Math 2057, Section 5

Material covered until Oct. 20

There is a test on Thursday, October 27. Material: Sections 14.6 - 15.2. There will be no question from section 15.1 except possibly a problem involving the average value, that we be discussed on Tuesday, Oct. 25.

Section 14. 6: Directional Derivatives and the Gradient Vector.

A direction is given by a vector $\mathbf{u}$ with $|\mathbf{u}| = 1$. That is, the vector has length one. The directional derivative of a function in the direction of $\mathbf{u}$ measure the rate of change in that direction.At a given point $\mathbf{x}$ it is denoted by $D_{\mathbf{v}}f(\mathbf{x})$ and given by

$$D_{\mathbf{u}}f(\mathbf{x}) = \lim_{t\to 0} \frac{f(\mathbf{x}+t\mathbf{u}) - f(\mathbf{x})}{t}$$

if the limit exists. If f is differentiable at the point $\mathbf{x} = (x, y)$ and $\mathbf{u} = (a, b) =< \cos(\theta), \sin(\theta) >$, then we have by the chain rule:

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b = \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta)\,.$$

The vector

$$\nabla f(x, y) =< \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} >$$

is called the gradient. We can then write

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}\,.$$

From this we get the following information:

(1) Denote the angle between ∇f and $\mathbf{u}$ by θ, then

$$D_{\mathbf{u}}f = |\nabla f| \cos\theta\,.$$

(2) The maximal rate of change is in the direction of ∇f and the rate of change in that direction is $|\nabla f|$.

(3) The minimal rate of change is in the direction of $-\nabla f$ and the rate of change in that direction is $-|\nabla f|$.

(4) The rate of change in the direction orthogonal to ∇f is zero.

Similar statements holds for functions of three or more variables.

Important:

(1) Make sure that the vector that you are using has length one. If necessary, you might have to divide by the length of the given vector to normalize the directional vector.

(2) Make sure you know how to find the directional vector if you are given a point $P(a, b)$ and asked to find the directional derivative at P in the direction to another point $Q(c, d)$. You find the vector $\mathbf{u} =< c, d > - < a, b >$, and then take $\mathbf{v} = \frac{1}{|\mathbf{u}|}\mathbf{u}$.

If the surface S is given by the equation $F(x, y, z) = 0$ where F is a differentiable function of three variables such that $\nabla F(x, y, z)$ is not the zero vector, then the equation of the tangent plane at a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ is given by

$$F_x(\mathbf{x}_0)(x - x_0) + F_y(\mathbf{x}_0)(y - y_0) + F_z(\mathbf{x}_0)(z - z_0) = 0\,.$$

If all the partial derivatives of F are non-zero, then the symmetricequation for the tangent plane is

$$\frac{x - x_0}{F_x(\mathbf{x}_0)} = \frac{y - y_0}{F_y(\mathbf{x}_0)} = \frac{z - z_0}{F_z(\mathbf{x}_0)}$$

Exercises from Section 14.6: 1, 5–25 odd, 29, 31, 39–43 odd

Section 14. 7: Maximum and Minimum Values.

Make sure you understand the difference between absolute maximum/minimum and local maximum/minimum . Recall also the definition of saddle point and critical point. A critical point is where either one of the partial derivatives does not exists or ∇f is the zero vector $\mathbf{0}$.
The test for local max/min is given by the following:

(1) Find the critical points. Thus, find the points where at least one of the partial derivatives does not exists. Solve the equations $f_x = f_y = 0$ (or in 3-dimensions $f_x = f_y = f_z = 0$).
(2) Then use the second derivative test: Suppose the partial derivatives of f are continuous around the point (a, b) and suppose that $f_x(a, b) = f_y(a, b) = 0$. Let

$$\begin{aligned} D &= f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2 \\ &= \det \begin{pmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{pmatrix} . \end{aligned}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(b) If $D < 0$ then (a, b) is a saddle point.

If D is a closed bounded set in $\mathbb{R}^n$ and f is a continuous function on D, then f attains an absolute value and a minimum value in D.
The question is then: How do we find those values?
Answer:

1) In the interior of D we find all the critical points as above.
2) We evaluate the function at the critical points.
3) We find the extreme values of f on the boundary (how to do that is discussed in the next section).
4) The largest of the numbers in (2) and (3) is the maximum value, and the smallest is the minimum value.

It is a common mistake only to find the critical points and not answer the actual question: What is the maximum and what is the minimum? So don't forget step 4!

Exercises from Section 14.7: 1–17 odd, 27–33 odd, 37, 41, 45

Section 14.8 Lagrange Multipliers

In step 3 in the last section the problem was to find the extreme values of f on the boundary of D. Often the boundary is given by an equation of the form $g(x, y) = 0$. Let us assume that $g_y \neq 0$.

Then we can solve this equation for $y = y(x)$ as a function of x. Then we have a function of one variable $h(x) = f(x, y(x))$. If $f(x, y(x))$ is an extreme value, we get by using the chain rule:

$$\begin{aligned} 0 &= h'(x) \\ &= f_x(x, y(x)) + f_y(x, y(x))y'(x) \\ &= f_x(x, y(x)) - f_y(x, y(x))\frac{g_x(x, y)}{g_y(x, y)} \end{aligned}$$

or, by multiplying through by $g_y(x, y)$:

$$\nabla f(x, y) \cdot < g_y(x, y), -g_x(x, y) >= 0\,.$$

Thus $\nabla f(x, y)$ is orthogonal to the vector $< g_y(x, y), -g_x(x, y) >$, or $\nabla f(x, y)$ and $\nabla g(x, y)$ are parallel. Thus, there exists a number λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y)\,.$$

Note, that in the end, we we need not to know the function $y(x)$!

Method of Lagrange Multipliers: Let f be a function of two or more variables. To find the maximum and minimum values of $f(\mathbf{x})$ subject to the constrain $g(\mathbf{x}) = k$ [assuming that these extreme values exists and $\nabla g \neq \mathbf{0}$ where $g(\mathbf{x}) = k$];

(1) Find all values of $\mathbf{x}$ and λ such that

$$\begin{aligned} \nabla f(\mathbf{x}) &= \lambda \nabla g(\mathbf{x}) \\ g(\mathbf{x}) &= k\,. \end{aligned}$$

(2) Evaluate f at all those points. The largest value is the maximum value, the smallest value is the minimum.

If we have two constrains $g(\mathbf{x}) = k$ and $h(\mathbf{x}) = c$, then, at the points where f takes the extreme values, we have that the vector $\nabla f(\mathbf{x})$ is in the plane spanned by $\nabla g(\mathbf{x})$ and $\nabla h(\mathbf{x})$. Therefore, there exists numbers λ and μ such that

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) + \mu \nabla h(\mathbf{x})\,. \qquad (1)$$

Method of Lagrange Multipliers, with two constrains: Let f be a function of two or more variables. To find the maximum and minimum values of $f(\mathbf{x})$ subject to the constrain $g(\mathbf{x}) = k$ and $g() = c$ [assuming that these extreme values exists and $\nabla g \neq \mathbf{0} \neq \nabla h$ where $g(\mathbf{x}) = k$ and $h() = c$];

(1) Find all values of $\mathbf{x}$, λ and μ that solve the equation (1) and such that $g(\mathbf{x}) = k$ and $h(\mathbf{x}) = c$.
(2) Evaluate f at all those points. The largest value is the maximum value, the smallest value is the minimum.

Exercises from Section 14.8: 1, 5–17 odd, 27–33 odd, 39 and 45

Section 15.1: Double integrals over Rectangles

We discussed shortly this section. Read it and understand the definition of the double integral. You should understand the basic idea behind the double integral. You will also need to know how we find the average value, p. 986. The average value of a function d of two variables defined on a rectangle R is

$$f_{\text{ave}} = \frac{1}{\text{A}} \iint_R f(x, y)\, dA$$

where A is the area of R.

Section 15.2 Iterated Integrals.

To evaluate the double integral of a continuous function over a rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \mid a \leq x \leq b,\ c \leq y \leq d\}$$

we use Fubini's Theorem:

Fubini's Theorem: If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx \\ &= \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy\, dx \end{aligned}$$

Exercises from Section 14.8: 1, 5–17 odd, 27–33 odd, 39 and 45

Problems from section 15.2: 1-29 every second odd numbered problem.