Integral Transforms
Math 2025

Gestur Ólafsson

Mathematics Department

Louisiana State University

Chapter 4

Linear Maps

We have all seen linear maps before. In fact, most of the maps we have been using in Calculus are linear.

Example. ***The Integral***

To integrate the function $f(x) = x^2 + 3x - cosx$ *over the interval* $[a, b]$, *we first find the antiderivative of* x^2, *that is* $\frac{1}{3}x^3$, *then the antiderivative of* x, *which is* $\frac{1}{2}x^2$, *and then multiply that by* 3 *to get* $\frac{3}{2}x^2$. *Finally, we find the antiderivative of* $cosx$, *which is* $sinx$, *and then multiply that by* -1 *to get* $-sinx$. *To finish the problem we insert the endpoints. Thus,*

Two Important Examples

Example. ***The Integral***
To integrate the function $f(x) = x^2 + 3x - cosx$ *over the interval* $[a, b]$, *we first find the antiderivative of* x^2, *that is* $\frac{1}{3}x^3$, *then the antiderivative of* x, *which is* $\frac{1}{2}x^2$, *and then multiply that by* 3 *to get* $\frac{3}{2}x^2$. *Finally, we find the antiderivative of* $cosx$, *which is* $sinx$, *and then multiply that by* -1 *to get* $-sinx$. *To finish the problem we insert the endpoints. Thus,*

$$\begin{aligned}\int_{-1}^{1} x^2 + 3x - cosx\,dx &= \int_{-1}^{1} x^2\,dx + 3\int_{-1}^{1} x\,dx \\ &\quad - \int_{-1}^{1} cosx\,dx \\ &= \left[\frac{1}{3}x^3\right]_{-1}^{1} + \left[\frac{3}{2}x^2\right]_{-1}^{1} - [sinx]_{-1}^{1} \\ &= \frac{2}{3} - sin1 + sin(-1).\end{aligned}$$

The Integral(cont.)

What we have used is the fact that the integral is a linear map $\mathcal{C}^1([a,b]) \longrightarrow \mathcal{C}([a,b])$ and that

The Integral(cont.)

What we have used is the fact that the integral is a linear map $\mathcal{C}^1([a,b]) \longrightarrow \mathcal{C}([a,b])$ and that

$$\int_a^b rf(x) + sg(x)\,dx = r\int_a^b f(x)\,dx + s\int_a^b g(x)\,dx.$$

Example. ***The Derivative***

Another example is differentiation $Df = f'$. *To differentiate the function* $f(x) = x^4 - 3x + e^x - cosx$, *we first differentiate each term of the function and then add:*

Two Important Examples(cont.)

Example. ***The Derivative***
Another example is differentiation $Df = f'$. *To differentiate the function* $f(x) = x^4 - 3x + e^x - cosx$, *we first differentiate each term of the function and then add:*

$$\begin{aligned} D(x^4 - 3x + e^x - cosx) &= Dx^4 - 3Dx + De^x \\ &\quad -Dcosx \\ &= 4x^3 - 3 + e^x + sinx. \end{aligned}$$

Definition. Let V and W be two vector spaces. A map $T : V \longrightarrow W$ is said to be linear if for all $v, u \in V$ and all $r, s \in \mathbb{R}$ we have:

$$T(rv + su) = rT(v) + sT(u).$$

Definition. Let V and W be two vector spaces. A map $T : V \longrightarrow W$ is said to be linear if for all $v, u \in V$ and all $r, s \in \mathbb{R}$ we have:

$$T(rv + su) = rT(v) + sT(u).$$

Remark: This can also be written by using two equations:

$$T(v + u) = T(v) + T(u)$$

$$T(rv) = rT(v).$$

Lemma. *Suppose that* $T : V \longrightarrow W$ *is linear. Then* $T(\vec{0}) = \vec{0}$.

Proof. We can write $\vec{0} = 0v$, where v is any vector in V. But then $T(\vec{0}) = T(0v) = 0T(v) = 0$ □

Example. *Let us find all the linear maps from* $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$. *Any arbitrary vector* $(x_1, x_2) \in \mathbb{R}^2$ *can be written as:*

$$(x_1, x_2) = x_1(1,0) + x_2(0,1).$$

Hence,

$$T(x_1, x_2) = x_1 T(1,0) + x_2 T(0,1).$$

Write $T(1,0)$ *and* $T(0,1)$ *as:*

$$T(1,0) = (a_{11}, a_{12}), T(0,1) = (a_{21}, a_{22}), \text{ where } \quad a_{ij} \in \mathbb{R}.$$

Then,

$$\begin{aligned} T(x_1, x_2) &= x_1(a_{11}, a_{12}) + x_2(a_{21}, a_{22}) \\ &= (x_1 a_{11} + x_2 a_{21}, x_1 a_{12} + x_2 a_{22}) \\ &= (x_1, x_2) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \end{aligned}$$

Thus, all the information about T is given by the matrix:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Example. *Next, let us find all the linear maps* $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$*. As before we write* $(x_1,x_2,x_3)\in\mathbb{R}^3$ *as:*

$$(x_1,x_2,x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$$

where,

$$T(1,0,0) = (a_{11},a_{12},a_{13})$$
$$T(0,1,0) = (a_{21},a_{22},a_{23})$$
$$T(0,0,1) = (a_{31},a_{32},a_{33}).$$

Examples(cont.)

Then,

$$\begin{aligned} T(x_1,x_2,x_3) &= x_1(a_{11},a_{12},a_{13}) + x_2(a_{21},a_{22},a_{23}) \\ &\quad + x_3(a_{31},a_{32},a_{33}) \\ &= (x_1a_{11}+x_2a_{21}+x_3a_{31}, x_1a_{12}+x_2a_{22} \\ &\quad + x_3a_{32}, x_1a_{13}+x_2a_{23}+x_3a_{33}) \\ &= (x_1,x_2,x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \end{aligned}$$

Examples(cont.)

Example. *All the linear maps from* $\mathbb{R}^3 \longrightarrow \mathbb{R}$. *Notice that* $\mathbb{R}$ *is also a vector space, so we can consider all the linear maps* $\mathbb{R}^n$ *to* $\mathbb{R}$. *We have :*

Examples(cont.)

Example. *All the linear maps from* $\mathbb{R}^3 \longrightarrow \mathbb{R}$. *Notice that* $\mathbb{R}$ *is also a vector space, so we can consider all the linear maps* $\mathbb{R}^n$ *to* $\mathbb{R}$. *We have :*

$$\begin{aligned} T(x_1, x_2, ..., x_n) &= x_1 T(1,0,...,0) + x_2 T(0,1,...,0) \\ &\quad + ... + x_n T(0,0,...,1) \\ &= x_1 a_1 + x_2 a_2 + ... + x_n a_n \end{aligned}$$

where,

$$T(1,0,...,0) = a_1, T(0,1,...,0) = a_2, T(0,0,...,1) = a_n.$$

Lemma. *A map* $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is linear if and only if there exists numbers* $a_{ij}, i = 1, ..., n, j = 1, ..., m$*, such that:*

$$\begin{aligned} T(x_1, x_2, ..., x_n) &= (x_1 a_{11} + x_2 a_{21} + ... + x_n a_{n1}, ..., \\ &\quad x_1 a_{1m} + x_2 a_{2m} + ... + x_n a_{nm}) \end{aligned}$$

Lemma. *A map* $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is linear if and only if there exists numbers* $a_{ij}, i = 1, ..., n, j = 1, ..., m$*, such that:*

$$T(x_1, x_2, ..., x_n) = (x_1 a_{11} + x_2 a_{21} + ... + x_n a_{n1}, ..., x_1 a_{1m} + x_2 a_{2m} + ... + x_n a_{nm})$$

This can also be written as:

$$T(x_1, x_2, ..., x_n) = (\sum_{i=1}^{n} x_i a_{i1}, \sum_{i=1}^{n} x_i a_{i2}, \sum_{i=1}^{n} x_i a_{im})$$

Lemma. *A map* $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ *is linear if and only if there exists numbers* $a_{ij}, i = 1, ..., n, j = 1, ..., m$, *such that:*

$$T(x_1, x_2, ..., x_n) = (x_1 a_{11} + x_2 a_{21} + ... + x_n a_{n1}, ..., x_1 a_{1m} + x_2 a_{2m} + ... + x_n a_{nm})$$

This can also be written as:

$$T(x_1, x_2, ..., x_n) = (\sum_{i=1}^{n} x_i a_{i1}, \sum_{i=1}^{n} x_i a_{i2}, \sum_{i=1}^{n} x_i a_{im})$$

or by using matrix multiplication:

$$T(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n) \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & . & a_{2m} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{pmatrix}.$$

Counterexample

Example. *The map* $T(x,y,z) = (2x + 3xy, z + y)$ *is not linear because of the factor* xy*. Notice that:*

$$T(1,1,0) = (5,0)$$

but

$$T(2(1,1,0)) = T(2,2,0) = (16,0)$$

and

$$2T(1,1,0) = (10,0) \neq (16,0)$$

Examples

Example. *Evaluate the given following maps at a given point:*

$$\begin{aligned} T(x,y) &= (3x+y,3y), \quad (x,y)=(1,-1) \\ T(1,-1) &= (3\cdot 1-1,3(-1))=(2,-3) \end{aligned}$$

Example. *Evaluate the given following maps at a given point:*

$$T(x,y) = (3x+y,3y), \quad (x,y)=(1,-1)$$
$$T(1,-1) = (3\cdot 1-1,3(-1)) = (2,-3)$$

$$T(x,y,z) = (2x-y+3z,2x+z), \quad (x,z,y)=(2,-1,1)$$
$$T(2,-1,1) = (4+1+3,4+1) = (8,5)$$

Examples(cont.)

Example. *Some examples involving differentiation and integration:*

$$D(3x^2 + 4x - 1) = 6x + 4$$

Examples(cont.)

Example. *Some examples involving differentiation and integration:*

$$D(3x^2 + 4x - 1) = 6x + 4$$

$$\begin{aligned}\int_1^2 x^2 - e^x\,dx &= \left[\frac{1}{3}x^3 - e^x\right]_1^2 \\ &= \frac{8}{3} - e^2 - \frac{1}{3} + e \\ &= \frac{7}{3} - e^2 + e\end{aligned}$$

Definition. Let V and W be two vector spaces and $T : V \longrightarrow W$ a linear map.

Kernel

Definition. Let V and W be two vector spaces and $T : V \longrightarrow W$ a linear map.

- A1 The set $Ker(T) = \{v \in V : T(v) = 0\}$ is called the **kernel** of T.

Definition. Let V and W be two vector spaces and $T : V \longrightarrow W$ a linear map.

- A1 The set $Ker(T) = \{v \in V : T(v) = 0\}$ is called the **kernel** of T.
- A2 The set $Im(T) = \{w \in W : \text{there exists a} v \in V : T(v) = w\}$ is called the **image** of T.

Kernel

Definition. Let V and W be two vector spaces and $T : V \longrightarrow W$ a linear map.

- A1 The set $Ker(T) = \{v \in V : T(v) = 0\}$ is called the **kernel** of T.
- A2 The set $Im(T) = \{w \in W :$ there exists a $v \in V : T(v) = w\}$ is called the **image** of T.

Remark: Notice that $Ker(T) \subseteq V$ and $Im(T) \subseteq W$.

Theorem. *The kernel of T is a vector space.*

Proof. Let $u, v \in Ker(T)$ and $r, s \in \mathbb{R}$ We have to show that $ru + sv \in Ker(T)$. Now,$u, v \in Ker(T)$ if and only if $T(u) = T(v) = 0$. Hence,

$$\begin{aligned} T(ru + sv) &= rT(u) + sT(v) \quad (T\text{is linear}) \\ &= r \cdot 0 + s \cdot 0 \quad (u, v \in Ker(T)) \\ &= 0 \end{aligned}$$

This shows that $ru + sv \in Ker(T)$. □

Remark: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map:

$$T(x, y) = (x^2 + y, x + y).$$

Remark

Remark: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map:

$$T(x, y) = (x^2 + y, x + y).$$

Then,

$$T(1, -1) = (1 - 1, 1 - 1) = (0, 0).$$

Remark: Let $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the map:

$$T(x, y) = (x^2 + y, x + y).$$

Then,

$$T(1, -1) = (1 - 1, 1 - 1) = (0, 0).$$

But $T(2(1, -1)) = T(2, -2) = (4 - 2, 2 - 2) = (2, 0) \neq (0, 0)$.

So if T is not linear, then the set $v \in V : T(v) = 0$ is in general not a vector space.

Example. *Let* $\mathbb{R}^2 \longrightarrow \mathbb{R}$ *be the map:* $T(x, y) = 2x - y$.
Describe the kernel of T.
We know that (x, y) *is in the kernel of* T *if and only if*
$T(x, y) = 2x - y = 0$.
Hence, $y = 2x$. *Thus, the kernel of* T *is a line through* $(0, 0)$ *with slope* 2.

Example. *Let* $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ *be the map:* $T(x, y, z) = (2x - 3y + z, x + 2y - z)$. *Describe the kernel of* T.

We have that $(x, y, z) \in Ker(T)$ *if and only if*

$$2x - 3y + z = 0 \quad \text{and} \quad x + 2y - z = 0.$$

The equations describe planes through $(0, 0, 0)$ *with normal vectors* $(2, -3, 1)$ *and* $(1, 2, -1)$ *respectively. The normal vectors are not parallel and therefore the planes are different. It follows that the intersection is a line.*

Examples(cont.)

Let us describe this line. Adding the equations we get:

$$3x - y = 0 \quad \text{or} \quad y = 3x.$$

Plugging this into the second equation we get:

$$0 = x + 2(3x) - z = 7x - z \quad \text{or} \quad z = 7x.$$

Hence, the line is given by: $x \cdot (1, 3, 7)$.

Theorem. *Let V and W be vector spaces, and $T : V \longrightarrow W$ linear. Then, $Im(T) \subseteq W$ is a vector space.*

Proof. Let $w_1, w_2 \in Im(T)$. Then we can find $u_1, u_2 \in V$ such that $T(u_1) = w_1, T(u_2) = w_2$. Let $r, s \in \mathbb{R}$. Then,

$$\begin{aligned} rw_1 + sw_2 &= rT(u_1) + sT(u_2) \\ &= T(ru_1 + su_2) \in Im(T)). \end{aligned}$$

□

Chapter 5

Inner Product

Let us start with the following problem. Given a point $P \in \mathbb{R}^2$ and a line $L \subseteq \mathbb{R}^2$, how can we find the point on the line closest to P?

Let us start with the following problem. Given a point $P \in \mathbb{R}^2$ and a line $L \subseteq \mathbb{R}^2$, how can we find the point on the line closest to P?

Answer: Draw a line segment from P meeting the line in a right angle. Then, the point of intersection is the point on the line closest to P.

Inner Product(cont.)

Let us now take a plane $L \subseteqq \mathbb{R}^3$ and a point outside the plane. How can we find the point $u \in L$ closest to P?

The answer is the same as before, go from P so that you meet the plane in a right angle.

In both examples we need two things:

Observation

In both examples we need two things:

- A1 We have to be able to say what the length of a vector is.

Observation

In both examples we need two things:

- A1 We have to be able to say what the length of a vector is.
- B1 Say what a right angle is.

Observation

In both examples we need two things:

- A1 We have to be able to say what the length of a vector is.
- B1 Say what a right angle is.

Both of these things can be done by using the dot-product (or inner product) in $\mathbb{R}^n$.

Definition

Definition. Let $(x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n) \in \mathbb{R}^n$. Then, the **dot-product** of these vectors is given by the number:

$$((x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n)) = x_1y_1+x_2y_2+...+x_ny_n.$$

Definition

Definition. Let $(x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n) \in \mathbb{R}^n$. Then, the **dot-product** of these vectors is given by the number:

$$((x_1, x_2, ..., x_n), (y_1, x_2, ..., y_n)) = x_1y_1 + x_2y_2 + ... + x_ny_n.$$

The **norm** (or length) of the vector $\vec{u} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the non-negative number:

$$\|u\| = \sqrt{(u, u)} = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}.$$

Example. ■ *a)* $((1, 2, -3), (1, 1, 1)) = 1 + 2 - 3 = 0$

■ *b)* $((1, -2, 1), (2, -1, 3)) = 2 + 2 + 3 = 7$

Because,

$$|x_1y_1 + x_2y_2 + ... + x_ny_n| \leq \sqrt{x_1^2 + x_2^2 + ... + x_n^2}\sqrt{y_1^2 + y_2^2 + ... + y_n^2}$$

or

$$|(u, v)| \leq \|u\| \cdot \|v\|$$

we have that (for $u, v \neq 0$)

$$-1 \leq \frac{(u, v)}{\|u\| \cdot \|v\|} \leq 1.$$

Perpendicular(cont.)

Hence we can define:

$$\cos(\angle(u,v)) = \frac{(u,v)}{\|u\| \cdot \|v\|}.$$

In particular, $u \perp v$ (u perpendicular to v) if and only if $(u,v) = 0$.

Example

Example. *Let* L *be the line in* $\mathbb{R}^2$ *given by* $y = 2x$. *Thus,*

$$L = \{r(1,2) : r \in \mathbb{R}\}.$$

Let $P = (2,1)$. *Consider the following questions.*

Question 1

Question 1: What is the point on L closest to P?

Question 1

Question 1: What is the point on L closest to P?

Answer: Because $u \in L$, we can write $\vec{u} = (r, 2r)$. Furthermore, $v - u = (2 - r, 1 - 2r)$ is perpendicular to L. Hence,

$$0 = ((1, 2), (2 - r, 1 - 2r)) = 2 - r + 2 - 4r = 4 - 5r.$$

Hence, $r = \frac{4}{5}$ and $\vec{v} = (\frac{4}{5}, \frac{8}{5})$.

Question 2

Question 2: What is the distance of P from the line?

Question 2

Question 2: What is the distance of P from the line?

Answer: The length of the vector $v-u$, i.e. $\|v-u\|$. First we have to find out what $v-u$ is. We have done almost all the work:

$$v-u=(2,1)-(\frac{4}{5},\frac{8}{5})=(\frac{6}{5},\frac{-3}{5}).$$

The distance therefore is:

$$\sqrt{\frac{36}{25}+\frac{9}{25}}=\frac{3\sqrt{5}}{5}.$$

Properties

- 1 (**positivity**)To be able to define the norm, we used that $(u, u) \geq 0$.

Properties

- 1 (**positivity**)To be able to define the norm, we used that $(u, u) \geq 0$.
- 2 (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u, u) = 0$ only if $u = 0$.

Properties

- 1 (**positivity**)To be able to define the norm, we used that $(u, u) \geq 0$.
- 2 (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u, u) = 0$ only if $u = 0$.
- 3 (**linearity**)If the vector $v \in \mathbb{R}^n$ is fixed, then a map $u \mapsto (u, v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is,

$$(ru + sw, v) = r(u, v) + s(w, v).$$

Properties

- 1 (**positivity**)To be able to define the norm, we used that $(u,u) \geq 0$.
- 2 (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u,u) = 0$ only if $u = 0$.
- 3 (**linearity**)If the vector $v \in \mathbb{R}^n$ is fixed, then a map $u \mapsto (u,v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is,

$$(ru + sw, v) = r(u,v) + s(w,v).$$

- 4 (**symmetry**) For all $u, v \in \mathbb{R}^n$ we have: $(u,v) = (v,u)$.

Properties

- 1 (**positivity**)To be able to define the norm, we used that $(u,u) \geq 0$.
- 2 (**zero length**)All non-zero vectors should have a non-zero length. Thus, $(u,u) = 0$ only if $u = 0$.
- 3 (**linearity**)If the vector $v \in \mathbb{R}^n$ is fixed, then a map $u \mapsto (u,v)$ from $\mathbb{R}^n$ to $\mathbb{R}$ is linear. That is,

$$(ru + sw, v) = r(u,v) + s(w,v).$$

- 4 (**symmetry**) For all $u, v \in \mathbb{R}^n$ we have: $(u,v) = (v,u)$.

We will use the properties above to define an inner product on arbitrary vector spaces.

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

- 1 (**positivity**) $(u, u) \geq 0$, for all $v \in V$.

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

- 1 (**positivity**) $(u,u) \geq 0$, for all $v \in V$.
- 2 (**zero length**) $(u,u) = 0$ only if $u = 0$.

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

- 1 (**positivity**) $(u, u) \geq 0$, for all $v \in V$.
- 2 (**zero length**) $(u, u) = 0$ only if $u = 0$.
- 3 (**linearity**)If $v \in V$ is fixed, then a map $u \mapsto (u, v)$ from V to $\mathbb{R}$ is linear.

Definition

Let V be a vector space. An inner product on V is a map $(.,.) : V \times V \longrightarrow \mathbb{R}$ satisfying the following properties:

- 1 (**positivity**) $(u, u) \geq 0$, for all $v \in V$.
- 2 (**zero length**) $(u, u) = 0$ only if $u = 0$.
- 3 (**linearity**)If $v \in V$ is fixed, then a map $u \mapsto (u, v)$ from V to $\mathbb{R}$ is linear.
- 4 (**symmetry**) $(u, v) = (v, u)$, for all $u, v \in V$.

Definition. We say that u and v are perpendicular if $(u, v) = 0$.

Definition

Definition. We say that u and v are perpendicular if $(u, v) = 0$.

Definition. If (.,.) is an inner product on the vector space V, then the norm of a vector $v \in V$ is given by:

$$\|u\| = \sqrt{(u, u)}.$$

Properties

Lemma. *The norm satisfies the following properties:*

- 1 $\|u\| \geq 0$, and $\|u\| = 0$ only if $u = 0$.
- 2 $\|ru\| = |r| \cdot \|u\|$.

Proof. We have that

$$\begin{aligned} \|ru\| &= \sqrt{(ru, rv)} \\ &= \sqrt{(r^2(u, v)} \\ &= |r|\sqrt{(u, v)} = |r| \cdot \|u\| . \end{aligned}$$

$\square$

Example. *Let* $a < b$, $I = [a, b]$, *and* $V = PC([a, b])$.

Define:

$$(f, g) = \int_a^b f(t)g(t)\, dt$$

Then, (.,.) is an inner product on V.

Proof. Let $r, s \in \mathbb{R}$, $f, g, h \in V$. Then:

- 1 $(f, f) = \int_a^b f(t)^2\, dt$. As $f(t)^2 \geq 0$, it follows that $\int_a^b f(t)^2\, dt \geq 0$.
- 2 If $(f, f) = 0$, then $f(t)^2 = 0$ for all t, i.e $f = 0$.
- 3 $\int_a^b (rfsg)(t)h(t)\, dt = \int_a^b rf(t)h(t) + sg(t)h(t)\, dt$

$$\begin{aligned} &= r\int_a^b f(t)h(t)\, dt + s\int_a^b g(t)h(t)\, dt \\ &= r(f, h) + s(g, h). \end{aligned}$$

  Hence, linear in the first factor.
- 4 As $f(t)g(t) = g(t)f(t)$, it follows that $(f, g) = (g, f)$.

Notice that the norm is:

$$\|f\| = \sqrt{\int_a^b f(t)^2\, dt}.$$

Examples(cont.)

Example. *Let* $a = 0$, $b = 1$ *in the previous example. That is,* $f(t) = t^2$ *and* $g(t) = t - 3t^2$. *Then:*

$$
\begin{aligned}
(f, g) &= \int_0^1 t^2(t - 3t^2)\, dt \\
&= \int_0^1 t^3 - 3t^4)\, dt \\
&= \frac{1}{4} - \frac{3}{5} \\
&= -\frac{7}{20}.
\end{aligned}
$$

Examples(cont.)

Also, the norms are:

$$\|f\| = \sqrt{\int_0^1 t^4\,dt} = \frac{1}{\sqrt{5}}.$$

$$\begin{aligned} \|g\| &= \sqrt{\int_0^1 (t-3t^2)^2\,dt} \\ &= \sqrt{\int_0^1 t^2 - 6t^3 + 9t^4\,dt} \\ &= \sqrt{\frac{1}{3} - \frac{3}{2} + \frac{9}{5}} \\ &= \sqrt{\frac{19}{30}}. \end{aligned}$$

Examples(cont.)

Example. *Let* $f(t) = \cos 2\pi t$ *and* $g(t) = \sin 2\pi t$*. Then:*

$$\begin{aligned}(f, g) &= \int_0^1 \cos 2\pi t \sin 2\pi t \, dt \\ &= \frac{1}{4\pi} \left[(\sin 2\pi t)^2 \right]_0^1 = 0.\end{aligned}$$

So, $\cos 2\pi t$ *is perpendicular to* $\sin 2\pi t$ *on the interval* $[0, 1]$*.*

Example. *Let* $f(t) = \chi_{[0,1/2)} - \chi_{[1/2,1)}$ *and* $g(t) = \chi_{[0,1)}$.
Then:

$$\begin{aligned} (f,g) &= \int_0^1 (\chi_{[0,1/2)}(t) - \chi_{[1/2,1)}(t))(\chi_{[0,1)})\,dt \\ &= \int_0^1 \chi_{[0,1/2)}(t)\,dt - \int_0^1 \chi_{[1/2,1)}(t)\,dt \\ &= \int_0^{1/2} dt - \int_{1/2}^1 dt = \frac{1}{2} = \frac{1}{2} = 0. \end{aligned}$$

One can also easily show that $\|f\| = \|g\| = 1$.

Problem

Problem: Find a polynomial $f(t) = a + bt$ that is perpendicular to the polynomial $g(t) = 1 - t$.

Problem: Find a polynomial $f(t) = a + bt$ that is perpendicular to the polynomial $g(t) = 1 - t$.

Answer: We are looking for numbers a and b such that:

$$\begin{aligned} 0 = (f, g) &= \int_0^1 (a + bt)(1 - t)\, dt \\ &= \int_0^1 a + bt - at - bt^2\, dt \\ &= a + \frac{b}{2} - \frac{a}{2} - \frac{b}{3} \\ &= \frac{a}{2} + \frac{b}{6}. \end{aligned}$$

Problem: Find a polynomial $f(t) = a + bt$ that is perpendicular to the polynomial $g(t) = 1 - t$.

Answer: We are looking for numbers a and b such that:

$$\begin{aligned} 0 = (f, g) &= \int_0^1 (a + bt)(1 - t)\,dt \\ &= \int_0^1 a + bt - at - bt^2\,dt \\ &= a + \frac{b}{2} - \frac{a}{2} - \frac{b}{3} \\ &= \frac{a}{2} + \frac{b}{6}. \end{aligned}$$

Thus, $3a + b = 0$. So, we can take $f(t) = 1 - 3t$.

Facts

We state now two important facts about the inner product on a vector space V. Recall that in $\mathbb{R}^2$ we have:

$$\cos(\theta) = \frac{(u, v)}{\|u\| \cdot \|v\|}.$$

where u, v are two non-zero vectors in $\mathbb{R}^2$ and θ is the angle between u and v. In particular, because $-1 \leq \cos\theta \leq 1$, we must have:

$$\|(u, v)\| \leq \|u\| \cdot \|v\|.$$

We will show now that this comes from the positivity and linearity of the inner product.

Theorem

Theorem. *Let V be a vector space with inner product (.,.). Then:*

$$|(u, v)| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$.

Proof. We can assume that $u, v \neq 0$ because otherwise both the LHS and the RHS will be zero. By the positivity of the inner product we get:

$$\begin{aligned} 0 &\leq (v - \frac{(v,u)}{\|u\|^2} u, v - \frac{(v,u)}{\|u\|^2} u) \quad (positivity) \\ &= (v,v) - \frac{(v,u)}{\|u\|^2}(u,v) - \frac{(v,u)}{\|u\|^2}(v,u) + \frac{(v,u)^2}{\|u\|^4}(u,u) \quad (linearity) \\ &= \|v\|^2 - 2\frac{(u,v)^2}{\|u\|^2} + \frac{(v,u)^2}{\|u\|^2} \quad (symmetry) \\ &= \|v\|^2 - \frac{(u,v)^2}{\|u\|^2}. \end{aligned}$$

Thus,

$$\frac{(u,v)^2}{\|u\|^2} \leq \|v\|^2 \quad \text{or} \quad \|(u,v)\| \leq \|u\| \cdot \|v\|.$$

□

Note

Notice that:

$$0 = (v - \frac{(u,v)}{\|u\|^2}u, v - \frac{(u,v)}{\|u\|^2}u)$$

only if

$$v - \frac{(u,v)}{\|u\|^2}u = 0$$

i.e.

$$v = \frac{(u,v)}{\|u\|^2}u.$$

Thus, v and u have to be on the same line through 0.

Lemma

We can therefore conclude:

Lemma. $\|(u,v)\| = \|u\| \cdot \|v\|$ *if and only if* u *and* v *are on the same line through* 0*.*

The following statement is generalization of Pythagoras Theorem.

The following statement is generalization of Pythagoras Theorem.

Theorem. *Let V be a vector space with inner product (.,.). Then:*

$$\|u+v\| \leq \|u\| \cdot \|v\|$$

for all $u, v \in V$. Furthermore, $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ if and only if $(u, v) = 0$.

Proof.

$$\begin{aligned} \|u+v\|^2 &= (u+v, u+v) \\ &= (u,u) + 2(u,v) + (v,v) \quad (*) \\ &\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2 \\ &= (\|u\| + \|v\|)^2. \end{aligned}$$

If $(u,v) = 0$, then (*) reads:

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

On the other hand, if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, we see from (*) that $(u,v) = 0$. □

Example

Example. *Let* $u = (1, 2, -1), v = (0, 2, 4)$. *Then:*

$$(u, v) = 4 - 4 = 0$$

and

$$\|u\|^2 = 1 + 4 + 1 = 6, \|v\|^2 = 4 + 16 = 20.$$

Also, $u + v = (1, 4, 3)$ *and finally:*

$$\|u + v\|^2 = 1 + 16 + 9 = 26 = 6 + 20 = \|u\|^2 + \|v\|^2 .$$

Chapter 6

Generating Sets and Bases

Generating Sets and Bases

Let V be the vector space $\mathbb{R}^2$ and consider the vectors $(1,0),(0,1)$. Then, every vector $(x,y) \in \mathbb{R}^2$ can be written as combination of those vectors. That is:

$$(x,y) = x(1,0) + y(0,1).$$

Similarly, the two vectors $(1,1)$ and $(1,2)$ do not belong to the same line, and every other vector can be written as a combination of those two vectors.

In particular:

$$(x, y) = a(1, 1) + (1, 2)$$

gives us two equations

$$a + b = x \quad \text{and} \quad a + 2b = y$$

Thus, by substituting the first equation to the second, we get

$$b = -x + y$$

Inserting this into the first equation we get

$$a = 2x - y$$

Intro(cont.)

Take for example the point $(4,3)$. Then:

$$\begin{aligned}(4,3) &= 5(1,1) + (-1)(1,2) \\ &= 5(1,1) - (1,2)\end{aligned}$$

Intro(cont.)

We have similar situation for $\mathbb{R}^3$ and all of the spaces $\mathbb{R}^n$.

In the case of $\mathbb{R}^3$, for example, every vector can be written as combinations of $(1,0,0), (0,1,0)$ and $(0,0,1)$, i.e.,

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1).$$

Or, as a combination of $(1,-1,0), (1,1,1)$ and $(0,1,-1)$, that is:

$$(x,y,z) = a(1,-1,0) + b(1,1,1) + c(0,1,-1).$$

The latter gives three equation:

$$\begin{aligned} a+b &= x \quad (1) \\ -a+b+c &= y \quad (2) \\ b-c &= z \quad (3). \end{aligned}$$

$(2)+(3)$ gives:

$$-a+2b = y+z \quad (4)$$

$(4)+(1)$ gives:

$$3b = x+y+z \text{ or } b = \frac{x+y+z}{3}.$$

Then (1) gives:

$$\begin{aligned} a &= x - b \\ &= x - \frac{x+y+z}{3} \\ &= \frac{2x-y-z}{3}. \end{aligned}$$

Finally, (3) gives:

$$c = b - z = \frac{x+y-2z}{3}$$

Hence, we get:

$$(x,y,z) = \frac{2x-y-z}{3}(1,-1,0) + \frac{x+y+z}{3}(1,1,1) + \frac{x+y-2z}{3}(0,1,-1).$$

In general

Notice that we get only one solution, so there is only one way that we can write a vector in $\mathbb{R}^3$ as a combination of those vectors. In general, if we have k vectors in $\mathbb{R}^3$, then the equation:

$$x = (x_1, x_2, ..., x_n) = c_1v_1 + c_2v_2 + ... + c_kv_k \quad (*)$$

gives n-equations involving the n-coordinates of $v_1, v_2, ..., v_k$ and the unknowns $c_1, c_2, ..., c_k$. There are three possibilities:

Possibilities

- A The equation(*) has no solution. Thus, x can not be written as a combination of the vectors $v_1, v_2, ..., v_k$.
- B The equation (*) has only one solution, so x can be written in exactly one way as a combination of $v_1, v_2, ..., v_k$.
- C The system of equations has infinitely many solutions, so there are more than one way to write x as a combination of $v_1, v_2, ..., v_k$.

Case C

Let us look at the last case a little closer. If we write x in two different ways:

$$\begin{aligned} x &= c_1 v_1 + c_2 v_2 + \ldots + c_k v_k \\ x &= d_1 v_1 + d_2 v_2 + \ldots + d_k v_k \end{aligned}$$

Then, by subtracting, we get:

$$0 = (c_1 - d_1) v_1 + (c_2 - d_2) v_2 + \ldots + (c_k - d_k) v_k$$

where some of the numbers $c_i - d_i$ are non-zero.

Case C (cont.)

Similarly, since we can write:

$$0 = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$$

and

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

then we also have:

$$x = (c_1 + a_1) v_1 + (c_2 + a_2) v_2 + \ldots + (c_k + a_k) v_k.$$

Thus, we can write x as a combination of the vectors $v_1, v_2, \ldots, v_k$ in several different ways (in fact ∞-many ways). We will now use this as a motivation for the following definitions.

Definition

Definition. Let V be a vector space and $v_1, v_2, ..., v_n \in V$.

- 1 Let $W \subseteq V$ be a subspace. We say that W is spanned by the vectors $v_1, v_2, ..., v_n$ if every vector in W can be written as a linear combination of $v_1, v_2, ..., v_n$. Thus, if $w \in W$, then there exist numbers $c_1, c_2, ..., c_n \in \mathbb{R}$ such that $w = c_1v_1 + c_2v_2 + ... + c_nv_n$.
- 2 The set of vectors $v_1, v_2, ..., v_n$ is linearly dependent if there exist $c_1, c_2, ..., c_n$, not all equal to zero, such that $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$.

Definition

Definition.
- 1 The set of vectors $v_1, v_2, ..., v_n$ is linearly independent if the set is not linearly dependent(if and only if we can only write $c_1v_1 + c_2v_2 + ... + c_nv_n = 0$ with all $c_i = 0$).
- 2 The set of vectors $v_1, v_2, ..., v_n$ is a basis for W, if $v_1, v_2, ..., v_n$ is linearly independent and spans W.

Before we show some examples, let us make the following observations:

Lemma. *Let V be a vector space with an inner product $(.,.)$. Assume that $v_1, v_2, ..., v_n$ is an orthogonal subset of vectors in V (thus $(v_i, v_j) = 0$ if $i \neq j$). If*

$v = c_1 v_1 + c_2 v_2 + ... + c_n v_n$, *then* $c_i = \frac{(v, v_i)}{\|v_i\|^2}, i = 1, ..., n$.

Proof. Assume that $v = c_1v_1 + c_2v_2 + \ldots + c_nv_n$. Take the inner product with v_1 in both sides of the equation. The LHS is (v, v_1). The RHS is:

$$\begin{aligned}(c_1v_1 + c_2v_2 + \ldots + c_nv_n, v_1) &= c_1(v_1, v_1) + c_2(v_2, v_1) \\ &\quad + \ldots + c_n(v_n, v_1) \\ &= c_1(v_1, v_1) \\ &= c_1 \left\|v_1\right\|^2 .\end{aligned}$$

Thus, $(v, v_1) = c_1 \left\|v_1\right\|^2$, or $c_1 = \frac{(v,v_1)}{\left\|v_1\right\|^2}$. Repeat this for $v_2, \ldots, v_n$. □

Corollary. *If the vectors $v_1, v_2, ..., v_n$ are orthogonal, then they are linearly independent.*

Example. *Let* $V = \mathbb{R}^2$. *The vectors* $(1,2)$ *and* $(-2,-4)$ *are linearly dependent because:*

$$(-2)(1,2) + 1(-2,-4) = 0.$$

The vectors $(1,2),(1,1)$ *are linearly independent. In fact,* $(1,2),(1,1)$ *is a basis for* $\mathbb{R}^2$.

Example(cont.)

Indeed, let $(x, y) \in \mathbb{R}^2$. Then,

$$\begin{aligned}(x, y) &= c_1(1, 2) + c_2(1, 1) \\ &= (c_1 + c_2, 2c_1 + c_2).\end{aligned}$$

Thus,

$$x = c_1 + c_2$$
$$y = 2c_1 + c_2.$$

Subtracting we get: $x - y = -c_1$, or $c_1 = y - x$.

Example(cont.)

Plugging this into the first equation we get:

$$c_2 = x - c_1 = x - (y - x) = 2x - y.$$

Thus, we can write any vector in $\mathbb{R}^2$ as a combination of those two. In particular, for $(0, 0)$ we get $c_1 = c_2 = 0$.

Example(cont.)

The vectors $(1,2),(-2,1)$ are orthogonal and hence linearly independent, and in fact a basis. Hence,

$$(x,y) = c_1(1,2) + c_2(-2,1).$$

Taking the inner product we get: $c_1 = \frac{x+2y}{\|v_1\|^2} = \frac{x+2y}{5}$ and $c_2 = \frac{-2x+y}{5}$.

Examples

Example. *Let* $V = \mathbb{R}^3$*. One vector can only generate a line, two vectors can at most span a plane, so we need at least three vectors to span* $\mathbb{R}^3$*. The vectors* $(1, 2, 1), (1, -1, 1)$ *are orthogonal but* *<u>not</u>* *a basis. In fact, those two vectors span the plane:*

$$W = (x, y, z) \in \mathbb{R}^3 : x - z = 0$$

(explain why).

On the other hand, the vectors: $(1, 2, 1), (1, -1, 1)$ *and* $(1, 0, -1)$ *are orthogonal, and hence a basis.*

Example(cont.)

We have, for example:

$$(4,3,1) = c_1(1,2,1) + c_2(1,-1,1) + c_3(1,0,-1)$$

with

$$c_1 = \frac{4+6+1}{1+4+1}$$
$$c_2 = \frac{4-3+1}{3}$$
$$c_3 = \frac{4-1}{2}$$

In general, we have:

$$(x,y,z) = \frac{x+2y+z}{6}(1,2,1) + \frac{x-y+z}{3}(1,-1,1) + \frac{x-z}{2}(1,0,-1)$$

Let us now discuss some spaces of functions:

a) Let $v_0(x) = 1, v_1(x) = x$ and $v_2(x) = x^2$. Then, v_o, v_1 and v_2 are linearly independent.

$$\begin{aligned} 0 &= c_0 v_0(x) + c_1 v_1(x) + c_2 v_2(x) \text{ for all } x \\ &= c_0 + c_1 x + c_2 x^2 \end{aligned}$$

Take $x = 0$, then we get $c_0 = 0$

Example(cont.)

Differentiate both sides to get:

$$0 = c_1 + 2c_2 x$$

Take again $x = 0$ to find $c_1 = 0$. Differentiate one more time to get that $c_2 = 0$. Notice that the span of v_0, v_1, v_2 is in the space of polynomials of degree ≤ 2. Hence, the functions $1, x, x^2$ form a basis for this space. Notice that the functions $1 + x, 1 - 2x, x^2$ are also a basis.

b) Are the functions $v_0(x) = x, v_1(x) = xe^x$ linearly independent/dependent on $\mathbb{R}$? Answer: No.

Assume that $0 = c_0 x + c_1 xe^x$. It does not help to put $x = 0$ now, but let us first differentiate both sides and get:

$$0 = c_0 + c_1 e^x + c_1 xe^x$$

Now, $x = 0$ gives:

$$0 = c_0 + c_1 \quad (1)$$

Differentiating again, we get:
$0 = c_1 e^x + c_1 e^x + c_1 xe^x$. Now, $x = 0$ gives $0 = 2c_1$, or $c_1 = 0$. Hence, (1) gives $c_1 = 0$.

c) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[\frac{1}{2},1)}$ are orthogonal and hence linearly independent. Let us show this directly. Assume that

$$0 = c_1\chi_{[0,\frac{1}{2})} + c_2\chi_{[\frac{1}{2},1)}.$$

An equation like this means that every x we put into the function on the RHS, the result is always 0.

Let us take $x = \frac{1}{4}$. Then:
$\chi_{[0,\frac{1}{2})}(\frac{1}{4}) = 1$, but $\chi_{[\frac{1}{2},1)}(\frac{1}{4}) = 0$. Hence,
$0 = c_1 \cdot 1 + c_2 \cdot 0$, or $c_1 = 0$. Taking $x = \frac{3}{4}$ shows that $c_2 = 0$.

d) The functions $\chi_{[0,\frac{1}{2})}, \chi_{[0,1)}$ are not orthogonal, but linearly independent.

$$0 = c_1\chi_{[0,1)} + c_2\chi_{[0,\frac{1}{2})}.$$

Take x so that $\chi_{[0,\frac{1}{2})}(x) = 0$ but $\chi_{[0,1)}(x) = 1$. Thus, any $x \in [0,1) \setminus [0,\frac{1}{2}) = [\frac{1}{2},1)$ will do the job. So, take $x = \frac{3}{4}$. Then, we see that:

$$0 = c_1 \cdot 1 + c_2 \cdot 0, \text{or} c_1 = 0.$$

Then take $x = \frac{1}{4}$ to see that $c_2 = 0$.

Chapter 7

Gram-Schmidt Orthogonalization

Gram-Schmidt Orthogonalization

The "best" basis we can have for a vector space is an orthogonal basis. That is because we can most easily find the coefficients that are needed to express a vector as a linear combination of the basis vectors $v_1, ..., v_n$:

$$v = \frac{(v, v_1)}{\|v_1\|^2} v_1 + ... + \frac{(v, v_n)}{\|v_n\|^2} v_n.$$

But usually we are not given an orthogonal basis. In this section we will show how to find an orthogonal basis starting from an arbitrary basis.

Procedure

Let us start with two linear independent vectors v_1 and v_2 (i.e. not on the same line through zero). Let $u_1 = v_1$. How can we find a vector u_2 which is perpendicular to u_1 and that the span of u_1 and u_2 is the same as the span of v_1 and v_2? We try to find a number $a \in \mathbb{R}$ such that:

$$u_2 = au_1 + v_2, \quad u_2 \perp u_1$$

Take the inner product with u_1 to get:

$$\begin{aligned} 0 = (u_2, u_1) &= a(u_1, u_1) + (v_2, u_1) \\ &= a \left\| u_1 \right\|^2 + (v_2, u_1) \end{aligned}$$

or

$$a = -\frac{(v_2, u_1)}{\left\| u_1 \right\|^2}$$

Procedure(cont.)

What if we have a third vector v_3? Then, after choosing u_1, u_2 as above, we would look for u_3 of the form:

$$u_3 = a_1 u_1 + a_2 u_2 + v_3$$

Take the inner product with u_1 to find:

$$0 = (u_3, u_1) = a_1 \|u_1\|^2 + (v_3, u_1)$$

or

$$a_1 = -\frac{(v_3, u_1)}{\|u_1\|^2}$$

or

$$a_3 = -\frac{(v_3, u_2)}{\|u_2\|^2}$$

Thus:

$$
\begin{aligned}
u_1 &= v_1 \\
u_2 &= v_2 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\
u_3 &= v_3 - \frac{(v_3, u_1)}{\|u_1\|^2} u_1 - \frac{(v_3, u_2)}{\|u_2\|^2} u_2
\end{aligned}
$$

Examples

Example. *Let* $v_1 = (1,1), v_2 = (2,-1)$. *Then, we set* $u_1 = (1,1)$ *and*

$$\begin{aligned} u_2 &= (2,-1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (2,-1) - \frac{2-1}{2}(1,1) \\ &= \frac{3}{2}(1,-1) \end{aligned}$$

Examples(cont.)

Example. *Let* $v_1 = (2, -1), v_2 = (0, 1)$. *Then, we set* $u_1 = (2, -1)$ *and*

$$\begin{aligned} u_2 &= (0,1) - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \\ &= (0,1) - \frac{-1}{5}(2,-1) \\ &= \frac{2}{5}(1,2) \end{aligned}$$

Note

We could have also started with $v_2 = (0,1)$, and get first basis vector to be $(0,1)$ and second vector to be:

$$(2,-1) - \frac{(2,-1)\cdot(0,1)}{\|(0,1)\|^2}(0,1) = (2,0)$$

Examples(cont.)

Example. *Let* $v_1 = (0,1,2), v_2 = (1,1,2), v_3 = (1,0,1)$*. Then, we set* $u_1 = (0,1,2)$ *and*

$$\begin{aligned} u_2 &= (1,1,2) - \frac{(0,1,2)\cdot(1,1,2)}{\|(0,1,2)\|^2}(0,1,2) \\ &= (1,1,2) - \frac{5}{2}(0,1,2) \\ &= (1,0,0) \end{aligned}$$

$$\begin{aligned} u_3 &= (1,0,1) - \frac{2}{5}(0,1,2) - (1,0,0) \\ &= \frac{1}{5}(0,-2,1) \end{aligned}$$

Examples(cont.)

Example. *Let* $v_0 = 1, v_1 = x, v_2 = x^2$*. Then,* v_0, v_1, v_2 *is a basis for the space of polynomials of degree* $\leq$*. But they are not orthogonal, so we start with* $u_0 = v_0$ *and* $u_1 = v_1 - \frac{(v_1, u_0)}{\|u_0\|^2} u_0$*. So we need to find:*

$$(v_1, u_0) = \int_0^1 x\, dx = [\frac{1}{2}x^2]_0^1 = \frac{1}{2}$$

$$\|u_0\|^2 = \int_0^1 1\, dx = [x]_0^1 = 1$$

Hence, $u_1 = x - \frac{1}{2}$. Then:

$$u_1 = v_2 - \frac{(v_2, u_0)}{\|u_0\|^2} u_0 - \frac{(v_2, u_1)}{\|u_1\|^2} u_1 \,.$$

We also find that:

$$\begin{aligned} (v_2, u_0) &= \int_0^1 x^2 \, dx = \frac{1}{3} \\ (v_2, u_1) &= \int_0^1 x^2 (x - \frac{1}{2}) \, dx = \frac{1}{12} \\ \|u_1\|^2 &= \int_0^1 (x - \frac{1}{2})^2 \, dx = \frac{1}{12}. \end{aligned}$$

Hence, $u_2 = x^2 - \frac{1}{3} - (x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$.

Theorem. *(Gram-Schmidt Orthogonalization)Let V be a vector space with inner product $(.,.)$. Let $v_1, ..., v_k$ be a linearly independent set in V. Then, there exists an orthogonal set $u_1, ..., u_k$ such that $(v_i, u_i) > 0$ and span$\{v_i, ..., v_i\} =$ span$\{u_1, ..., u_i\}$ for all $i = 1, ..., k$.*

Proof. See the book, p.$129 - 131$. □