Solutions to the homework, part 1

<u>8.2</u> Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{E}^n$. Then $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$

<u>Solution</u>: Use the triangle inequality $\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$ for all $\vec{a}, \vec{b} \in \mathbb{E}^n$. We get

$$\|\vec{x} - \vec{z}\| = \|\vec{x} + (-\vec{y}) + \vec{y} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$

<u>8.5</u> Let $a_1, a_2 \in \mathbb{R}$. Then $\langle \vec{x}, \vec{y} \rangle_{\vec{a}} = a_1 x_1 y_1 + a_2 x_2 y_2$ is a scalar (or inner) product on $\mathbb{R}^2$ if and only if $a_1 > 0, a_2 > 0$.

<u>Solution</u> If both a_1 and a_2 are zero, then $\langle \vec{x}, \vec{y} \rangle_{\vec{a}} = 0$ for all $\vec{x}, \vec{y}$. Assume that one of a_1 or a_2 is zero. If $a_1 = 0$ then $\langle (1,0), (1,0) \rangle_{\vec{a}} = 0$ but $(1,0) \neq (0,0)$. If $a_2 = 0$ then $\langle (0,1), (0,1) \rangle_{\vec{a}} = 0$. If one of the numbers, say a_1, is negative, then again $\langle (1,0), (1,0) \rangle_{\vec{a}} = a_1 < 0$, contradicting (iii). Similarly for $a_2 < 0$. Assume now that $a_1 > 0$ and $a_2 > 0$.

(i)
$$\begin{aligned}
\langle a\vec{x} + \vec{y}, \vec{z} \rangle &= \langle (a x_1 + y_1, a x_2 + y_2), (z_1, z_2) \rangle \\
&= a_1 (a x_1 + y_1) z_1 + a_2 (a x_2 + y_2) z_2 \\
&= a_1 a x_1 z_1 + a_1 y_1 z_1 + a_2 (a x_2 z_2 + y_2 z_2) \\
&= a (a_1 x_1 z_1 + a_2 x_2 z_2) + a_1 y_1 z_1 + a_2 y_2 z_2 \\
&= a \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle
\end{aligned}$$

(ii)
$$\begin{aligned}
\langle \vec{x}, \vec{y} \rangle &= a_1 x_1 y_1 + a_2 x_2 y_2 \\
&= a_1 y_1 x_1 + a_2 y_2 x_2 \\
&= \langle \vec{y}, \vec{x} \rangle_a
\end{aligned}$$

(iii) $\langle \vec{x}, \vec{x} \rangle = a_1 x_1^2 + a_2 x_2^2 \geq 0$ and $= 0$ if and only if $x_1 = x_2 = 0$.

8.7| a) $\vec{x}^{(j)} = (\frac{1}{j}, (1+\frac{1}{j})^j) \to (0, e)$

b) $x^{(j)} = (\frac{1}{j}, \frac{e^j}{j^{100}}) \to (0, \infty)$ so the limit does not exists.

8.11|

a) Show that every convergent sequence $\{\vec{x}^j\}$ is bounded.
Assume that $\vec{x}^j \to x \in \mathbb{E}^n$. Let $M_1 = \|\vec{x}\|$. As $\vec{x}^j \to \vec{x}$, there exist $N \in \mathbb{N}$ such that for all $j \geq N$ we have

$$\|\vec{x} - \vec{x}_j\| < 1.$$

But then $\|\vec{x}^j\| \leq \|\vec{x} - \vec{x}^j\| + \|\vec{x}\|$

$$< 1 + M$$

for all $j \geq N$. Now let

$$M_2 = \max_{j=1,\dots,N} \|x_j\|$$

and $M = \max\{M_1, M_2\}$.

b) $x_j = ((-1)^j, 0)$.

c) Was done in class.

8.14| Show that $\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2$

Solution:

$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$

$= \|x\|^2 + 2\langle x, y\rangle + \|y\|^2 + \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$

$= 2(\|x\|^2 + \|y\|^2)$.

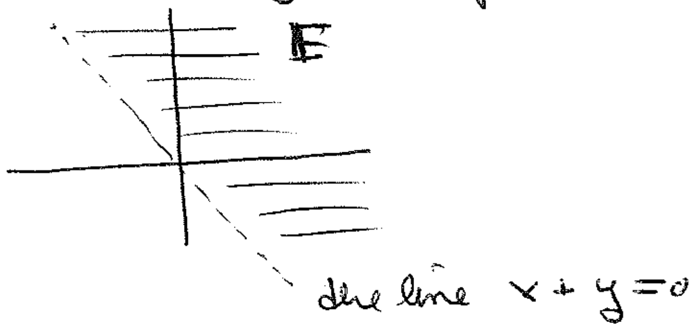
8.17| We write vectors as $\vec{x} = (x, y)$

(a) $A = \{\vec{x} \in \mathbb{E}^2 \mid x_1 \neq 0\}$ open (If $(x, 0), \overset{=\vec{x}}{}$ ~~let~~ in A, then $B_{\frac{|x|}{2}}(\vec{x}) \subseteq A$.

(b) $B = \{(x, y) \mid x = 0\}$ closed because $\mathbb{E}^2 \setminus B = A$ is open.

(c) $C = \{\vec{x} \in \mathbb{E}^2 \mid x \geq 0,\ y > 0\}$, neither open nor closed

(d) $E = \{\vec{x} \in \mathbb{E}^2 \mid x + y > 0\}$ open.



(e) $D = A \cup B = \mathbb{E}^2$ is open

(f) $F = \{\vec{x} \in \mathbb{E}^2 \mid x + y \leq 0\}$ closed, because $F^c = E$,

(g) $G = \{\vec{x} \in \mathbb{E}^2 \mid 1 < x^2 + y^2 < 4\}$ open.

8.20| $\mathcal{O}$ open. Let $\vec{p} \in \mathcal{O}$, then there exist $r_p > 0$ s.t. $B_{r_p}(\vec{p}) \subseteq \mathcal{O}$
but then
$$\mathcal{O} = \bigcup_{p \in} \mathcal{O}_p .$$

8.31| $\mathbb{Q}^n$ is dense in $\mathbb{E}^n$. (We know that $\mathbb{Q} \subseteq \mathbb{E} = \mathbb{R}$: If $r \in \mathbb{R}$ then any open interv. $(r-\varepsilon, r+\varepsilon)$ contains a rational number.) Let $(x_1, \dots, x_n) \in \mathbb{E}^n$. Let ~~...~~
For $m \in \mathbb{N}$ there exists ~~...~~ rational numbers x_j^m s.t. $|x_j - x_j^m| < \frac{1}{m} \frac{1}{\sqrt{n}}$. Let $\vec{q}_m = (x_1^m, \dots, x_n^m) \in \mathbb{Q}^n$. Then
$$\|\vec{x} - \vec{q}_m\| < \frac{1}{m}$$
Hence $\vec{q}_m \to \vec{x}$ and $\vec{x} \in \overline{\mathbb{Q}^n}$.

8.4 $\|\vec{x}\|_t = |x_1| + |x_2|$ satisfies the req. from Def. 2.4.4 to be a norm.

i. $\|\vec{x}\|_t = 0 \iff \vec{x} = (0,0)$: $\|\vec{x}\| = |x_1| + |x_2| = 0 \iff x_1 = x_2 = 0$

(note that $|x_j| \le \|\vec{x}\|_t$)

ii) $\|a\vec{x}\|_t = |a|\,\|\vec{x}\|_t$: $\|a\vec{x}\|_t = \|(ax_1, ax_2)\|_t$

$$= |ax_1| + |ax_2|$$
$$= |a|(|x_1| + |x_2|)$$
$$= |a|\,\|\vec{x}\|_t .$$

iii) $\|\vec{x} + \vec{y}\|_t \le \|\vec{x}\|_t + \|\vec{y}\|_t$. We have

$$\|\vec{x} + \vec{y}\|_t = \|(x_1 + y_1, x_2 + y_2)\|_t$$
$$= |x_1 + y_1| + |x_2 + y_2|$$
$$\le |x_1| + |y_1| + |x_2| + |y_2|$$
$$= (|x_1| + |x_2|) + (|y_1| + |y_2|)$$
$$= \|\vec{x}\|_t + \|\vec{y}\|_t \quad \blacksquare$$

8.40) $f : [a,b] \to \mathbb{R}$, $G_f = \{(x, f(x)) \mid x \in [a,b]\}$.

If f is continuous. Then $f([a,b]) = [m, M]$ where $m = \inf f([a,b])$ and $M = \sup f([a,b])$

It follows that

$$G_f \subseteq [a,b] \times [m, M]$$

is bounded. Let (x_j, y_j) be a sequence in G_f, such that $(x_j, y_j) \longrightarrow (x,y) \in \mathbb{E}^2$.

As $(x_j, y_j) \in G_f$ we get $y_j = f(x_j)$.
Also $x_j \to x$. Hence, because f is continuous:
$y = \lim y_j = \lim f(x_j) = f(\lim x_j) = f(x)$.
Hence $(x, y) = (x, f(x)) \in G_f$ so G_f is closed.

b) $g(x) = \begin{cases} \sin(\frac{\pi}{x}), & x \in (0, \pi] \\ 0, & x = 0 \end{cases}$

Then G_f is not compact. Solution: Let $r \in [-1, 1]$, $r \neq 0$. Then there exist $\delta \in [-\frac{1}{2}, \frac{1}{2}]$ such that

$$\sin(\pi\delta) = r.$$

Let $x_k = \frac{1}{2k + \delta}$, $k = 1, 2, \ldots$. Then $\sin(\frac{\pi}{x_k})$
$= \sin(2\pi k + \delta\pi) = \sin(\delta\pi) = r$. In particular
$(x_k, r) \in G_f$. But $(x_k, r) \to (0, r) \notin G_f$.
Hence G_f is not closed.

8.46) Show that the union of the x-axis and the y-axis is connected.

Solution: Let $A = \{(x, 0) \mid x \in \mathbb{R}\}$, $B = \{(0, y) \mid y \in \mathbb{R}\}$
Then both A and B are connected (Why?).
As $A \cap B = \{(0, 0)\} \neq \emptyset$ it follows from Thm. 8.4.1 that $A \cup B$ is connected.

8.51) $E = \{(x,y) \in \mathbb{E}^2 \mid x^2 = 1\}$ is not connected.
Note first that

$$E = \{(1,y) \in \mathbb{E}^2\} \cup \{(-1,y) \mid y \in \mathbb{R}\}.$$

Let $O_1 = \{(x,y) \mid y \in \mathbb{R},\ x > 0\}$ open

$O_2 = \{(x\,y) \mid y \in \mathbb{R},\ x < 0\}$ open.

Then $O_1 \cap O_2 = \emptyset$, $E \subset O_1 \cup O_2$, and $E \cap O_1,\ E \cap O_2 \neq \emptyset$.

8.55) $E \subseteq \mathbb{E}^n$ connected, $\vec{p}$ a cluster point. Then $F = E \cup \{\vec{p}\}$ is connected.

Solution: If $\vec{p} \in E$, then there is nothing to prove because $E \cup \{\vec{p}\} = E$. So assume that $\vec{p} \notin E$ and that $E \cup \{\vec{p}\}$ is not connected. Then there exist $A, B \subset \mathbb{E}^n$ open, $F \subseteq A \cup B$ and $F \cap A \cap B \neq \emptyset$. As $E \subset F$ it follows that either $A \cap E \neq \emptyset$ or $B \cap E \neq \emptyset$. Assume that $E \cap A \neq \emptyset$. Then $E = (A \cap E) \cup (B \cap E)$ is a disjoint decomposition of relatively open sets. As E is connected it follows that $E \subset A$. As $B \cap F \neq \emptyset$ it follows $\vec{p} \in B$. But $\vec{p}$ is a cluster point of E and hence $B \cap E \neq \emptyset$ a contradiction to $B \cap E \cap A = B \cap E \neq \emptyset$. ∎

9.4) $$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) $f(\vec{x}) \to 0$ along each of the coordinate axes. This is clear because
$$f(x,0) = f(y,0) = 0.$$

b) $y = kx$: $$f(x,kx) = \frac{kx^3}{x^4+k^2x^2} = \frac{kx}{x^2+k^2} \to 0$$
as $x \to 0$

c) $\lim_{\vec{x}\to 0} f(\vec{x})$ does not exist: If it exists it has to be zero. Let $y = x^2$. Then
$$f(x,y) = \frac{x^4}{x^4+x^4} = \frac{1}{2} \quad \text{so}$$
$$f(x,x^2) \longrightarrow \frac{1}{2} \quad \text{as } x \to 0.$$

d) The limit $\lim_{\vec{x}\to\infty} f(\vec{x})$ does not exists

i) Take the sequence $\vec{x}_n = (n, n^2) \to \infty$. Then
$$f(\vec{x}_n) = 1/2$$

ii) Take now $\vec{x}_k = (n, kn)$ $(k \neq 0)$. Then
$$f(\vec{x}_k) = \frac{kn}{n^2+k^2} \longrightarrow 0 \quad \text{as } n \to \infty.$$

9.5) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Then $\lim_{\vec{x}\to 0} f(x,y) = 0$: Let $x = r\cos\theta$, $y = r\sin\theta$

Then for $r \neq 0$

$$f(r\cos\theta, r\sin\theta) = \frac{r^3\cos\theta\sin^2\theta}{r^4\cos^2\theta + r^2\sin^2\theta}$$

$$= r\,\frac{\cos\theta\sin^2\theta}{\sin^2\theta + r^2\cos^2\theta}$$

Note, if $\sin\theta = 0$, then $f(r\cos\theta, r\sin\theta) = 0$. For other values of θ we have

$$|f(r\cos\theta, r\sin\theta)| \le r\left|\frac{\cos\theta\sin^2\theta}{\sin^2\theta}\right|$$

$$= r|\cos\theta|$$

$$\le r.$$

So, if $\varepsilon > 0$ is given, take $\delta = \varepsilon$.

9.14) Define

$f(x,y) = \begin{cases} \left(xy, \dfrac{x^2y}{x^4+y^2}\right) & (x,y) \neq (0,0) \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$

Then f is not continuous at zero: Write $f = (f_1, f_2)$. Then f is continuous iff both f_1 and f_2 are continuous. Recall if we case $f_2(x,y) = \dfrac{x^2y}{x^4+y^2}$ is not continuous by 9.4. But in our