Math 2057, Section 5

Material covered until Sept. 8

Section 14. 1: Functions of Several Variables.

- Definition of function of two variables. P. 887
- You should be able to determine the domain of simple functions like
  (1) Functions involving ln like $\ln(x^2-1)$;
  (2) Rational functions like $\dfrac{x^2+y-1}{x+y+3}$;
  (3) Functions involving $\sqrt{}$ like $\sqrt{1-x^2-y^2}$;
- Definition of the graph of a function, p. 890;
- **Very important** The section on level curves.
- Functions of three or more variables.

List of problems from Section 14.1: 1, 7, 11–31 every second odd problem, 37–45 every odd problem, 53 Section 14.2: Limit and Continuity.

This is a very important section, lot of what follows depends on taking limits. The main concept is the definition on page 902:

Definition 0.1. Let f be a function of two variables whose domain D includes points arbitrary close to (a,b). Then we say that the **limit of $\mathbf{f(x,y)}$ as $\mathbf{(x,y)}$ approaches $\mathbf{(a,b)}$** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every positive number $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon$$

whenever $(x,y) \in D$ and $0 < \sqrt{(x-a)^2+(y-b)^2} < \delta$.

Other notations are:

$$f(x,y) \underset{(x,y)\to(a,b)}{\longrightarrow} L$$

and

$$f(x,y) \to L, \qquad (x,y) \to (a,b)\,.$$

Note that the limit is unique and does not depend on **the curve we the points $\mathbf{(x,y)}$ get closer to the point $\mathbf{(a,b)}$.**
If you are working out the limit as $(x,y)\to(0,0)$ then it is often good to use polar coordinates $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The limit $(x,y)\to(0,0)$ corresponds then to $r \to 0$.
Example Determine of the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$$

exists.
Solution Using polar coordinates and the fact that
$$x^2 + y^2 = r^2$$
and
$$x^2 - y^2 = r^2(\cos^2(\theta) - \sin^2(\theta)) = r^2 \cos(2\theta)$$
we get
$$\frac{x^2 - y^2}{x^2 + y^2} = \frac{r^2 \cos(2\theta)}{r^2} = \cos(2\theta)$$
which depends on θ.
Another way to solve this problem is to take different lines. If we take the line $x = 0$, then the limit is -1. If we use the line $y = 0$, then the limit is 1.
Recall the definition of a **continuous** function on page 906:

Definition 0.2. A function of two variables is called **continuous** at the point (a, b) if
$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)\,.$$
The function f is continuous in its domain of definition D if f is continuous at all points in D.

Note a similar definition of a continuous function of more than two variables.

List of problems from Section 14.2: 3, 5–21 every odd problem, 27–33 every odd problem.

Section 14.3: Partial Derivatives.

In one variable/dimension we can only approach a point from left or from the right. We can therefore only study the rate of change in **one direction**. Recall the definition of the derivative:
$$f'(a) = Df(a) = \frac{df}{dx}(a) = \lim_{h\to 0} \frac{f(a + h) - f(a)}{h}$$
if the limit exists.
In two and more dimension we have **infinitely many** directions and curves to approach a point. The definition of the derivative (or rate of change) will therefore become more complicated. The first step is to only look at the rate of change in the x-direction (thus keeping the y fixed) or in the y-direction. This gives rise to the partial derivatives
$$f_x(a, b) = D_x f(a, b) = D_1 f(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h\to 0} \frac{f(a + h, b) - f(a, b)}{h}$$
and
$$f_y(a, b) = D_y f(a, b) = D_2 f(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h\to 0} \frac{f(a, b + h) - f(a, b)}{h}$$
if the limits exists.

- What is the geometric interpretation of the partial derivatives?
- How do you find the partial derivatives of functions that are defined implicitly?
- What are the higher order partial derivatives?

Make sure that you understand the definition and are able to differentiate simple functions. Then go over to more complicated functions.
Note also the definition for function of more than two variables on page 914.
Problems from Section 14.3: 5, 9, 13-31 odd, 35-53 odd, 57 and 61.

Section 14.1: Tangent Planes and Linear Approximations.

We started to discuss this section on Thursday, Sept. 8 and will finish it on Tuesday, Sept. 13. The main concepts here are:

- For function of one variable: Recall that the tangent line at the point (a, b) is given by $y - b = f'(a)(x - a)$.
- This can also be read **as linear approximation**

$$f(x) \sim b + f'(a)(x - a)\,.$$

- In two variables we need to replace **line** by **plane**. The equation of a plane, containing the point (x_0, y_0, z_0) in three dimensions is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0\,.$$

  If $C \neq 0$ then we can solve for $z - z_0$ and write

$$z - z_0 = a(x - x_0) + b(y - y_0)\,.$$

Definition 0.3. Suppose the function f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is given by

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)\,.$$

Homework due on Tuesday, Sept. 13

Section 14.2 problems 10 and 36 and from Section 14.3 problems 14 and 20.