SOLUTIONS

MATH 2025 — Fall 2009

Test 2, Thursday, Oct 22, 2009

For partial credit, show all your work!

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1) Assume that the $2D$-wavelet transform applied to the matrix $\mathbf{f} = \begin{pmatrix} s_{0,0} & s_{0,1} \\ s_{1,0} & s_{1,1} \end{pmatrix}$ results in the matrix $\begin{pmatrix} 6 & 1 \\ -2 & 2 \end{pmatrix}$.

a[8P]) Apply the inverse Haar wavelet transform to find the matrix $\mathbf{f} = \begin{pmatrix} 7 & 1 \\ 7 & 9 \end{pmatrix}$.

$\begin{pmatrix} 6 & 1 \\ -2 & 2 \end{pmatrix} \to \begin{pmatrix} 4 & 3 \\ 8 & -1 \end{pmatrix}$ (inverse transform on each column)

$\to \begin{pmatrix} 7 & 1 \\ 7 & 9 \end{pmatrix}$ (inverse transform on each row)

b[4P]) What is the meaning of the number 6? The average of the numbers in the original matrix

c[4P]) What is the meaning of the number -2? Average change going from row 1 to row 2.

2 [5P]) Write the inverse of $3 + 4\sqrt{3} \in \mathbb{Q}(\sqrt{3})$ in the form $a + b\sqrt{3}$. $(3 + 4\sqrt{3})^{-1} = -\frac{1}{13} + \frac{4}{39}\sqrt{3}$

$$(3+4\sqrt{3})^{-1} = \frac{1}{3+4\sqrt{3}} \cdot \frac{3-4\sqrt{3}}{3-4\sqrt{3}}$$

$$= \frac{3-4\sqrt{3}}{9-16\cdot 3} = \frac{3-4\sqrt{3}}{-39} = -\frac{3}{13} + \frac{4}{39}\sqrt{3}$$

3[8P]) a) Fill out the missing parts in the multiplication table for $\mathbf{Z}_5$:

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

b) What is the inverse of 3 in $\mathbf{Z}_5$? $\underline{\underline{3^{-1} = 2}}$

Fact: If V and W are vector spaces and $T: V \to W$ is linear, then $\{v \in V \mid T(v) = 0\}$ is a vector space.

4[20P]) Which of the following sets is a vector space and which is not a vector space. Remember, the correct argument counts for 1/2 of the points!

a) $\{(x, y, z) \in \mathbb{R}^3 \mid 2x + 3y - z = 0\}$. Is a vector space, because $(x,y,z) \mapsto 2x+3y-z$ is a linear map $\mathbb{R}^3 \to \mathbb{R}$.

b) $\{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$. Yes, is a vector space, because $f \mapsto f(0)$ is linear $C^1(\mathbb{R}) \to \mathbb{R}$.

c) $\{(x, y, z) \in \mathbb{R}^3 \mid 3xy - z = 0\}$. Not linear because of the factor xy.
Take $(1,0,0), (0,1,0) \in \{(x,y,z) \in \mathbb{R} \mid 3xy - z = 0\}$ but $(1,1,0) = (1,0,0) + (0,1,0)$ is not in the set. So not closed under addition

d) $\{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$.
Not a vector space

e) $\{f \in C([-1, 1]) \mid \int_{-1}^{1} f(t) \cos(t)\, dt = 0\}$. The integral is linear so $f \mapsto \int_{-1}^{1} f(t)\cos(t)dt$ is linear. Hence a vector space

5[20P]) Which of the following maps are linear and which are not. Remember, the correct argument counts for 1/2 of the points!

a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T((x, y, z)) = (3x + 2y + z - 1, 2x - y)$. Not linear
$T(0,0,0) = (-1,0) \neq (0,0)$.

b) $T : C^3(\mathbb{R}) \to C(\mathbb{R})$, $T(f) = f''' + 3ff'' + f$. Not linear

c) $T : C([-1, 1]) \to \mathbb{R}$, $T(f) = \int_{-1}^{1} f(t) \sin(t)\, dt$. The integral is linear
$T(rf+sg) = \int_{-1}^{1} (rf+sg)(t)\sin(t)dt = r\int_{-1}^{1} f(t)\sin(t)dt + s\int_{-1}^{1} g(t)\sin(t)dt$
$= rT(f) + sT(g)$.

d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y - z \\ 2x + z \\ xy + 1 \end{pmatrix}$ Not linear, $T(\vec{0}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

e) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y - z \\ 2x - 4y + z \\ 2x - y + z \end{pmatrix}$ linear $= \begin{pmatrix} 3 & 2 & -1 \\ 2 & -4 & 1 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

6[15P]) Define an inner product on $\mathbb{R}^3$ by $((x,y,z),(u,v,w)) = 2xu + yv + 3zw$.

a) Evaluate $((1,-2,3),(2,0,1)) =$ $2\cdot 1\cdot 2 + (-2)\cdot 0 + 3\cdot 3\cdot 1 = 4+9 = \underline{\underline{13}}$

b) Find the norm of the vector $(1,-1,1)$. $\|(1,-1,1)\| =$ $\sqrt{6}$

$\|(1,-1,1)\|^2 = 2\cdot 1^2 + (-1)^2 + 3\cdot 1^2 = 2+1+3 = 6$

c) Are the vectors $(1,1,-1)$ and $(1,1,1)$ orthogonal? Yes

$((1,1,-1),(1,1,1)) = 2\cdot 1\cdot 1 + 1\cdot 1 + 3\cdot(-1)\cdot 1 = 2+1-3 = \underline{0}$

7[10P]) On $C([-1,1])$ define an inner product by $(f,g) = \int_{-1}^{1} f(t)g(t)\,dt$.

a) Find $(t^2,t) =$ 0

$(t^2,t) = \int_{-1}^{1} t^2\cdot t\,dt = \int_{-1}^{1} t^3\,dt = \frac{1}{4}t^4\Big]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$

b) Find the norm of the function $t \mapsto t^2$. $\|t^2\| =$ $\sqrt{2/5}$

$\|t^2\|^2 = \int_{-1}^{1} t^4\,dt = \frac{1}{5}t^5\Big]_{-1}^{1} = \frac{2}{5}$

8[6P]) Let V and W be vector spaces and let $T: V \to W$ be linear. Show that if $u, v \in V$ are such that $T(u) = T(v) = 0_W$ and $r, s \in \mathbb{R}$. Then $T(ru+sv) = 0_W$.

$T(ru+sv) = rT(u) + sT(v)$

$= r0_W + s0_W = 0_W + 0_W = 0_W$

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c[4P]) What is the meaning of the number -2?

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c) $\{(x,y,z) \in \mathbb{R}^3 \mid 3xy - z = 0\}$.

d) $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$.

e) $\{f \in C([-1,1]) \mid \int_{-1}^{1} f(t)\cos(t)\,dt = 0\}$.

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d) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 3x + 2y - z \\ 2x + z \\ xy + 1 \end{pmatrix}$

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