrow WT(O_i, ^-) \\ (T, \beta) &\longmapsto (\Phi''_i(T), \Phi''_i(\beta)). \end{aligned}$$

where $\Phi_i''(T) = \mathrm{Hom}_{\Lambda_S}(L_i, T)$. For $f, g \in \Phi_i(T)$ we define $\Phi_i''(\beta)(f, g)$ as the composite homomorphism

$$L_i \xrightarrow{g} T \xrightarrow{\beta} \hat{T} \xrightarrow{\hat{f}} \mathrm{Hom}_{\Lambda_i}(L_i, A_i/\Lambda_i) = V_i^*/L_i^* \xrightarrow{b_i^{-1}} V_i/L_i$$

(where we identify $\mathrm{Hom}_{\Lambda_i}(L_i, V_i/L_i)$ with $\mathrm{End}_{A_i}(V_i)/\mathrm{End}_{\Lambda_i}(L_i) = E_i/O_i$).

We leave to the reader to check that the formalism of transfer as described in [12] Chap. VII Section 4 works well in this situation and that the maps Φ_i,Φ_i' and Φ_i'' give isomorphisms. Clearly the diagram (6) is commutative.

Corollary 3.6 *We keep the notations of Theorem* 3.5. *There is an isomorphism*

$$W(\Lambda) \xrightarrow{\simeq} W(R) \oplus \bigoplus_{i=1}^{r} W(O_i, {}^-). \tag{7}$$

Proof. Apply Theorem 3.1 and Theorem 3.5. □

Corollary 3.7. *Suppose in addition that p has only one prime divisor in K. Then the restriction homomorphism $W(\Lambda) \longrightarrow W(R)$ induces an isomorphism of torsion subgroups*

$$W(\Lambda)_{\mathrm{tor}} \longrightarrow W(R)_{\mathrm{tor}}.$$

Proof. We see from the decomposition (7) that it is enough to show that $W(O, {}^-)$ is free. In effect, let (M, h) be in the kernel of the signature homomorphism $\sigma : W(O, {}^-) \longrightarrow \mathbb{Z}^m$. In particular M has even rank over O. It is easy to see that for an inert prime $\mathfrak{q}$ of E_i, the rank map

$$\rho : W((O_i)_{\mathfrak{q}}, {}^-) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

is an isomorphism. Thus $(M_v, h) = 0$ in $W((O_i)_v, {}^-)$ for every place (finite and infinite) v not dividing p. Now, since p has only one prime divisor, say $\mathfrak{p}$, it follows from Hilbert Reciprocity that $(M_{\mathfrak{p}}, h)$ is also metabolic. Hence, by the Hasse Principle, (M, h) is metabolic as well. □

Corollary 3.8. *Let K be a totally real field having only one dyadic prime and odd class number. Suppose in addition that K contains units with independent sign and that p has only one prime divisor in K. Then $W(\Lambda)$ is a finitely generated free abelian group.*

Proof. Our hypotheses imply that $W(R)$ is torsion-free (see [5] Chap IV Corollary 4.3). Hence, by Corollary 3.7, $W(\Lambda)$ is torsion-free as well. □

REMARK. Corollary 3.8 was observed by Alexander-Conner-Hamrick for $K = \mathbb{Q}$ and $\Gamma = C_{p^\nu}$ (see [1] Chapter III).

THEOREM 3.9. *The canonical map* $\partial : W_0(A) \longrightarrow W_0T(\Lambda)$ *is surjective.*

PROOF. By Theorem 3.5, it is sufficient to prove that $\partial : W(E_i, ^-) \longrightarrow WT(O_i, ^-)$ is surjective. Let (T, β) be a torsion form over O_i whose underlying O-module T is simple. Then $T \simeq O_i/\mathfrak{q}$, where $\mathfrak{q}$ is a prime S-ideal of E fixed by the involution. Since $\mathfrak{q}$ is unramified, it can be seen as an ideal of the field F_i fixed by the involution. By surjectivity of the norm map $N_{E/F} : C(E_i) \longrightarrow C(F_i)$, there exists an ideal $\mathfrak{a}$ of E_i and $\lambda \in F_i$ such that $\mathfrak{q} = \lambda\mathfrak{a}\bar{\mathfrak{a}}$. Let $h(x, y) = \lambda x\bar{y}$. It is now easy to see that $\partial(E_i, h) = (T, \beta)$ (see proof of Theorem 2.3) □

COROLLARY 3.10. *The cokernel of* $\partial : W(A) \longrightarrow WT(\Lambda)$ *is isomorphic to* $C(K)/C(K)^2$.

PROOF. We know (see for instance [12] Chapter 6 theorem 6.11) that coker(∂ : $W(K) \longrightarrow WT(R)$) is isomorphic to $C(K)/C(K)^2$. Corollary 3.10 follows inmmediately from this fact together with Theorem 3.9.

4. **A general approach.** In this section K is a number field, R is the ring of integers of K, A is a semisimple K-algebra with K-involution and Λ an R-order preserved by the involution on A.

Let $G_0(\Lambda)$ be the Grothendieck group of the category of finitely generated Λ-modules (for the definition and elementary properties we refer to [3] Chap. 5). According to Theorem 38.42 in [3], we can also see $G_0(\Lambda)$ as the Grothendieck group of the subcategory of Λ-*lattices*. We shall take this viewpoint to define an action of the cyclic group C_2 of order two on $G_0(\Lambda)$. In effect, let σ be the generator of C_2 and L a Λ-lattice. We define

$$[L]^\sigma = [L^*]$$

where $L^* = \mathrm{Hom}_\Lambda(L, \Lambda)$. The following lemma shows how C_2 acts on torsion modules.

LEMMA 4.1. *Let T be a finitely generated torsion Λ-module. Then the following holds in $G_0(\Lambda)$:*

$$[T]^\sigma = -[\hat{T}]$$

where $\hat{T} = \mathrm{Hom}_\Lambda(T, A/\Lambda)$.

PROOF. Take a presentation $0 \longrightarrow L_1 \longrightarrow L_0 \longrightarrow T \longrightarrow 0$ of T, where L_i $(i := 0, 1)$ is a Λ-lattice. By definition of the action of C_2 we have in $G_0(\Lambda)$

$$\begin{aligned}[T]^\sigma &= ([L_0] - [L_1])^\sigma = [L_0]^\sigma - [L_1]^\sigma \\ &= [L_0^*] - [L_1^*] = -([L_1^*] - [L_0^*]) \\ &= -[\hat{T}]\end{aligned}$$

where the last equality comes from the dual sequence $0 \longrightarrow L_0^* \longrightarrow L_1^* \longrightarrow \hat{T} \longrightarrow 0$. □

PROPOSITION 4.2. *The map* $\varphi : WT(\Lambda) \longrightarrow H^1(C_2, G_0(\Lambda))$ *given by* $\varphi(T,h) = [T]$ *is a well-defined homomorphism and satisfies* $\varphi\partial = 0$.

PROOF. Let (T,h) be a torsion form. Since T is by definition self-dual, we have $[T]^\sigma + [T] = -[\hat{T}] + [T] = 0$. Hence $[T]$ is a 1-cocycle. Suppose now that $[T,h]$ is metabolic, that is there exists a submodule $S \subset T$ such that $S^\perp = S$. The submodule S provides an exact sequence $0 \longrightarrow S \longrightarrow T \longrightarrow \hat{S} \longrightarrow 0$, which interpreted in $G_0(\Lambda)$ gives $[T] = [S] + [\hat{S}] = [S] - [S]^\sigma$. Thus $[T]$ is a 1-coboundary and φ is well defined.

It is left to show that $\varphi\partial = 0$. By the definition of ∂ we have $(\varphi\partial)(V,h) = [L^\sharp/L]$ for some lattice $L \subset V$. Since $L^\sharp \simeq L^*$ we have $(\varphi\partial)(V,h) = [L^\sharp/L] = [L^*] - [L] = [L]^\sigma - [L] = 0$ in $H^1(C_2, G_0(\Lambda))$. □

EXAMPLE 4.3. Let $\Lambda = R$. Then $H^1(C_2, G_0(\Lambda)) \simeq C(K)/C(K)^2$ and $\mathrm{Im}(\partial) = \ker(\varphi)$.

PROOF. Since $G_0(R)$ is isomorphic to $K_0(R)$ for a Dedekind domain, we have $G_0(R) \simeq \mathbb{Z} \oplus C(K)$ (see [7] Corollary 1.11). Moreover, one sees easily that this isomorphism can be chosen C_2-equivariant (C_2 acts trivially on $\mathbb{Z}$ and by $[\mathfrak{a}]^\sigma = [\mathfrak{a}^{-1}]$ on $C(K)$). Thus $H^1(C_2, G_0(\Lambda)) \simeq C(K)/C(K)^2$ as claimed. The statement that $\mathrm{Im}(\partial) = \ker(\varphi)$ follows from the exactness of (1) (see [12] Chapter 6 Theorem 6.11). □

Propositions (4.2) and Example (4.3) suggest that it is not unreasonable to conjecture that the sequence $W(A) \xrightarrow{\partial} WT(\Lambda) \xrightarrow{\varphi} H^1(C_2, G_0(\Lambda))$ is exact . It can be checked directly that $\mathrm{Im}(\varphi) = 0$ if $\Lambda = \mathbb{Z}\Lambda$ with Λ abelian and $\mathrm{Im}(\varphi) \simeq C(K)/C(K)^2$ for $\Lambda = R\Gamma$ with Γ a p-group, as expected from Theorem 2.3 and Corollary 3.10.

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