Solution hints for more of the Exercises

Recall Thm 10.2.4 loosely stated as $f \in C^1(D, \mathbb{E}^m)$ $\iff$ each $\frac{\partial f_i}{\partial x_j}$ exists and is continuous. Recall also, that in this case $Df(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]$.

10.25

a) $f(x) = (\cos(x), \sin(x))$. Then $\frac{\partial f_1}{\partial x} = -\sin(x)$ and $\frac{\partial f_2}{\partial x} = \cos(x)$. So both derivatives exist and are continuous. Hence $Df(x)$ exists for all $x \in \mathbb{E}^1$

b) $f(x) = (x, \sqrt{x^2}) = (x, |x|)$ differentiable on $\mathbb{E}^1 \setminus \{0\}$ becaus $f(x) = (x, x), x > 0$, $f(x) = (x, -x), x < 0$. Not differentiable at $x = 0$ becaus $f_2(x) = |x|$ does not exists at that point.

10.26, Only (a) $f(x) = (x_1 \cos x_2, x_1 \sin x_2)$. Hence $f_1(x_1, x_2) = x_1 \cos x_2$, $f_2(x_1, x_2) = x_1 \sin(x_2)$. It follows that

$$Df(x_1, x_2) = \left[\frac{\partial f_i}{\partial x_j}(x_1, x_2)\right] = \begin{bmatrix} \cos x_2 & -x_1 \sin(x_2) \\ \sin(x_2) & x_1 \cos(x_2) \end{bmatrix}.$$

10.27) $f(\vec{x}) = (e^{x_1}\cos(x_2), e^{x_1}\sin(x_2))$.

(a) $f \in C^1(\mathbb{E}^2, \mathbb{E}^2)$ because $\frac{\partial f_i}{\partial x_j}$ exists and is continuous on $\mathbb{E}^2$.

(b) $$Df(x_1, x_2) = \left[\frac{\partial f_i}{\partial x_j}(x_1, x_2)\right] = \begin{bmatrix} e^{x_1}\cos(x_2) & -e^{x_1}\sin(x_2) \\ e^{x_1}\sin(x_2) & e^{x_1}\cos(x_2) \end{bmatrix}$$

(c) $$\det Df(x_1, x_2) = e^{2x_1}\cos^2(x_2) + e^{2x_1}\sin^2(x_2) = e^{2x_1} > 0.$$

d) $D_{(1,\sqrt{3})} f(x) = Df(x)\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

$$= e^{x_1}\begin{bmatrix} \cos(x_2) - \sqrt{3}\sin(x_2) \\ \sin(x_2) + \sqrt{3}\cos(x_2) \end{bmatrix}.$$

10.28) $Df(1,2) = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Then

$$D_v f(1,2) = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1-6 \\ -2-8 \end{pmatrix} = \begin{pmatrix} -5 \\ -10 \end{pmatrix} = -5\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

10.31) $Df(c\vec{x}_1 + \vec{x}_2) \neq c\,Df(x_1) + Df(\vec{x}_2)$ in general. Let $f(x) = x^3$, $f: \mathbb{E}^1 \to \mathbb{E}^1$. Then $f'(x) = 3x^2$ and in general we do not have

$$3(cx+y)^2 = 3cx^2 + 3y^2$$

(Take $c=2$, $x=1$, $y=0$).

10.38)
$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) All the partial derivatives exist and are continuous on $\mathbb{E}^2 \setminus \{(0,0)\}$. Hence f is differentiable on this open set. For $(x,y) = (0,0)$ we have

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = 0.$$

b) f can not be differentiable at $(0,0)$ because f is not continuous at $(0,0)$

$$\lim_{r\to 0} f(r(\cos\theta, \sin\theta)) = \cos\theta\sin(\theta).$$

10.43 $\quad f(0) = \begin{pmatrix}1\\1\end{pmatrix}, \; Df(0) = \begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix}$

$g(0) = -\begin{pmatrix}1\\1\end{pmatrix}, \; Dg(0) = \begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$

Define $\varphi(x) = \vec{f}(x)\cdot\vec{g}(x)$, Then (see Thm. 10.2.5)

$$D\varphi(x)h = (Df(x)\vec{h})\cdot\vec{g}(x) + \vec{f}(x)\cdot(Dg(\vec{x})\vec{h})$$

Take $h = e_1$. Then we get

$$\frac{\partial\varphi}{\partial x_1}(0,0) = \begin{pmatrix}1\\-2\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-1\end{pmatrix}$$
$$= (-1+2+2-1) = 2$$

Take $h = e_2$:

$$\frac{\partial\varphi}{\partial x_2}(0,0) = \begin{pmatrix}2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\-1\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}$$
$$= -2-1+1+2 = 0$$

so $D\varphi(0,0) = (2,0)$.

10.48 Recall: $A = (a_{\nu\mu})$, $B = (b_{ij})$ and $C = AB = (c_{ij})$

Then

$$\boxed{c_{ij} = \sum_k a_{ik}b_{kj}}$$

We have $D(f\circ g)(x) = Df(g(x))Dg(x)$. Hence

$$\frac{\partial(f\circ g)_i}{\partial x_j}(\vec{x}) = \sum_{k=1}^m \frac{\partial f_i}{\partial x_k}(g(x))\frac{\partial g_k}{\partial x_j}(x).$$

10.46 $\quad D\vec{f}(\vec{0}) = \begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}$

$(a,b)\in E^2$. Let $\varphi(x) = (a,b)\cdot\vec{f}(x)$. Then $D\varphi(\vec{x})\vec{h} = (a,b)\cdot(Df(\vec{x})\vec{h})$. For the partial derivatives we get:

$$\frac{\partial \varphi}{\partial x_1}(\vec{0}) = (a,b)\cdot\begin{pmatrix}1\\4\end{pmatrix} = a+4b$$

$$\frac{\partial \varphi}{\partial x_2}(\vec{0}) = (a,b)\cdot\begin{pmatrix}2\\5\end{pmatrix} = 2a+5b$$

$$\frac{\partial \varphi}{\partial x_3}(\vec{0}) = (a,b)\cdot\begin{pmatrix}3\\6\end{pmatrix} = 3a+6b$$

Thus $D\varphi(\vec{0}) = (a+4b, 2a+5b, 3a+6b)$

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<u>10.49</u> $f(t) = (\cos(t), \sin(t))$. Then $f(2\pi) - f(0) = (0,0)$

But $Df(t)(2\pi - 0) = 2\pi(-\sin(t), \cos(t)) \neq (0,0)$.

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<u>10.50</u> $\|Df(x)\| \leq M$ bounded on a convex set D.

Recall Thm. 10.3.2: D convex $\|Df(x)\|$ bounded $\leq M$. Then for all $\vec{a}, \vec{b} \in D$:

$$\|f(\vec{b}) - f(\vec{a})\| \leq M\|\vec{b} - \vec{a}\|.$$

show that f is uniformly continuous: Given $\varepsilon > 0$ let $\delta = \dfrac{\varepsilon}{M+1}$. Then $\|f(\vec{b}) - f(\vec{a})\| < \varepsilon$ if $\|\vec{b} - \vec{a}\| < \delta$.

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<u>10.52</u> (a) If $Df(x) = 0$ for all x then $\|f(b) - f(a)\| \leq 0 \cdot \|b - a\| = 0$, so f is constant

(b) D connected. Let $a \in D$ and $\vec{y} = f(a)$. Let

$U = \{x \in D \mid f(x) = \vec{y}\}$. <u>Claim</u>: U is open.

Let $x \in U$. As D is open, there exists $r > 0$ such that $B_r(x) \subseteq D$. But $B_r(x)$ is convex and $Df(y) = 0$ on $B_r(x)$. It follows that f is constant on $B_r(x)$. As $f(x) = y$ it follows that $f(z) = y$ for all $z \in B_r(x)$.

Hence $B_r(x) \subseteq U$. As $D \setminus U = f^{-1}(\mathbb{E}^m \setminus \{y\})$ and $\mathbb{E}^m \setminus \{y\}$ is open it follows that $D \setminus U$ is open in D. As $U \cup (D \setminus U) = D$ and D is connected, it follows that $D \setminus U = \emptyset$.

10.55, $f(x_1, x_2) = \sin(x_1 + x_2)$. We have (p.302)

$$R_N(b) = \sum_{|k|=N+1} \frac{\partial^{|k|} f}{\partial \vec{x}^k}(\mu) \frac{(\vec{b} - \vec{a})^k}{k_1! \cdots k_n!}$$

In our case we have

$$\frac{\partial^{|k|} f}{\partial \vec{x}^k}(\mu) = \begin{cases} (-1)^n \sin(\mu_1 + \mu_2), & |k| = 2n \\ (-1)^n \cos(\mu_1 + \mu_2), & |k| = 2n+1 \end{cases}$$

Hence $\left|\frac{\partial^{|k|} f}{\partial \vec{x}^k}(\mu)\right| \leq 1$ for all $\vec{\mu}$. We also have $\frac{|x|^k}{k!} \to 0$, $k \to \infty$, for all $x \in \mathbb{R}$. More generally we have $\sum_{|k|=N+1} \left| \frac{b_1^{k_1}}{k_1!} \cdots \frac{b_n^{k_n}}{k_n!} \right| \to 0$ as $N \to \infty$. This shows that that $|R_N(b)| \to 0$.

10.60, [only (a) & C].

(a) $f(x_1, x_2) = (x_1 \cos x_2, x_1 \sin x_2) = (y_1, y_2)$

$$Df(x_1, x_2) = \begin{pmatrix} \cos x_2 & -x_1 \sin x_2 \\ \sin x_2 & x_1 \cos x_2 \end{pmatrix}$$

$\det Df(x_1, x_2) = x_1 \quad (\neq 0$ for all $x_1 \neq 0)$.

Let $f(x_1^0, x_2^0) = (y_1^0, y_2^0) = x_1^0(\cos(x_2^0), \sin(x_2^0))$

Define sign : $\mathbb{R}\setminus\{0\} \to \{1,-1\}$ by

$$\text{sign}(t) = \begin{cases} 1 & , t>0 \\ -1 & , t<0 \end{cases}$$

Then, if $x_1^0 \neq 0$ we define

$$x_1 = \text{sign}(x_1^0)\sqrt{y_1^2 + y_2^2}$$ differentiable

If $\cos(x_2) \neq 0$, then

$$\tan(x_2) = \frac{y_2}{y_1}, \quad x_2 = \tan^{-1}\left(\frac{y_2}{y_1}\right),$$ differentiable.

as long as $y_1 \neq 0$. So we have to take r so that $x_1 \neq 0$ and $\cos(x_2) \neq 0$ for $(x_1, x_2) \in B_r(x_0)$.
Similar for $\sin(x_2) \neq 0$. [one can give more exact values, $0 < r < |x_1^0|$, $0 < r < \min_{n\in\mathbb{Z}}\{|\pi n - \frac{\pi}{2} - x_2^0|\}$]

(c) Is the same as (a) because x_3 only shows up in the last coordinate so

$$Df(\vec{x}) = \begin{pmatrix} \cos x_2 & -x_1\sin(x_2) & 0 \\ \sin x_2 & x_1\cos(x_2) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

10.59] Use: If $T: \mathbb{E}^n \to \mathbb{E}^m$ is a linear isomorphism then $n=m$. We have

$$I = f^{-1}\circ f : D \to \mathbb{E}^n$$

Thus

$$I = Df^{-1}(f(x))\, Df(x) : \mathbb{E}^n \to \mathbb{E}^n$$

similarly

$$I = Df(f^{-1}(y))\, Df^{-1}(y) : \mathbb{E}^m \to \mathbb{E}^m$$

The first equation implies that $m \geq n$ and the second that $n \geq m$ (The ~~sec~~ first equation says that $Df(x)$ is injective and the second that $Df(x)$ is surj.)

<u>10.62</u> See problem 10.60 (a).

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