Print Your Name Here: ____________________

Show all work in the space provided. *Indicate clearly* if you continue on the back. Write your name at the top of the *scratch* sheet *if it is to be graded.* No books or notes are allowed. A scientific calculator is OK—but not needed. The maximum total score is 200.

Part I: Short Questions. Answer 12 of the following 18 questions for 8 points each. Circle the numbers of the 12 questions listed below that you want counted—*no more than 12!* Detailed explanations are not required, but they may help with partial credit and are *risk-free!* Maximum score: 96 points. ***All matrices on this test refer to the standard basis for Euclidean space*** $\mathbb{E}^n$.

1. Give an example of a sequence of open sets $O_n \subset \mathbb{E}^p$ such that $\bigcap_{n\in\mathbb{N}} O_n$ is not open.

$O_n = (-\frac{1}{n}, \frac{1}{n})$, $\bigcap O_n = \{0\}$

2. True or ~~False~~: Every subset of $\mathbb{E}^n$ can be expressed as a union of closed sets.

$A = \bigcup_{x\in A} \{x\}$

3. True or Give a Counterexample: If a sequence $x^j \in \mathbb{E}^n$ is *bounded* then $\{x^j \mid j \in \mathbb{N}\}$ is a *compact* set.

$x^j = 1/j$, $j=1,\dots,$ $\{x^j\}$ is not closed

4. True or False: The graph $G_f = \{(x, f(x)) \mid x \in [0,1]\}$ is a connected subset of $\mathbb{E}^2$ *if and only if* f is continuous.

$$f(x) = \begin{cases} \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

5. Let $f : \mathbb{E}^2 \to \mathbb{E}^1$ be defined by $f(x) = \begin{cases} \frac{x_1 x_2^2}{x_1^4 + x_2^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ True or False: f is continuous at $\mathbf{0}$.

$$f(r\cos\theta, r\sin\theta) = r\,\frac{\cos\theta\sin^2\theta}{r^2\cos^4\theta + \sin^2\theta} \longrightarrow 0$$

6. If $f : \mathbb{E}^2 \to \mathbb{E}^1$ is given by $f(x) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ find the directional derivative $D_v f(0,0)$, in the direction of the vector $v = (1,1)$.

$$\lim_{h\to 0} \frac{f(hv) - f(0)}{h} = \lim_{h\to 0} \frac{1}{h^2+1} = 1$$

7. True or False: If $f \in C(\mathbb{E}^n, \mathbb{E}^m)$ then $\{x \mid f(x) = 0\}$ is a *closed* set.

8. Find a bijection $f : [0,1) \xrightarrow{\text{onto}} S^1$, the unit circle, with $f \in C[0,1)$ but $f^{-1} \notin C[S^1]$.

$f(t) = (\cos(2\pi t), \sin(2\pi t))$

9. True or False: If $f \in C(\mathbb{E}^2, \mathbb{E}^2)$ such that there are a and $b \in \mathbb{E}^2$ with $f(a) = (1,1)$ and $f(b) = (-1,-1)$, then there exists a point $c \in \mathbb{E}^2$ such that $f(c) = 0$.

$f(x,y) = e^x(\cos(2\pi y), \sin(2\pi y))$

10. True or False: The set $\mathcal{GL}(n,\mathbb{R}) = \{T \in \mathcal{L}(\mathbb{E}^n) \mid \det T \neq 0\}$ is an *open* subset of the normed vector space $\mathcal{L}(\mathbb{E}^n)$.

det is continuous, so $\det^{-1}(\mathbb{R}\setminus\{0\})$ is open

11. True or False: If $f : \mathbb{E}^n \to \mathbb{E}^m$ is such that $\frac{\partial f_i}{\partial x_j}$ exists for all i, j, then f is differentiable.

All $\frac{\partial f_i}{\partial x_j}$ have to exist and be continuous

12. Suppose $f : [0,1] \times \mathbb{E} \to \mathbb{E}^2$ by $f(x) = (e^{x_1} \cos x_2, e^{x_1} \sin x_2)$. True or False: For all a and $b \in \mathbb{E}^2$ we have $\|f(b) - f(a)\| \le e\|b - a\|$.

True, because $\sup_{(x,y)\in[0,1]\times\mathbb{E}} \|Df(x,y)\| = M < \infty$

Use Thm. 10.3.2.

13. Which hypothesis of the Magnification Theorem fails to be satisfied at $x = 0$ by $f : \mathbb{E}^1 \to \mathbb{E}^1$ defined by $f(x) = \begin{cases} 2x + 4x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$

f' not continuous at 0.

14. Let $f \in C^1(\mathbb{E}^4, \mathbb{E}^1)$ be such that the matrix $[f'(x)]_{1\times 4} = [x_2^2 - 1,\ x_1x_3,\ x_4^2 - 1,\ x_2x_3]$. Find all $x_0 = (x_1, x_2, x_3, x_4)$ at which the implicit function theorem guarantees that there is a *local* C^1 solution of the equation $f(x) = f(x_0)$ for x_3 in terms of the other three variables.

?

15. Suppose that $\mathbf{f} = (f_1, f_2) \in C^1(\mathbb{E}^3, \mathbb{E}^2)$ and $[\mathbf{f}'(\mathbf{x}_0)] = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 2 \end{pmatrix}$. If $\mathbf{f}(x_1(x_2), x_2, x_3(x_2)) = 0$ with x_1 and x_3 being C^1 functions of x_2 in a neighborhood of $x_0 = (a, b, c)$, find $x_1'(b)$ and $x_3'(b)$. (Hint: Apply the Chain Rule to differentiate $f(x_1(x_2), x_2, x_3(x_2)) = 0$ with respect to x_2 on both sides.

We get the system $\left.\begin{aligned} 2\frac{\partial x_1}{\partial x_2} + 3\frac{dx_3}{dx_2} &= -1 \\ -\frac{dx_1}{dx_2} + 2\frac{dx_3}{\partial x_2} &= -1 \end{aligned}\right\}$ which leads to $\frac{dx_1}{dx_2} = 1/7$, $\frac{dx_3}{dx_2} = -\frac{3}{7}$

16. Let $f : \mathbb{E}^2 \to \mathbb{E}^2$ by $f(x) = (x_1 \cos x_2, x_1 \sin x_2)$. Find the matrix $[(f^{-1})'(0,1))]$ of the local inverse at $(0,1)$ in the standard basis. Hint: $f(1, \frac{\pi}{2}) = (0,1)$. Use the Inverse Function Theorem.

We always have

$D(f^{-1})(f(x))\,Df(x) = I$ or $D(f^{-1})(f(x)) = Df(x)^{-1}$.

In this case

$Df(1, \pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so $Df^{-1}(0,1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

17. Let $A \in \mathcal{L}(\mathbb{E}^4)$ such that $AA' = 2I$, where $A = A'$, the *transpose* of the matrix of A. Find all possible values of $\det A$.

$(\det A)^2 = 2^4 = 16$ so $\det A = \pm 4$

18. Let $A \in \mathcal{L}(\mathbb{E}^4)$ such that $\det A = -5$. If $B = [-1,1]^4$, a cubical box in $\mathbb{E}^4$, find $\int_{A(B)} 1\,dx$.

By the change of variable formula:

$\int_{A(B)} 1\,dx = \int_B |\det A|\,dx = 5 \cdot 2^4 = \underline{80}$

Part II: Proofs. ***Prove carefully*** 4 of the following 6 theorems for 26 points each. **Circle** the *letters* of the 4 proofs to be counted in the list below—*no more than 2!* You may write the proofs below, on the back, or on scratch paper. Maximum total credit: 104 points.

A. Define the *closure* $\bar{S}$ of $S \subseteq \mathbb{E}^n$ to be the intersection of all closed sets that contain S. *Explain* why $\bar{S}$ is a closed set, and *prove* that $\bar{S} = S \cup C$, where C is the set of all cluster points of S.

B. Let f be a real-valued function defined on a closed finite interval $[a,b]$ in $\mathbb{E}^1$. Define the *graph* $G_f = \{(x_1, x_2) \in \mathbb{E}^2 \mid x_2 = f(x_1),\ x_1 \in [a,b]\}$.

(i) Prove: If $f \in C[a,b]$, then G_f is a compact subset of $\mathbb{E}^2$. (Hint: Use the Heine–Borel Theorem.)

(ii) Let $g(x_1) = \begin{cases} \sin\frac{\pi}{x_1} & \text{if } x_1 \in (0,1], \\ 0 & \text{if } x_1 = 0. \end{cases}$ Is the graph G_g a compact subset of $\mathbb{E}^2$? Prove your conclusion.

C. A subset $D \subseteq \mathbb{E}^n$ is called *convex* if and only if for each pair of points $\boldsymbol{a}$ and $\boldsymbol{b} \in D$ the straight-line segment $S(\boldsymbol{a},\boldsymbol{b}) = \{\phi(t) \mid \phi(t) = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a}),\ t \in [0,1]\}$ is contained in D. Prove that every convex subset of $\mathbb{E}^n$ is connected.

D. Define $f : \mathbb{E}^2 \to \mathbb{E}^1$ by $f(\boldsymbol{x}) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & \text{if } \boldsymbol{x} \in \mathbb{E}^2 \setminus \{\boldsymbol{0}\}, \\ 0 & \text{if } \boldsymbol{x} = \boldsymbol{0}. \end{cases}$

(i) Prove: $\frac{\partial f}{\partial x_1}$ and $\frac{\partial f}{\partial x_2}$ exist at *every point* $\boldsymbol{x} \in \mathbb{E}^2$.

(ii) Prove: f is not differentiable at $\boldsymbol{0}$.

E. Let $S\mathcal{L}(2,\mathbb{R})$ denote the set of all two-by-two real matrices $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ for which $\det X = 1$. If $X \in S\mathcal{L}(2,\mathbb{R})$, *show by applying the Implicit Function Theorem* that there are at least *two* of the four coordinates x_1, x_2, x_3, x_4, either of which can be expressed locally as a continuously differentiable function of the three other coordinates.

F. Define $\mathcal{O}(2) = \{X \in \mathcal{L}(\mathbb{E}^2) \mid XX^t = I\}$ where X^t denotes the transpose of X and I is the identity transformation. Denote the matrix $[X] = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ so that X is determined uniquely by the corresponding vector $\boldsymbol{x} = (x_1, x_2, x_3, x_4) \in \mathbb{E}^4$.

(i) Prove that $X \in \mathcal{O}(2) \Leftrightarrow f(\boldsymbol{x}) = \boldsymbol{0} \in \mathbb{E}^3$, where $f : \mathbb{E}^4 \to \mathbb{E}^3$ is given by $f(\boldsymbol{x}) = (x_1 x_3 + x_2 x_4,\ x_1^2 + x_2^2 - 1,\ x_3^2 + x_4^2 - 1)$ for each $\boldsymbol{x} \in \mathbb{E}^4$.

(ii) Show that $[X] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{O}(2)$ and prove that in a neighborhood of X there exists at least 1 coordinate x_i of $\boldsymbol{x}$ such that the three remaining coordinates $x_j, j \neq i$, are locally C^1-functions of x_i.