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Triple wavelet sets

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§0 Outline

1. Wavelets, multiresolution and wavelet sets

2. Coxeter groups and foldable figures

3. Fractal functions and "Coxeter wavelet sets"

4. Tripple wavelet sets.

§1 Wavelets, multiresolution and wavelet sets

- Let A be an $n \times n$ real matrix + expansive

$\Leftrightarrow$ All eigenvalues have $|\text{modulus}| > 1$

- Let $\Gamma \subseteq \mathbb{R}^n$ discrete set
- $\psi \in L^2(\mathbb{R}^n)$, $\psi \neq 0$

Then we can form the system

$$\{ \psi_{A^n, \gamma} = \psi_{n,\gamma} : x \longmapsto |\det A|^{n/2} \psi(A^T x - \gamma) \}$$

$n \in \mathbb{Z}$, $\gamma \in \Gamma$

[Can also replace $\underbrace{\{ A^n \mid n \in \mathbb{Z} \}}_{}^{= \mathcal{D}}$ by more general subsets of $GL(n, \mathbb{R})$]

Def. ψ is a (orthonormal) (A,Γ) wavelet if the system $\{\psi_{n,\gamma} \mid n \in \mathbb{Z}, \gamma \in \Gamma\}$ forms an (orthonormal) basis of $L^2(\mathbb{R}^n)$.

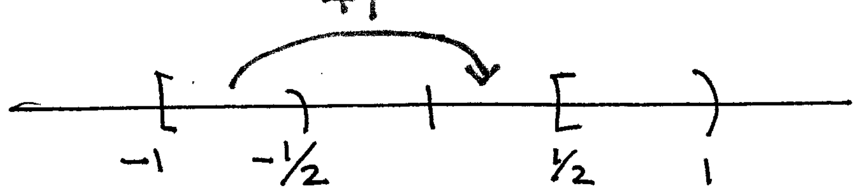
Def. A measurable set $\Omega \subseteq \mathbb{R}^n$ is a <u>wavelet set</u> if the function

$$\psi = \frac{1}{|\Omega|}\mathcal{F}^{-1}\chi_\Omega$$

is an orthogonal wavelet.

Ex $n = 1$, A = multiplication by 2 and $\Gamma = \mathbb{Z}$. Let $\Omega = [-1,-\tfrac{1}{2}) \cup [\tfrac{1}{2}, 1)$

Translation of the first half by 1 moves $\Omega \to [0,1)$



This implies that the exponential functions $e_n : x \mapsto e^{2\pi i n x}$ form an orthonormal basis of $L^2(\Omega)$.

$\Rightarrow$ Let $\psi = \mathcal{F}^{-1}\chi_{\Omega}$. Then

$$T_n \psi : x \mapsto \psi(x-n)$$

form an orthogonal basis ~~of~~

$$L^2_{\Omega}(\mathbb{R}) = \{ f \in L^2 \mid \operatorname{Supp} \hat{f} \subseteq [-1,-\tfrac{1}{2}) \cup [\tfrac{1}{2},1) \}$$

Next we note, that $\{A^n\Omega \mid n \in \mathbb{Z}\}$ forms a measurable tiling of $\mathbb{R}$, i.e.

$$\mathbb{R}' = \bigcup A^n \Omega$$

up to measure zero and $|A^n\Omega \cap A^m\Omega| = 0$ if $n \neq m$.

$$\Rightarrow \quad L^2(\mathbb{R}^n) \simeq \bigoplus L^2_{A^n\Omega}(\mathbb{R})$$

which shows that Ω is a wavelet set.

Thm 1) Assume that $\Gamma \subseteq \mathbb{R}^n$ is a co-compact discrete subgroup. Then Ω is a wavelet set $\iff$

- Ω is a multiplicative A tile i.e. $\mathbb{R}^n = \dot{\bigcup} A^n \Omega$ up to zero sets
- Ω is Γ tile for $\mathbb{R}^n$ i.e. $\{\Omega + \gamma\}_{\gamma \in \Gamma}$ is a measurable tiling of $\mathbb{R}^n$

Fuglede '74

$\iff$ $\{e_\gamma\}_{\gamma \in \Gamma}$ orthogonal basis for $L^2(\Omega)$.

$\rightsquigarrow$ analytic problem into geometric problem.

Thm Same conditions: Wavelet sets exists.

2 Coxeter groups and foldable figures

* So we have the following situation

→ M = manifold / set

→ μ a measure on M

→ Two groups acting unitarily on $L^2(M,\mu)$

by "dilation" + "translation"

$$\psi \to D_g T_\gamma \psi$$

→ wavelet : $\{D_g T_\gamma \psi\}$ orthonormal basis.

→ wavelet set : Fourier transform $\mathcal{F}$

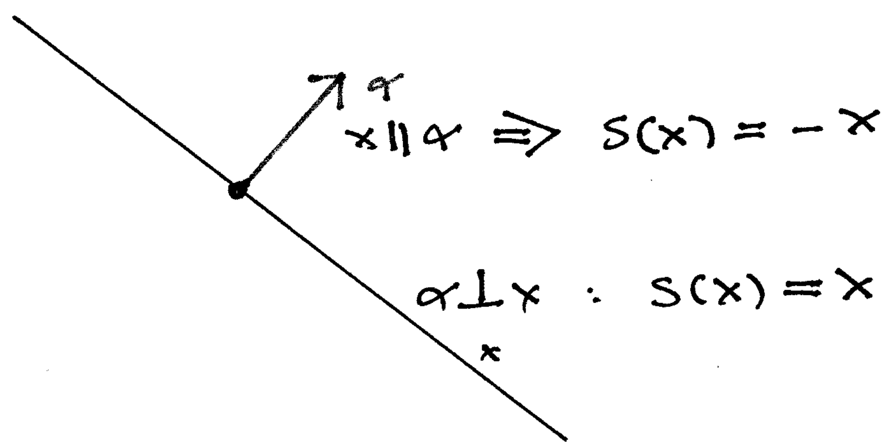
+ set Ω s.t. $\psi = \mathcal{F}^{-1}\chi_{\Omega}$ wavelet.

Why only usual translation and dilations ?

Massopust + Larson: Replace translation by T by "translation" by an affine Coxeter/Weyl group.

* Let $s : \mathbb{R}^n \to \mathbb{R}^n$ linear, then s is a reflexion if there exists a $\alpha \in \mathbb{R}^n \setminus \{0\}$ s.t.

$$s(x) = x - \frac{2(x,\alpha)}{|\alpha|^2}\alpha$$



* A finite reflexion group W is a finite subgroup of $O(n)$ such that there are finitely many reflexion such that $W = \langle s_1, \dots, s_\ell \rangle$.

~~O(n~~

* Classification : Irreducible root systems, simple Lie algebras, ...

$$\underbrace{A_n, B_n, C_n, D_n}_{\text{infinite sequence}}, E_6, E_7, E_8, F_4, G_2 .$$

* Next we can consider the corresponding affine Weyl-group $\widetilde{W}$ generated by reflections about affine hyperplanes

$$s_{r,k}(x) = x - \frac{2((x,r)-k)}{|r|^2}\, r$$

$$= s_r(x) + k\, \underbrace{r^\vee}_{\frac{2}{|r|^2} r}$$

<u>Thm</u> $\widetilde{W} \simeq W \ltimes \Gamma$

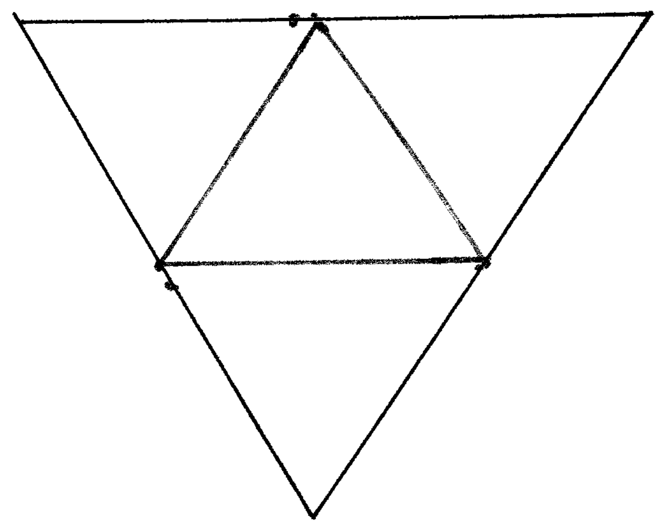
The finite reflection group

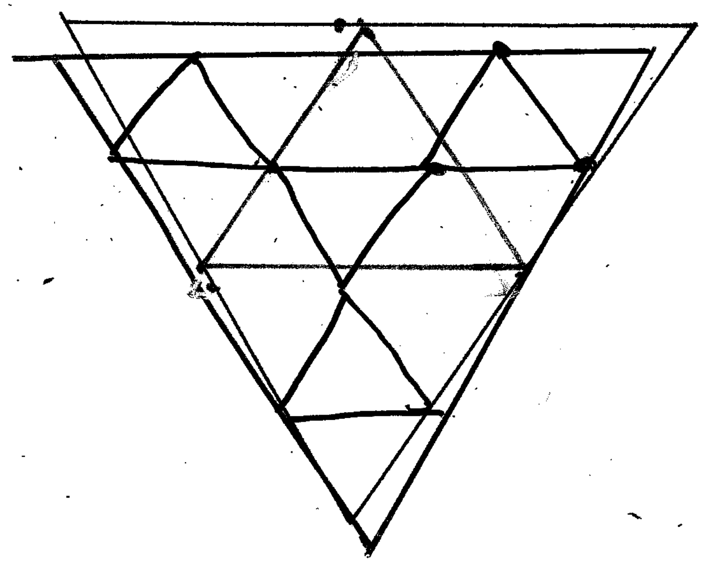
The lattice (= co-compact subgroup) generated by the $r^\vee$.

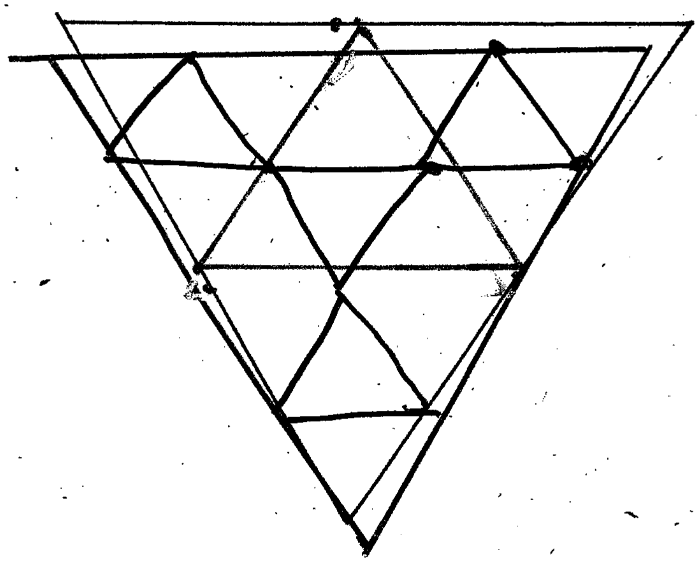
[if no subspace is pointwise fixed]

Now the Geometry

Def A compact connected subset $F \subseteq \mathbb{R}^n$ is a foldable figure if there exists a finite set S of affine hyperplanes that cuts F into finitely many congruent subfigurs $F_1, \ldots, F_m$ each similar to F, so that the reflection in any of the cutting hyperplanes in S bounding a subfigure $F_k \to$ some F_j.

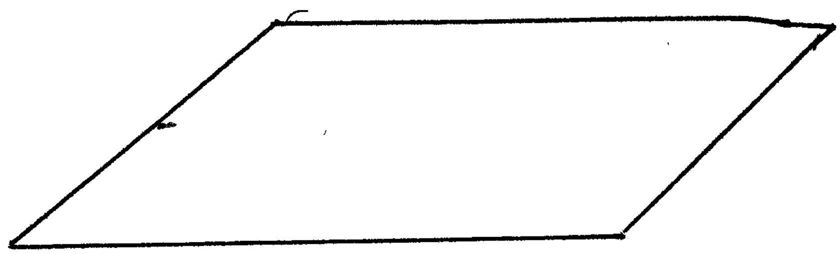




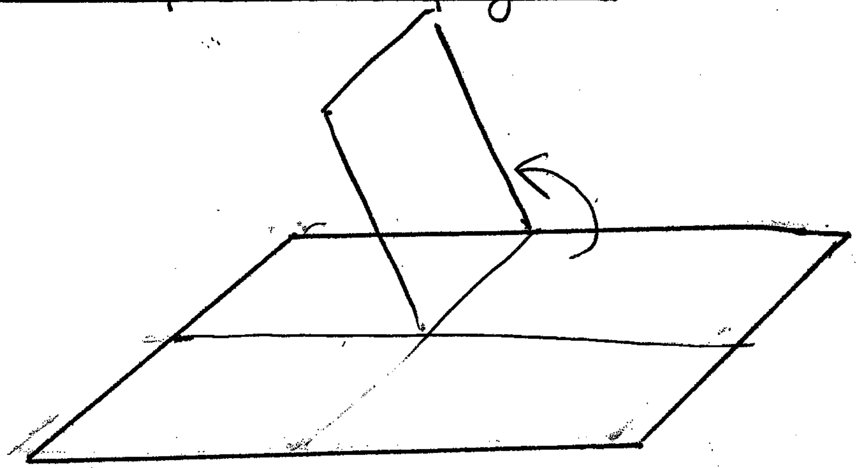


NOT A SPECTRAL SET!

Not a foldable figure



Not a foldable figure



But a spectral set

<u>Theorem</u> 1) The reflection group generated by the reflections about the bounding hyperplanes of a foldable figure F is an affine Weyl group $\tilde{W}$, which has F as a fundamental domain.

~~2) Fundam~~

2) Affine Weylgroups (some conditions)

$\Longleftrightarrow$ Foldable figures

is a bijection.

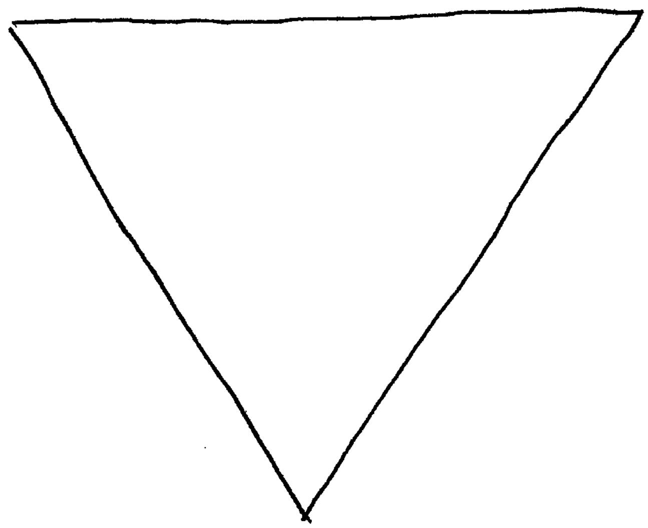
A expancive

$\rightsquigarrow$ Fractal construction : Fractal functions

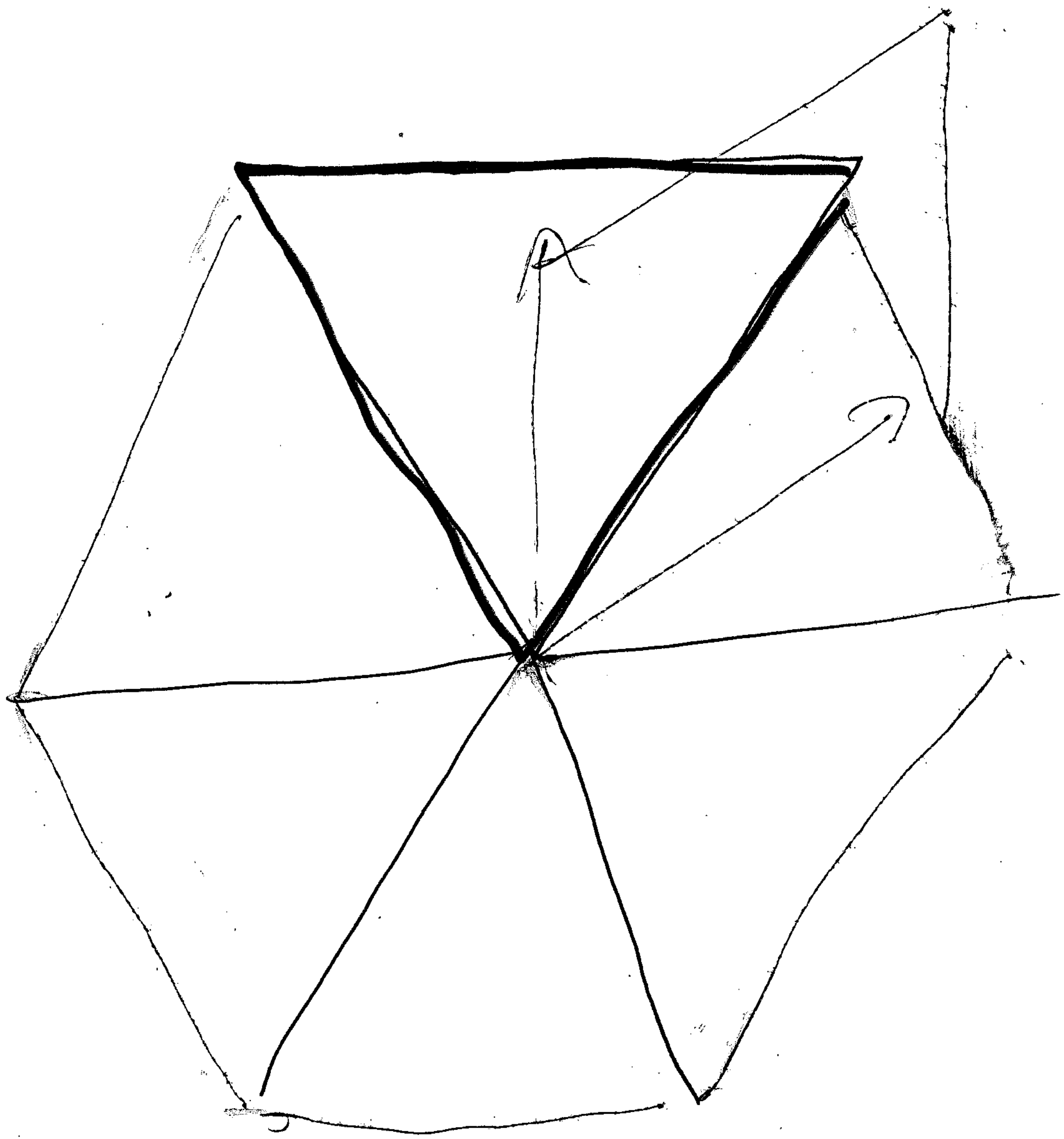
$\rightsquigarrow$ Coxeter Wavelet sets.

replacing translations T by translations by $\tilde{W}$!

Note



Note



Spectral set !

<u>Thm</u> (L+M) Given an affine Weyl group $\widetilde{W}$ with corresponding foldable figure F ($0 \in F^0$) and given an exansive matrix A, then there exists a set Ω such that

1) Ω is $\widetilde{W}$ congruent to F
(i.e. $\exists$ partition F_w of F s.t.
$\Omega = \dot{\bigcup\limits_{w}} w F_w$ ($\Rightarrow \Omega$ also a fundamental domain)

2) Ω is an A-tile.

<u>Q</u>: Given $\Gamma_1 \leq \mathbb{R}^n$ discrete co-compact, can we also find Ω such that it is (A, Γ_1) wavelet set?

In general NO, but in $\mathbb{R}^2$ the answere is always yes for T_1 a suitable multiple of the T in the Weyl group.