PROBLEMS TO STUDY BEFORE Q#2

Linear span of vectors

(1) Which of the following set of vectors span the given vector space

1) $V = \mathbb{R}^2$ $(1,2), (2,1)$
2) $V = \mathbb{R}^2$ $(1,1), (1,-1)$
3) $V = \mathbb{R}^3$ $(1,1,0), (0,1,0), (1,0,1)$
4) $V = \mathbb{R}^3$ $(1,1,3), (2,1,1), (4,3,7)$
5) $V =$ Space of polynomials of degree ≤ 4

$1, x, x^2, 1-x^3, x^3+x^4$

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(2) a) Write the vector $(2,5)$ as a linear combination of the vectors $(1,1)$ and $(2,-1)$.

b) Write the vector (x,y) as a linear combination of $(1,1)$ and $(2,-1)$.

3) a) Write the vector (x, y) as a linear combination of $(2,1)$ and $(-1,2)$

b) write the vector $(4,3)$ as a linear combination of the vectors $(2,1)$ and $(-1,2)$.

4) Let $v_1 = (1,1,2)$ and $v_2 = (1,0,-3)$.

a) Show that v_1 and v_2 are linearly independent.

b) The vectors v_1 and v_2 span a plane L in $\mathbb{R}^3$. Find a pair of orthogonal vector that span the same plane L.

c) Write the vector $(5,3,0)$ as a linear combination of v_1 and v_2

5) Show that the following sets of vectors are linearly independent.

a) $(1,2), (2,-1)$

b) $(1,2,0), (1,0,1)$

c) $(1,2,1), (1,0,-1), (1,-1,1)$.

d) $f(x) = x, \quad g(x) = 1 + x - x^2$. on $[-1,1]$

e) $V = C([-\pi,\pi]),\ \cos(x),\ \sin(x)$.

The inner product is

$$(f,g) = \int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

e) $\psi_0^{(2)}, \psi_1^{(2)}, \psi_2^{(2)}, \psi_3^{(2)}$

f) $(1,1,1), (1,0,1), (0,1,1)$.

6) Which of the following sets of vectors is a basis for the given vector space.

a) $V = \mathbb{R}^2$, $(1,1), (0,1)$

b) $V = \mathbb{R}^2$, $(1,1)(1,-1)(0,1)$

c) $V = \mathbb{R}^3$, $(1,0,1), (1,0,-1), (0,1,0)$

d) $V = \mathbb{R}^3$, $(1,1,1), (1,2,-1), (2,3,0)$

e) $V = \mathbb{R}^3$, $(1,2,0), (0,1,1), (0,1,2), (3,1,2)$

f) Let V be the plane $\{(x,y,z) \in \mathbb{R}^3 \mid 2x+y=0\}$

i) $(1,-2,0), (0,0,1)$

ii) $(2,-4,3)$

iii) $(1,-2,1), (0,0,1), (2,-4,3)$.

7) Find an orthogonal set of vectors spanning the same space as the given set of vectors.

a) $(1,2,0),\ (0,1,1)$

b) $(1,1,0,0),\ (0,1,0,1),\ (0,0,1,1)$

c) $(2,1,1)\quad (0,1,0)$

d) $1+x,\ x$ in $C[0,1]$ with innerproduct $\int_0^1 f(x)g(x)dx = (f,g)$.

Solutions

① (1) No $(2,4) = 2\cdot(1,2)$ so the vectors are linearly dependent and only give us a line.

(2) Yes: The vectors are orthoganal and hence linearly dependent, and thus a basis

OR: $(x,y) = \frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)$.

(3) Yes: Linearly independent, 3 vectors $\Rightarrow$ basis.

OR

$$(x,y,z) = c_1(1,1,0) + c_2(0,1,0) + c_3(1,0,1)$$

$$\left.\begin{aligned} x &= c_1 + c_3 \\ y &= c_1 + c_2 \\ z &= c_3 \end{aligned}\right\} \quad \begin{aligned} c_1 &= x - z \\ c_2 &= y - c_1 = y - x + z. \end{aligned}$$

(4) No: $(4,3,7) = 2(1,1,3) + (2,1,1)$ so the set of three vectors only span a plane.

(5) The space of polynomials of degree ≤ 4 is the space of polynomials of the form

$$c_0 \cdot 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

The question is: Given such a polynomial can we find numbers d_0, d_1, d_4 such that

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

$$= d_0 \cdot 1 + d_1 x + d_2 x^2 + d_3 (1 - x^3) + d_4 (x^3 + x^4)$$

collecting the powers of x

$$= (d_0 + d_3) + d\,x + d_2 x^2 + (d_4 - d_3) x^3 + d_4 x^4$$

This gives us the equations

(1) $d_0 + d_3 = c_0$

(2) $d_1 = c_1$

(3) $d_2 = c_2$

(4) $d_4 - d_3 = c_3$

(5) $d_4 = c_4$

(2), (3), and (5) give

$$d_1 = c_1$$

$$d_2 = c_2$$

$$d_4 = c_4$$

Using (4) and (5) gives

$$d_3 = d_4 - c_3 = c_4 - c_3$$

Using (1) and the value of d_3 gives

$$d_0 = c_0 - d_3 = c_0 - c_4 + c_3$$

So the answer is YES

(2) a) We have to solve the system

$$(2,5) = a(1,1) + b(2,-1)$$
$$= (a+2b, a-b)$$

So

$$a + 2b = 2$$
$$a - b = 5$$

Thus multiply the second equation by 2 and adding gives

$$3a = 12 \text{ or } a = 4$$

Then the second equation gives

$$b = a - 5 = 4 - 5 = -1$$

Thus

$$(2,5) = 4(1,1) - (2,-1).$$

b) We have to find a and b so that

$$(x,y) = a(1,1) + b(2-1)$$

This gives

$$a + 2b = x$$
$$a - b = y$$

Thus $3a = x + 2y$ or

$$a = \frac{x+2y}{3}$$

Then $b = a - y = \dfrac{x+2y}{3} - y = \dfrac{x-y}{3}$

Hence

$$(x,y) = \frac{x+2y}{3}(1,1) + \frac{x-y}{3}(2,-1)$$

3) a) The vectors $(2,1)$ and $(-1,2)$ are orthogonal. and $\|(2,1)\|^2 = \|(-1,2)\|^2 = 5$. Thus

$$(x,y) = \frac{(x,y)\cdot(2,1)}{5}(2,1) + \frac{(x,y)\cdot(-1,2)}{5}(-1,2)$$

$$= \frac{2x+y}{5}(2,1) + \frac{2y-x}{5}(-1,2).$$

b) Use (a) with $x=4$, $y=3$:

$$(4,3) = \frac{11}{5}(2,1) + \frac{2}{5}(-1,2).$$

4) a) If $c_1 v_1 + c_2 v_2 = 0$ then

$$\left.\begin{aligned} c_1 + c_2 &= 0 \\ c_1 &= 0 \\ 2c_1 - 3c_2 &= 0 \end{aligned}\right\} \rightsquigarrow c_2 = -c_1 = 0$$

$c_1 = 0 \rightsquigarrow c = 0$

So both c_1 and c_2 are zero.

b) Use Gram-Schmidt:

$$w_1 = v_1 = (1,1,2)$$

$$w_2 = v_2 - \frac{(v_2, w_1)}{\|w_1\|^2} w_1$$

$$= (1,0,-3) - \frac{-5}{6}(1,1,2)$$

$$= (1,0,-3) + \frac{5}{6}(1,1,2)$$

$$= \left(\frac{11}{6}, \frac{5}{6}, -\frac{4}{3}\right).$$

c) $(5,3,0) = a(1,1,2) + b(1,0,-3)$

$$\left.\begin{array}{l} a+b=5 \\ a=3 \rightsquigarrow a=3 \\ 2a-3b=0 \end{array}\right\} \rightsquigarrow b = 5-a = 5-3 = \underline{\underline{2}}$$

This solves also the last equation. So

$(5,3,0) = 3(1,1,2) + 2(1,0,-3)$.

5) a) The vectors are orthogonal.

b) If $a(1,2,0) + b(1,0,1) = (0,0,0)$. Then

$$\left.\begin{array}{l} a+b=0 \\ 2a=0 \\ b=0 \end{array}\right\} \text{ so both } a \text{ and } b \text{ are zero } (a=b=0)$$

c) The vector are paarwise orthogonal

d) If $af + bg = 0$ then

$b + (a+b)x - bx^2 = 0$

for all x in $[-1,1]$. Set $x=0$. Then $b=0$. So $ax=0$. No take any number $x \neq 0$ to see that $a=0$

e) $\cos(x)$ and $\sin(x)$ are orthogonal

[you can also do it directly:

$a\cos(x) + b\sin(x) = 0$.

Insert $x=0$. Then $\cos(x)=1$ and $\sin(x)=0$

So $a=0$. Then take $x=\frac{\pi}{2}$ to see $b=0$.]

e) The functions are pairwise orthogonal.

f) linearly independent.

$$a(1,1,1)+b(1,0,1)+c(0,1,1)=(0,0,0)$$

gives

$$\left.\begin{aligned} a+b &= 0 \\ a+c &= 0 \end{aligned}\right\} \quad \boxed{b=c}$$
$$a+b+c=0$$

Substracting the second equation from the last one gives

$$\boxed{b=0}$$

Then $c=0$ and the firs equation gives $a=0$. So $\boxed{a=b=c=0}$

6) a) Two linearly independent vectors in $\mathbb{R}^2$ form a basis. $(1,1)$ & $(0,1)$ are linearly independent and hence a basis.

b) No three vectors in $\mathbb{R}^2$ are linearly dependent $\left(\frac{1}{2}(1,1)-\frac{1}{2}(1,-1)=(0,1)\right)$.

c) Three linearly independent vectors in $\mathbb{R}^3$ form a basis. The given vectors are orthogonal and hence linearly independent.
[one can also show this directly
$$(x,y,z) = \frac{x+z}{2}(1,0,1) + \frac{x-z}{2}(1,0,-1) + y(0,1,0).]$$

d) No, linearly dependent
$$(2,3,0) = (1,1,1) + (1,2,-1)$$

e) No, 4 vectors in $\mathbb{R}^3$ are linearly independent.

f) i) Note first that both vectors are in the plane. We need two linearly independent vectors in the plane. But the given vectors are orthogonal, hence linearly independent.

ii) No, we need two vectors.

iii) No, linearly dependent
$$(2,-4,3) = 2(1,-2,1) + (0,0,1).$$

7) We use Gram-Schmidt

a) $w_1 = (1,2,0)$
$$w_2 = (0,1,1) - \frac{2}{5}(1,2,0)$$
$$= (0,1,1) - \left(\frac{2}{5},\frac{4}{5},0\right) = \left(\frac{2}{5},\frac{1}{5},1\right)$$

$$\boxed{w_1 = (1,2,0),\ w_2 = (2,1,5)}$$ (why?)

S-9

b) $w_1 = (1,1,0,0)$

$w_2 = (0,1,0,1) - \frac{1}{2}(1,1,0,0)$

$= \left(-\frac{1}{2}, \frac{1}{2}, 0, 1\right)$ (You can then take $(-1,1,0,2)$)

$w_3 = (0,0,1,1) - 0 \cdot w - \frac{1}{3/2}\left(-\frac{1}{2}, \frac{1}{2}, 0, 1\right)$

$= (0,0,1,1) - \left(-\frac{1}{3}, \frac{1}{3}, 0, \frac{2}{3}\right)$

$= \left(\frac{1}{3}, -\frac{1}{3}, 1, \frac{1}{3}\right)$

$\boxed{(1,1,0,0),\ (-1,1,0,2),\ (1,-1,3,1)}$

c) $w_1 = (0,1,0)$

$w_2 = (2,1,1) - \frac{1}{1}(0,1,0) = (2,0,1)$

$w_1 = (0,1,0),\ w_2 = (2,0,1)$

d) $w_1 = x$

$\|x\|^2 = \int_0^1 x^2\, dx = \frac{1}{3}$

$(1+x, x) = \int_0^1 x + x^2\, dx = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$w_2 = 1 + x - \frac{5/6}{1/3} x$

$= 1 + x - \frac{5}{2} x = 1 - \frac{3}{2} x$