Math 7390, Section 1

Applied Harminic Analysis

The heat equation on $\mathbb{R}^n$ and generalizations

Textbook: Lecture notes by G. Ólafsson.
Time: 1:40-2:30, Monday, Wednesday and Friday in 290 Prescott)
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SYLLABUS

This course can be described a the interplay between real analysis on $\mathbb{R}^n$ and complex analysis on $\mathbb{C}^n$, as well as the study of Hilbert spaces of holomorphic functions on $\mathbb{C}^n$ (minimal knowledge of complex analysis of one variable is only required). We will also introduce some basic facts on analysis on infinite dimensional spaces. We might even discuss some basic facts of representation theory of the Heisenberg group.
The heat equation is one of the more important partial differential equations on $\mathbb{R}^n$. We will discuss its relation to complex and infinite dimensional analysis. The main topic are:

- Basic Introduction to the heat equation on $\mathbb{R}^n$.
- Short introduction to Fourier analysis.
- The solution of the heat equation using Fourier analysis. The heat kernel.
- Holomorphic functions and the Fock space of holomorphic functions on $\mathbb{C}^n$.
- The Segal-Bargmann transform as an unitary isomorphism of L^2-spaces into to the Fock space.
- The infinite dimensional case.
- If there is still time, then we will discuss some generalizations, in particular related to root systems, multiplicity functions, finite reflection groups.

The heat equation on $\mathbb{R}^n$ is

$$\Delta u(x,t) = \frac{\partial}{\partial t} u(x,t)\,, \quad \lim_{t\to o^+} u(x,t) = f(x)\,. \tag{1}$$

We will assume that f is either in $L^2(\mathbb{R}^n)$ or that $f \in L^2(\mathbb{R}^n, h_t(x)dx)$ where h_t is the heat kernel

$$h_t(x) = (4\pi t)^{-n/2} e^{-\|x\|^2/(4t)}\,.$$

The solution to (1) is given by

$$u(x,t) = (2\pi t)^{-n/2} \int_{\mathbb{R}^m} f(x) e^{-\|x-y\|^2/(4t)}\, dx = \int_{\mathbb{R}^n} e^{-t\|\lambda\|^2} \widehat{f}(\lambda) e^{ix\cdot\lambda}\, d\lambda$$

where $\widehat{f}$ stands for the Fourier transform of f. This explicte form for the solution shows that $H_t f(x) = u(x,t)$ maps L^2 into L^2 and that $H_t f$ extends to a holomorphic function on $\mathbb{C}^n$. We will give a description of the image.

The natural domain for the heat transform $f \mapsto H_t f$ is not $L^2(\mathbb{R}^n)$ but the space $L^2(\mathbb{R}^n, h_t(x)dx)$. Note, that $h_t(x)dx$ is a probability measure on $\mathbb{R}^n$. We will then discuss how the heat transform on $L^2(\mathbb{R}^n, h_t(x)dx)$ extends to the limit $\mathbb{R}^\infty = \lim\limits_{\leftarrow} \mathbb{R}^n$.

The Laplacian and hence the heat transform is well defined on all Riemannian spaces. But instead of discussing the generalziations in that direction we will consider the action on finite reflection groups on $\mathbb{R}^n$, root systems and multiplicity functions. Associated to each multiplicity function is a generalization of (1) where the Laplacian is replaced by the singular differential operator of the form

$$\Delta + \sum_{\alpha\in\Delta^+} m(\alpha) \frac{1+e^{-2\alpha}}{1-e^{-\alpha}} \partial(h_\alpha)\,.$$

We will give a short overview over this theory, but without all the details. Other directions will be given depending on the time and the audience.

The lectures will follow the first part of my overview talks
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and
http://www.math.lsu.edu/ olafsson/pdf_files/heatequationMIT.pdf
The main topic in this class are: