MATH 1552-3, Summer 2011 Name: Solution

Test 1, Monday, June 20, 2010. For partial credit, show all your work!

1[8P]) Calculate the Simpson's S_4 approximation for the integral $\int_0^1 x^3\,dx$:

$S_4 = \frac{1}{4}$. We have $\Delta x = 1/4$. Hence $S_4 = \frac{1}{12}(0 + 4\frac{1}{4^3} + 2\frac{1}{2^3} + 4\frac{3^3}{4^3} + 1) = 1/4$.

2[6P]) For the function $f(x) = \frac{6}{1+x}$ on the interval $[0,1]$ use the Error Bound to find a value of N for which Error$(T_N) \leq 10^{-4}$. $N \geq 100$. We use the formula

$$\text{Error}(T_N) \leq \frac{K_2(b-a)^3}{12N^2} = \frac{K_2}{12N^2}\,.$$

To find K_2 we note that $f'(x) = -6/(1+x)^2$ and $f''(x) = 12/(1+x)^3$. We see that $f''(x)$ is decreasing so the maximum is taking at the left endpoint $x = 0$. Thus $K_2 = 12$. Hence Error$(T_N) \leq 1/N^2 \leq 10^{-4}$ or $N \geq 100$.

3[36P]) Evaluate the following integrals:

a) $\int x\ln(x)\,dx = \frac{x^2}{2}(\ln(x) - 1) + C$. Take $v' = x$ and $u = \ln(x)$. Then $v = \frac{1}{2}x^2$ and $u' = 1/x$. Thus partial integration gives

$$\int x\ln x dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,dx\frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C\,.$$

b) $2\int x^3\cos(x^2)\,dx = x^2\sin x^2 + \cos x^2 + C$ First let $y = x^2$ so $dy = 2xdx$. Then we get

$$\begin{aligned} 2\int x^3\cos(x^2)\,dx &= \int y\cos y\,dy \\ &= y\sin y - \int \sin y\,dy \text{ partial integration} \\ &= y\sin y + \cos y + C \\ &= x^2\sin x^2 + \cos x^2 + C\,. \end{aligned}$$

c) $\int e^{2x}\cos(x)\,dx = \frac{e^{2x}}{5}(\sin(x) + 2\cos(x))$. We use partial integration two times. The first time we have $u = e^{2x}$ and $u' = \cos(x)$, so $u' = 2e^{2x}$ and $v = \sin(x)$. Second time we use the

$u = 2e^{2x}$ and $v' = \sin(x)$. We get:

$$\int \int e^{2x} \cos(x)\, dx = e^{2x} \sin(x) - 2 \int e^{2x} \sin(x)\, dx = e^{2x} \sin(x) + 2e^{2x} \cos(x) - 4 \int e^{2x} \cos(x)\, dx, .$$

Thus $5 \int \int e^{2x} \cos(x)\, dx = e^{2x}(\sin(x) + 2\cos(x))$ or

$$\int \int e^{2x} \cos(x)\, dx = \frac{e^{2x}}{5}(\sin(x) + 2\cos(x))\,.$$

d) $\int \frac{x}{(x+1)(x^2+4)} = -\frac{1}{5} \ln|x+1| + \frac{1}{10} \ln(x^2+4) + \frac{2}{5} \tan^{-1}\left(\frac{x}{2}\right) + C$. Use partial fractions

$$\frac{x}{(x+1)(x^2+4)} = -\frac{1}{5}\frac{1}{x+1} + \frac{1}{5}\frac{x}{x^2+4} + \frac{4}{5}\frac{1}{x^2+4}\,.$$

Then use

$$\begin{aligned} -\frac{1}{5}\int \frac{1}{x+1}\, dx &= -\frac{1}{5} \ln|x+1| + C \\ \frac{1}{5}\int \frac{x}{x^2+4}\, dx &= \frac{1}{10} \ln(x^2+4) + C \qquad \text{use substitution } u = x^2+4,\ du = 2xdx \\ \frac{4}{5}\int \frac{1}{x^2+4}\, dx &= \frac{2}{5} \tan^{-1}(x/2) + C\,. \end{aligned}$$

4[27P]) Evaluate the following trigonometric integrals:

a) $\int \sin^3(x) \cos^2(x)\, dx = -\frac{1}{3} \cos^3(x) + \frac{1}{5} \cos^5(x) + C$. Use $\sin^2(x) = 1 - \cos^2(x)$ and the substitution $u = \cos(x)$, $du = -\sin(x)dx$ to get

$$\begin{aligned} \int \sin^3(x) \cos^2(x)\, dx &= -\int (1-u^2)u^2\, du \\ &= \int u^4 - u^2\, du \\ &= \frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= -\frac{1}{3} \cos^3(x) + \frac{1}{5} \cos^5(x) + C\,. \end{aligned}$$

b) $\displaystyle\int \tan(x)\sec(x)\,dx = \sec(x) + C$. Use the substitution $u = \sec(x)$, $du = \tan(x)\sec(x)dx$.

c) $\displaystyle\int \tan^3(x)\,dx = \frac{1}{2}\tan^2(x) + \ln|\cos(x)| + C$. Use the reduction formula for $\int \tan^n(u)\,du$ and then that $\int \tan(x)\,dx = -\ln|\cos(x)| + C$.

5[12P]) What substitution would you use in the following integrals and what is then dx?

a) $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ $\qquad x = 2\sin(\theta)$ and $dx = 2\cos(\theta)d\theta$

b) $\displaystyle\int \frac{dx}{\sqrt{x^2-1}} =$ $\qquad x = \sec(\theta)$ and $dx = \tan(\theta)\sec(\theta)d\theta$

6[16P]) Consider the integral $\displaystyle\int \frac{x^2-8x-2}{(x^3-4x^2+3x)^2\,(x^4-81)^2}\,dx$. Determine whether or not a term of the given type occurs in the general form of the complete partial fractions decomposition of the integrand. Circle **T** if it does and circle **F** if it does not. Below $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ denote constants.

T / F		
T / **F**	**1.**	$\dfrac{B_7}{(x+3)^2}$
T / **F**	**2.**	$\dfrac{B_1}{x-1}$
T / **F**	**3.**	$\dfrac{A_3x+B_3}{(x^2-9)^2}$
T / **F**	**4.**	$\dfrac{B_4}{(x+3)^3}$
T / **F**	**5.**	$\dfrac{B_5}{(x-1)^3}$
T / **F**	**6.**	$\dfrac{A_2x+B_2}{(x^2+9)^2}$
T / **F**	**7.**	$\dfrac{A_8x+B_8}{(x-3)^2}$

T / F	8. $\frac{B_6}{x}$

Use that

$$\left(x^3 - 4x^2 + 3x\right)^2 \left(x^4 - 81\right)^2 = x^2(x-1)^2(x-3)^4(x+3)^2(x^2+9)^2$$

and none of those factors is included in the numerator.

7[9P]) Evaluate the improper integral $\int_0^\infty xe^{-2x}\, dx = \frac{1}{4}$. We use integration by parts to find the antiderivative

$$\begin{aligned} \int xe^{-2x}\, dx &= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\, dx \quad u = x \text{ and } v' = e^{-2x} \text{ so } v = -\frac{1}{2}e^{-2x} \\ &= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C\,. \end{aligned}$$

Hence

$$\int_0^\infty xe^{-2x}\, dx = \lim_{R\to\infty}(-\frac{1}{2}Re^{-2R} - \frac{1}{4}e^{-2R}) + \frac{1}{4} = \frac{1}{4}\,.$$

8[18P]) Determine which of the following improper integrals exists. Give a (short) argument why. (The argument counts for 1/2 of the points.)

a) $\int_0^\infty \frac{x^2-1}{(x^2+x+1)^{3/2}}\, dx$ divergent. p test, $p = 1$.

b) $\int_1^\infty \frac{1}{x^2+1}\, dx$ convergent p-test, $p = 2 > 1$.

c) $\int_2^5 \frac{dx}{\sqrt{x-2}}$. convergent, p-test, $p = 1/2 < 1$.

9[18P]) Determine if the following sequences are convergent (then write *convergent*) or divergent (then write *divergent*. If the sequence converges determine the limit.

a) $a_n = \frac{2n^2+3n-1}{5n^2+2}$. Convergent with limit 2/5.

$$\frac{2n^2+3n-1}{5n^2+2} = \frac{n^2(2+3/n-1/n^2)}{n^2(5+2/n^2)} = \frac{2+3/n-1/n^2}{5+2/n^2} \to 2/5\,.$$

b) $a_n = 2n\sin(1/n)$. Convergent with limit 2. Write

$$2n\sin(1/n) = 2\frac{\sin(1/n)}{1/n}.$$

Then we see that

$$\begin{aligned}\lim_{n\to\infty} 2n\sin(1/n) &= 2\lim_{y\to 0}\frac{\sin(y)}{y}\\ &= 2\lim_{y\to 0}\frac{\cos(y)}{1}\\ &= 2.\end{aligned}$$

c) $\sqrt{n+2}-\sqrt{n}$. Convergent with limit 0. Write

$$\sqrt{n+2}-\sqrt{n} = \sqrt{n}(\sqrt{1+2/n}-1) = \frac{\sqrt{1+2/n}-1}{\sqrt{1/n}}.$$

Then

$$\begin{aligned}\lim_{n\to\infty}\sqrt{n+2}-\sqrt{n} &= \lim_{n\to\infty}\frac{\sqrt{1+2/n}-1}{\sqrt{1/n}}\\ &= \lim_{y\to 0}\frac{\sqrt{1+2y^2}-1}{y} \qquad \text{use } y=\sqrt{1/n}\\ &= \lim_{y\to 0}\frac{\frac{2y}{\sqrt{1+2y^2}}}{1}\\ &= \lim_{y\to 0}\frac{2y}{\sqrt{1+2y}}\\ &= 0.\end{aligned}$$

Formulas that you can use

1. **Simpson's Rule:** $S_N = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + \ldots + 4y_{N-1} + y_N\right)$.

2. Error(T_N)$\le \frac{K_2(b-a)^3}{12N^2}$ where K_2 is a number such that $|f''(x)| \le K_2$ for all $x \in [a, b]$.

3. Error(M_N)$\le \frac{K_2(b-a)^3}{24N^2}$ where K_2 is a number such that $|f''(x)| \le K_2$ for all $x \in [a, b]$.

4. Error(S_N)$\le \frac{K_4(b-a)^5}{180N^4}$ where K_4 is a number such that $|f^{(4)}(x)| \le K_4$ for all $x \in [a, b]$.

5. $\displaystyle\int \sin^n(u)\,du = -\frac{1}{n}\sin^{n-1}(u)\cos(u) + \frac{n-1}{n}\int \sin^{n-2}(u)\,du$

6. $\displaystyle\int \cos^n(u)\,du = \frac{1}{n}\cos^{n-1}(u)\sin(u) + \frac{n-1}{n}\int \cos^{n-2}(u)\,du$

7. $\displaystyle\int \sec^n(u)\,du = \frac{1}{n-1}\tan(u)\sec^{n-2}(u) + \frac{n-2}{n-1}\int \sec^{n-2}(u)\,du$

8. $\displaystyle\int \tan^n(u)\,du = \frac{1}{n-1}\tan^{n-1}(u) - \int \tan^{n-2}(u)\,du$