PROBLEMS - VECTOR SPACES.

1) Which of the following sets is a vector space and which are not.

a) $\{(x,y,z) \in \mathbb{R}^3 \mid 2xy + z = 0\}$

b) $\{(x,y) \in \mathbb{R}^2 \mid 2x + 3y = 0\}$

c) $\{(x,y,z) \in \mathbb{R}^3 \mid 2x + 3y + z + 2 = 0\}$

d) $\left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x + 2y + z = 0 \\ 3x - y + 2z = 0 \end{matrix}\right\}$

e) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0\}$

f) $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 5\}$

g) $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$

h) $\{(x,y) \in \mathbb{R}^2 \mid x\cos(y) = 0\}$

2) Which of the following sets is a vector space and which are not.

a) $\{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$

b) $\{f \in C^1(\mathbb{R}) \mid f(0) = 1\}$

c) $\{f \in C([0,1]) \mid \int_0^1 f(t)\,dt = 0\}$

d) $\{f \in C([0,1]) \mid \int_0^1 f(t)\,dt = 3\}$

e) $\{f \in C^1(\mathbb{R}) \mid f(0)f'(0) = 0\}$

f) $\{f \in C([-\pi,\pi]) \mid \int_{-\pi}^{\pi} f(t)\cos(t)\,dt = 0\}$

g) The set of all polynomials.

h) $\{f \in C^4(\mathbb{R}) \mid 3f'''' + 2f''' - 3f'' + 2f' - f = 0\}$

i) $\{f \in C^4(\mathbb{R}) \mid 3f'''' + 2f''' - 3f'' + 2f' - f = 1\}$

j) V and W vector spaces, $T: V \to W$ linear. Is the space $\{v \in V \mid T(v) = 0\}$ a vector space?

3) What is the definition of a linear map?

4) Which of the following maps are linear and which are not?

a) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+2y+z \\ xy+z \end{pmatrix}$

b) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+2y+z \\ x+y+z \end{pmatrix}$

c) $T: \mathbb{R}^3 \to \mathbb{R}^2$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x+2y+z-1 \\ x+y+z \end{pmatrix}$

d) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3x+2y-z$

e) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xyz$

f) $T: \mathbb{R} \to \mathbb{R}$, $T(x) = x^2$

g) $T: \mathbb{R}^3 \to \mathbb{R}$, $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \dfrac{1}{x+3y+z}$

5) What are the linear maps $\mathbb{R}^n \to \mathbb{R}^m$?

6) Which of the following maps are linear and which are not:

a) $T: C^1(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = f' + f$

b) $T: C^1(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = f f'$

c) $T: C([-\pi,\pi]) \to \mathbb{R},\ T(f) = \int_{-\pi}^{\pi} f(t)\,dt$

d) $T: C([-\pi,\pi]) \to \mathbb{R},\ T(f) = \int_{-\pi}^{\pi} f(t)\sin(t)\,dt$

e) $T: C([-1,1]) \to \mathbb{R},\ T(f) = f(0.5)$

f) $T: C((-1,1)) \to \mathbb{R},\ T(f) = f(0)^2$

g) $T: C^4(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = 3f'''' - 2f' + f$

h) $T: C(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = e^{f(x)}$

i) $T: C(\mathbb{R}) \to C(\mathbb{R}),\ T(f) = f^2 + f$

7) Evaluate the inner product

a) $((x,y,z),(u,v,w)) = 2xu + yv + 3zw$

- $(x,y,z) = (1,2,-1)$, $(u,v,w) = (2,-3,4)$
- What is now $\|(2,1,-1)\|$

b) $V = C([0,\pi])$, $(f,g) = \int_0^{\pi} f(t)g(t)\,dt$

$f(t) = \cos(t)$, $g(t) = t$

$(f,g) =$ ______________________

c) $V = C([-\pi,\pi])$, $(f,g) = \int_{-\pi}^{\pi} f(t)g(t)\,dt$

- $(\cos(t), \sin(t)) =$ ______________
- $\|\cos(t)\| =$ ______________________
- $\|x^2\| =$ __________________________
- $(x^2, x) =$ ______________________________