Solutions

1) $ty' + y = t$, $y(1) = 1$. $(t>0)$

- $y' + \frac{1}{t}y = 1$, $P(t) = \ln(t)$, $\mu(t) = t$
- $\int \mu f dt = \int t\,dt = \frac{1}{2}t^2$.

$y(t) = \frac{1}{t}(\frac{1}{2}t^2 + C)$, $y(1) = \frac{1}{2} + C = 1$, $C = \frac{1}{2}$.

$$y(t) = \frac{1}{2}(t + 1/t).$$

2) $$y' = \frac{y^2 - yt}{t^2 + yt} = \frac{(y/t)^2 - (y/t)}{1 + (y/t)} = \frac{v^2 - v}{1+v} = tv' + v$$

- $$tv' = \frac{v^2 - v}{1+v} - v = \frac{\cancel{v^2} - v - v - \cancel{v^2}}{1+v} = -\frac{2v}{1+v}$$

$\cancel{2\frac{v\,dv}{1+v}}$

$$\frac{(1+v)}{v}dv = (1 + \frac{1}{v})dv = -\frac{2}{t}$$

$$v + \ln|v| = -2\ln|t| + C = \ln(1/t^2) + C$$

$$ve^v = C/t^2$$

$$ye^{y/t} = Ct^{-1}, \quad y \equiv 0 \text{ is also a solution.}$$

3) $$(t-y)y' = t + y \rightsquigarrow y' = \frac{t+y}{t-y} = \frac{1+v}{1-v} = tv' + v$$

- $$\frac{1+v}{1-v} - v = \frac{1 + v - v + v^2}{1-v} = \frac{1+v^2}{1-v}$$

$$\left(\frac{1}{1+v^2} - \frac{v}{1+v^2}\right)dv = \frac{1}{t}dt$$

$$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln|t| + C$$

or
$$\frac{e^{\tan^{-1}(v)}}{\sqrt{1+v^2}} = Ct$$

$$\frac{e^{\tan^{-1}(y/t)}}{\sqrt{1 + y^2/t^2}} = Ct$$

$$\frac{e^{\tan^{-1}(y/t)}}{\sqrt{t^2+y^2}} = C.$$

4) General solution to $ty' + y = e^t, t > 0$.

- $y' + \frac{1}{t}y = \frac{1}{t}e^t$.
- $\mu(t) = t$.

$y(t) = \frac{1}{t}\left(\int e^t\,dt + C\right) = \frac{1}{t}(e^t + C)$.

5) (a) $y' - 4y = 0$, $y(t) = 2e^{4t}$

$y(0) = 2$.

(b) $y' + 2y = 3e^t$, $y(0) = 0$.

I will now use the Laplace, but you can use other meth.

$(s+2)Y(s) = \frac{3}{s-1}$ or

$$Y(s) = \frac{3}{(s+2)(s-1)} = \frac{1}{s-1} - \frac{1}{s+2}$$

$$\boxed{y(t) = e^t - e^{-2t}}$$

(c) $y'' + 3y' + 2y = 0$, $y(0) = 3$, $y'(0) = -6$

$(s^2 + 3s + 2) = (s+2)(s+1)$

$y(t) = c_1 e^{-t} + c_2 e^{-2t}$ | $c_1 + c_2 = 3$

$y'(t) = -c_1 e^{-t} - 2c_2 e^{-2t}$ | $-c_1 - 2c_2 = -6$

$-c_2 = -3$, $c_2 = 3$

$c_1 = 0$.

$$y(t) = 3e^{-2t}.$$

(d) $y'' + 8y' + 16y = 0$, $y(0) = 0$; $y'(0) = 1$

- $s^2 + 8s + 16 = (s+4)^2$.

$y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$, $y(0) = 0 \Rightarrow c_1 = 0$

$y(t) = c_2 t e^{-4t}$

$y'(t) = c_2 e^{-4t} - 4c_2 t e^{-4t}$, $y'(0) = c_2 = 1$.

$$y(t) = t e^{-4t}.$$

(e) $y'' + 2y' + y = e^{-t}, \; y(0) = y'(0) = 0$

$s^2 + 2s + 1 = (s+1)^2$

$y_h(t) = c_1 e^{-t} + c_2 t e^{-t}$

We try (why?) $y_p(t) = At^2 e^{-t}$

$y_p'(t) = -At^2 e^{-t} + 2Ate^{-t}$

$y_p''(t) = At^2 e^{-t} - 4Ate^{-t} + 2Ae^{-t}$

$L(y_p) = 2Ae^{-t} \stackrel{!}{=} e^{-t}, \; A = \frac{1}{2}$

$y(t) = c_1 \cancel{e}^{-t} + c_2 t e^{-t} + \frac{1}{2} t^2 e^{-t}, \; y(0) = 0 = c_1$

$y'(t) = c_2 e^{-t} - c_2 t e^{-t} + te^{-t} - \frac{1}{2} t^2 e^{-t}, \; y'(0) = c_2 = 0.$

$$\underline{\underline{y(t) = \frac{1}{2} t^2 e^{-t}}}$$

(f) $y'' + 2y' + 5y = t + e^t, \; y(0) = 0, y'(0) = 1$

$s^2 + 2s + 5 = (s+1)^2 + 4.$

$y_h(t) = c_1 \cos 2t + c_2 \sin(2t).$

$y_p(t) = At + Be^t + C$

$y_p'(t) = A + Be^t$

$y_p''(t) = Be^t$

$L(y_p) = 8Be^t + 5At + (5C + 2A) = t + e^t$

$B = 1/8, \; A = \frac{1}{5}, \; C = -\frac{2}{5}A = -\frac{2}{25}$

$y(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{1}{8} e^t + \frac{1}{5} t - \frac{2}{25}.$

Then use the initial values to solve for

$c_1 = -\frac{9}{200}$ and $c_2 = \frac{27}{80}$

(g) you have to try $t(A\cos 2t + B\sin 2t) = Y_p(t)$.

(h) try $Y_p(t) = At^2 + \cancel{Bt} + \cancel{C} + Bt + Ct^2e^{-t}$.

General solution:

$s^3 + 2s^2 + s = s(s^2 + 2s + 1) = s(s+1)^2$

$y_h(t) = c_1 + c_2e^{-t} + c_2te^{-t}$

7) $y_1' = y_2$
$y_2' = -y_1$ and $\vec{y}(0) = \begin{pmatrix}1\\0\end{pmatrix}$

$\vec{y}'(t) = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\vec{y}(t)$, $\vec{y}(0) = \begin{pmatrix}1\\0\end{pmatrix}$

$A = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

$sI - A = \begin{bmatrix}s & -1\\1 & s\end{bmatrix}$, $\det(sI-A) = 1+s^2$.

$(sI-A)^{-1} = \begin{bmatrix}\frac{s}{s^2+1} & +\frac{1}{s^2+1}\\ \frac{-1}{s^2+1} & \frac{s}{s^2+1}\end{bmatrix}$

$\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$

$\vec{y}(t) = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos t\\-\sin t\end{bmatrix}$

8) $\vec{y}' = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$ $\vec{y}(0) = \begin{bmatrix}1\\1\end{bmatrix}$

$A = \begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}$

$sI - A = \begin{bmatrix}s-1 & -2\\0 & s-1\end{bmatrix}$, $\det(sI-A) = (s-1)^2$.

$$\mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s-1} & \frac{2}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} e^t & 2te^t \\ 0 & e^t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t + 2te^t \\ e^t \end{bmatrix}.$$

10) a) $\frac{s}{(s+1)(s-1)} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s-1}\right)$, $\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) = \frac{1}{2}(e^t + e^{-t})$

$= \cancel{\tfrac{1}{2}} \cosh(t).$

b) $\frac{s+2}{s(s^2+1)} = \frac{2}{s} + \frac{-2s+1}{s^2+1}$

$\rightarrow 2 - 2\cos t + \sin t.$

c) $\frac{s-1}{s^2+4s+4} = \frac{s-1}{(s+2)^2} = \frac{1}{s+2} - \frac{3}{(s+2)^2}$

$\rightarrow e^{-2t} - 3te^{-2t}.$

11) a) $t * e^t = \int_0^t \cancel{t} u e^{t-u}\, du = e^t \int_0^t u e^{-u}\, du =$

$= e^t \left[-ue^{-u}\Big|_0^t + \int_0^t e^{-u}\, du \right]$

$= -t + e^t \left[-e^{-u} \right]_0^t = -t - 1 + e^t.$

12) (a) $(D^4 + 5D^2 + 4)(e^{2t}) = 40e^{2t}$

13) $\vec{y}' = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \vec{y}, \quad \vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(sI - A) = \begin{pmatrix} s-3 & 4 \\ -1 & s+1 \end{pmatrix}, \quad \det(sI-A) = (s-1)^2$

$$(sI-A)^{-1} = \begin{bmatrix} \frac{s+1}{(s-1)^2} & \frac{-4}{(s-1)^2} \\ \frac{+1}{(s-1)^2} & \frac{s-3}{(s-1)^2} \end{bmatrix}$$

$$\mathcal{L}^{-1}(sI-A)^{-1} = \begin{bmatrix} e^t + 2te^t & -4te^{+t} \\ e^t & e^t - 2te^t \end{bmatrix} = e^{tA}.$$

$$\vec{y}(t) = e^{tA} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -4te^t \\ e^t - 2te^t \end{bmatrix}$$

14) $t = e^x$ gives $z''(t) + 4z' + 4z = 0$

$s^2 + 4s + 4 = (s+2)^2$

$z(x) = c_1 e^{-2x} + c_2 x e^{-2x}$

$y(t) = c_1 t^{-2} + c_2 \ln(t)\, t^{-2}$.

15) $\left[\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 2 & 2 & 1 & 3 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 3 \\ 3 & 2 & 1 & 4 \end{array}\right]$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & -1 & -2 & 4 \end{array}\right]$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $R_2 \leftrightarrow R_3$, then new R_2 multip. by -1

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $R_1 \to R_1 - R_3$, $R_2 \to R_2 - 2R_3$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$ $R_1 \to R_1 - R_2$.

$\boxed{x = 1,\ y = 2,\ z = -3}$

17) To find $e^{t\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}}$: $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$sI - A = \begin{pmatrix} s & -1 \\ 0 & s-1 \end{pmatrix}$, $\det(sI - A) = s(s-1)$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & \frac{+1}{s(s-1)} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} 1 & 1 + e^t \\ 0 & e^t \end{bmatrix}$$

18) $$\vec{y}\,'(t) = \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^t \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} e^t - e^{3t} \\ 2e^t - e^{3t} \end{bmatrix} = \begin{bmatrix} e^t - 3e^{3t} \\ 2e^t - 3e^{3t} \end{bmatrix}$$

Hence $\vec{y}\,'(t) = A\vec{y}$. We also have

$$\vec{y}(0) = \begin{bmatrix} e^t - \cancel{3}e^{3t} \\ 2e^t - \cancel{3}e^t \end{bmatrix}_{t=0} = \begin{bmatrix} \text{[illegible]} \\ +1 \end{bmatrix}$$