Chapter 6

Basic Representation Theory

6.1 Invariant integrals and measures

In this section we shall state the existence of a Haar integral on a locally compact Hausdorff group and use some general results from measure theory. For details of the definitions and proofs, see for example the book by Gerald B. Folland *A Course in Abstract Harmonic Analysis*. We then discuss action measures on homogeneous spaces and group action.

6.1.1 Measure theory, an overview

Let X be a locally compact Hausdorff space and let $C_c(X)$ be the space of continuous complex valued functions on X having compact support. We start by recalling some standard results regarding integrals and measures on X. First, the σ-algebra $\mathcal{B}_a$ of Baire sets is the smallest σ-algebra on X such that each $f \in C_c(X)$ is measurable. It is generated by the sets

$$K_{f,\alpha} = \{x \in X \mid f(x) \geq \alpha\}$$

$\alpha \in \mathbb{R}$ and $f \in C_c(X)$, real valued. Note that each of the sets $K_{f,\alpha}$, $\alpha > 0$, is a compact G_δ set. Furthermore, every compact G_δ is Baire measurable. A **Radon measure** is a measure $\mu : \mathcal{B}_a \to [0, \infty]$ such that $\mu(K) < \infty$ for every compact G_δ set K.

The σ-algebra generated by the open sets is called the **Borel sigma algebra** is denoted by $\mathcal{B}_o(X)$. Note that every compact G_δ-set is contained in $\mathcal{B}_o$ and hence $\mathcal{B}_a \subseteq \mathcal{B}_o$.

Moreover, every Radon measure μ extends to a unique Borel measure which we also call μ that has the following properties:

(a) $\mu(K) < \infty$ for all compact sets K

(b) If E is Borel and $\mu(E) < \infty$ or E is open, then

$$\mu(E) = \sup\{\mu(K) \mid K \text{ is compact}, K \subseteq E\}.$$

(c) $\mu(E) = \inf\{\mu(U) \mid U \text{ is open}, E \subseteq U\}$ for each Bore set E.

If X is second countable, then even more holds:

Lemma 6.1.1. *Let X be a second countable locally compact Hausdorff space. Then $\mathcal{B}_a = \mathcal{B}_o$.*

Proof. Let K be a compact subset of X. Since X is second countable and locally compact, there exists a countable base U_i, $i = 1, 2, \ldots$ for the topology of X such that each $\bar{U}_i$ is compact. Now consider the collection of those U_i such that $\bar{U}_i$ misses K. This is countable and if $y \notin K$, then there is an open set U having compact closure such that $y \in \bar{U} \subseteq X - K$. In particular, there must be a U_i whose closure is compact and disjoint from K that contains y. Thus $X - K$ is the union of those U_i (call the corresponding index set of i's, I_0) with $\bar{U}_i \subseteq X - K$. Thus $K = \cap_{i \in I_0}(G - \bar{U}_i)$ is a G_δ. So every compact set is Baire. Since every open set is a countable union of compact sets, every open set is Baire. Thus every Borel set is Baire. □

A Borel measure will be called **regular** if μ satisfies (a), (b), and (c). We will from now on always view a Radon measure as a regular Borel measure on X without comments. If μ is a Radon-measure and $1 \leq p \leq \infty$ then we denote by $L^p(X)$ the corresponding L^p-space. Then $C_c(X)$ is dense in $L^p(X)$ for $1 \leq p < \infty$.

A Radon on Borel measure defines a positive linear form on $C_c(X)$ by

$$I(f) = \int_X f(x)\, d\mu(x)\,.$$

A linear functional $I : C_c(X) \to \mathbb{C}$ is said to be **positive** if $I(f) \geq 0$ for all positive $f \in C_c(X)$, i.e., $f(x) \geq 0$ for all $x \in X$. Note, that this implies that $I(f) \in \mathbb{R}$ for all real valued functions $f \in C_c(X)$. The following theorem states, that we can also define Radon measure by using positive linear form on $C_c(X)$.

Theorem 6.1.2 (Riesz-Markov). *Let $I : C_c(X) \to \mathbb{C}$ be a positive linear functional. Then there is a unique Borel measure μ such that*

$$I_\mu(f) = I(f) = \int_X f(x)\, d\mu(x)$$

for all $f \in C_c(X)$. If μ is chosen regular, then μ is unique.

6.1.2 Invariance properties of measures

Assume from now on that X is locally compact and Hausdorff space X and that G is a locally compact Hausdorff topological group acting separately continuously on X.

For $f \in C_c(X)$ and $a \in G$ define

$$\lambda(a)f(x) = a \cdot f(x) = f(a^{-1}x)$$

and note that $\lambda(ab) = \lambda(a)\lambda(b)$. Then $a \cdot f \in C_c(X)$. If μ is a Radon measure on X we define $a \cdot \mu$ by

$$I_{a\cdot\mu}(f) = I(a^{-1}f)$$

Thus

$$\int_X f(x)\, d(a\cdot\mu)(x) = \int_X f(ax)\, d\mu(x)\,.$$

Then $a \cdot \mu$ is a Radon measure on X.

Definition 6.1.3. *Let X be a locally compact Hausdorff topological space and G a locally compact topological Hausdorff group acting separately continuously on X. Let μ be a Radon or Borel measure on X. The measure μ is said to be*

(a) ***invariant*** *if for all $a \in G$ we have $a \cdot \mu = \mu$,*

(b) ***relatively invariant*** *if there exists a continuous homomorphism $\Delta_X : G \to \mathbb{R}^+$ such that $a \cdot \mu = \Delta_X(a)^{-1}\mu$ for all $a \in G$, or*

$$\int_X f(ax)\, d\mu(x) = \Delta_X(a) \int_X f(x)\, d\mu(x)\,,$$

(c) ***strongly quasi-invariant*** *if there exists a continuous function $\Delta_X : G \times X \to \mathbb{R}^+$ such that*

$$\int_X f(x)\Delta_X(a^{-1}, x)\, d\mu(x) = \int_X f(x)\, d(a\cdot\mu)(x)$$

for all $f \in C_c(X)$.

(d) ***quasi-invariant*** *if for all $a \in G$ the measure $a\cdot\mu$ is absolutely continuous with respect to μ,*

Remark 6.1.4. Note that (a) ⇒ (b) ⇒ (c) ⇒ (d). But, if X is not assumed σ-finite then (d) does not even necessarily imply that the Radon-Nikodym derivative $\frac{d(a\cdot\mu)}{d\mu}(y)$ exists even if $a\cdot\mu$ is absolute continuous with respect to μ. A nice discussion on that can be found in the book by Folland *A Course in Abstract Harmonic Analysis*. Note that μ is strongly quasi-invariant if and only if the Radon-Nikodym derivative $\Delta_X(a, x) = \frac{d(a\cdot\mu)}{d\mu}(y)$ exists and is continuous.

Lemma 6.1.5. *Let μ be a Radon measure on X. Let E be measurable and $a \in G$. Then $a \cdot \mu(E) = \mu(a^{-1} \cdot E)$.*

Proof. Let χ_E denote the indicator function of E. Then

$$a \cdot \mu(E) = \int_X \chi_E(ax)\, d\mu = \int_X \chi_{a^{-1}\cdot E}\, d\mu\,.$$

□

Lemma 6.1.6. *Let μ be a positive relatively invariant Radon measure on X.*

1. *$\emptyset \neq U \subseteq X$ open. Then $\mu(U) > 0$.*

2. *If $f \in C_c(X)$ is non-zero and positive, then $I(f) > 0$.*

Proof. (1) Assume that $\mu(U) = 0$. Let $K \subseteq X$ be compact. Then there are $a_1, \dots, a_n \in G$ such that

$$K \subseteq \bigcup_{j=1}^{n} a_j^{-1} U \,.$$

Hence

$$\mu(K) \leq \mu(\bigcup_{j=1}^{n} a_j^{-1} U) \leq \sum_{j=1}^{n} \mu(a_j^{-1} U) = \sum_{j=1}^{n} a_j \cdot \mu(U) \,.$$

But $a_j \mu$ is absolutely continuous with respect to μ and hence $a_j \mu(U) = 0$ and hence K has measure zero. It follows that $I(f) = 0$ for all $f \in C_c(X)$ and hence $\mu = 0$.

(2) Let $x \in X$ be such that $f(x) > 0$. Let $U = \{y \in X \mid f(y) > f(x)/2\}$. Then U is open, non-empty and $\mu(U) < \infty$ as $\overline{U} \subseteq \operatorname{supp}(f)$. It follows that

$$0 < \mu(U) \leq \frac{f(x)}{2} I(f)$$

and hence $I(f) > 0$. □

We will from now on assume that X is completely regular, and note that this is always the case if $X = G/H$ for some closed subgroup H of G.

Lemma 6.1.7. *Assume that $g, h \in C(X)$ and that $I(fg) = I(fh)$. Then $g = h$.*

Proof. We can assume that $h = 0$ by replacing g by $g - h$. Assume that there exists $x \in X$ such that $g(x) \neq 0$. We can assume that $g(x) > 0$. Then $U = \{y \in X \mid g(y) > g(x)/2\}$ is open and $\overline{U}$ is compact and hence of finite measure. As X is completely regular there exists a continuous function f such that $f(x) = 1$ and $f = 0$ outside U. We can assume that $f(y) \geq 0$ for all y, by replacing f by $|f|$ if necessary. In particular, fg is non-zero and compactly supported and hence $I(fg) > 0$, a contradiction. □

The following lemma explains the use of the inverse in the definition of relatively invariant and strongly quasi-invariant measure:

Lemma 6.1.8. *Assume that μ is strongly quasi-invariant. Let $a, b \in G$. Then the following holds:*

(a) $\mu_X(e, x) = 1$ for all $x \in X$;

(b) (The cocycle relation) $\mu_X(ab, x) = \mu_X(a, bx)\mu_X(b, x)$;

(c) Let $a \in G$ and $f \in C_c(X)$, then

$$\int_X f(ax)\mu_X(a,x)\,d\mu(x) = \int_X f(x)\,d\mu(x)\,.$$

Proof. (a) This is obvious as $e \cdot \mu = \mu$.

(b) Let $a, b \in G$ and $f \in C_c(X)$, then

$$\begin{aligned}
\int_X f(x)\mu_X(ab,x)\,d\mu(x) &= \int_X f((ab)^{-1}\cdot x)\,d\mu(x) \\
&= \int_X \lambda(b)f(a^{-1}x)\,d\mu(x) \\
&= \int_X \lambda(b)f(x)\mu_X(a,x)\,d\mu(x) \\
&= \int_X f(b^{-1}x)\mu_X(a,b(b^{-1}x))\,d\mu(x) \\
&= \int_X f(x)\mu_X(a,bx)\mu_X(b,x)\,d\mu(x)\,.
\end{aligned}$$

The claim now follows from Lemma 6.1.7.

(c) Note, that by (a) and (b) we have that $\mu_X(a,a^{-1}x)\mu_X(a^{-1},x) = 1$. Hence

$$\begin{aligned}
\int_X f(ax)\mu_X(a,x)\,d\mu(x) &= \int f(ax)\mu_X(a,a^{-1}(ax))\,d\mu(x) \\
&= \int_X f(x)\mu_X(a,a^{-1}x)\mu_X(a^{-1},x)\,d\mu(x) \\
&= \int_X f(x)\,d\mu(x)\,.
\end{aligned}$$

□

6.2 The Haar measure on G

We consider now the special case where $G = X$. The following theorem is fundamental in extending harmonic analysis on locally Euclidean spaces to general locally compact Hausdorff groups. Recall $\lambda(a)f(x) = f(a^{-1}x)$ if f is a function on a group G.

Theorem 6.2.1 (Haar). *Let G be a locally compact Hausdorff group. Then there is a nonzero positive integral I on $C_c(G)$ such that $I(\lambda(a)f) = I(f)$ for each $f \in C_c(X)$ and $a \in G$. Moreover, if J is another such integral, there is a constant $c > 0$ such that $J = cI$.*

The resulting Radon measure $m_G = m$ is called a left Haar measure for the left invariant integral I. It has the property

$$m(aE) = m(E)$$

for all a and all Baire sets E. Moreover, it has a unique regular extension to the Borel sets having the same invariance property. This measure is also called a left Haar measure. We will sometimes write dx instead of $dm(x)$.

Recall from Lemma 6.1.1, that if G is second countable, then $\mathcal{B}_a = \mathcal{B}_o$. Hence, any left invariant measure which is finite on compact sets and nonzero is a left Haar measure. For example, Lebesgue measures on the line or on $\mathbb{R}^n$ are Haar meaures for these groups.

We now establish some facts for a left invariant Haar integral I and its corresponding Haar measure m. For $f \in C_c(G)$ and $a \in G$ let $\rho(a)f(x) = f(xa)$.

Proposition 6.2.2. *Let I be a left Haar integral for locally compact Hausdorff group G and let m be the corresponding left invariant Haar measure on the σ-algebra of Borel subsets of G.*

(a) $m(U) > 0$ *for every nonempty open set* U*.*

(b) Let $f \in C_c(G)$ *be positive and non-zero. Then* $I(f) > 0$.

(c) $\int f(gx)\,dm(x) = \int f(x)\,dm(x)$ *for each nonnegative Borel function* f*.*

(d) There exists a continuous homomorphism $\Delta = \Delta_G : G \to \mathbb{R}^+$ *such that for all* $g \in G$ *we have*

1. $\int_G f(xa)\,d\mu(x) = \Delta(a)^{-1} \int_G f(x)\,d\mu(x)$ *for all* $f \in L^1(G)$*,*
2. $m(Ea) = \Delta(a)m(E)$ *for all Borel sets* E*.*

(e) $\int f(x^{-1})\Delta(x^{-1})\,dm(x) = \int f(x)\,dm(x)$ *for all* $f \in L^1(G)$*.*

Proof. (a) and (b) follows from Lemma 6.1.6

For (c), note one can show using $m(gE) = m(E)$ for any $g \in G$ and and any Borel set E that $\int s(gx)\,dm(x) = \int s(x)\,dm(x)$ for simple nonegative Borel functions s. Now if $f \geq 0$ is Borel, $f(x) = \lim s_n(x)$ for all x where s_n is a pointwise increasing sequence of simple Borel functions. Thus the Monotone Convergence Theorem gives $\int f(gx)\,dm(x) = \int f(x)\,dm(x)$.

To see (d), let m be a left Haar measure. Define a linear positive functional on $C_c(X)$ by $J(f) = I(\rho(a)f)$. Then $J(\lambda(b)f) = I(\rho(a)\lambda(b)f) = I(\lambda(b)(\rho(a)f)) = I(\rho(a)f) = J(f)$ as $\rho(a)$ and $\lambda(b)$ commutes. It follows that J is left invariant. Denote the corresponding measure by m'. Then, by Theorem 6.2.1 there exists a $\Delta(a) > 0$ such that Then $m' = \Delta(a)^{-1}m$ which translates to

$$m(Ea) = \int_X \chi_E(xa^{-1})\,d\mu(x) = \Delta(a)\mu(E)\,.$$

Thus $\int_X s(xa)\,dm = \Delta(a)^{-1} \int_X s\,dm$ for all simple measurable functions, and hence

$$\int_G f(xa)\,d\mu(x) = \Delta(a)^{-1} \int_X f(x)\,d\mu(x)$$

for all $f \in L^1(G)$.

We already know $\Delta(a) > 0$ for all $a \in G$, and clearly $\Delta(e) = 1$. Since $\Delta(ab)m(E) = m(Eab) = \Delta(b)m(Ea) = \Delta(b)\Delta(a)m(E)$ for Borel sets E. Hence $\Delta(ab) = \Delta(a)\Delta(b)$ and Δ is a homomorphism.

To see continuity, by Lemma **??**, we only need to show Δ is continuous at e. Using Lemma **??**, we can find compact neighborhoods U and V of e and a function $f \in C_c(G)$ such that $f = 1$ on U, $0 \le f \le 1$, and $\text{supp} f \subseteq UV$. Recall $\rho(y)f(x) = f(xy)$ for $x, y \in G$. Let $\epsilon > 0$. By right uniform continuity, there is an open neighborhood W of e contained in U^{-1} such that $|f(xy) - f(x)| < \frac{\epsilon I(f)}{m(UVU)}$ for all x and for $y \in W$. Note the support of f and $\rho(y)f$ are both contained in UVU. Consequently, if $y \in W$, then

$$\begin{aligned}|I(\rho(y)f) - I(f)| &= |I(\rho(y)f - f)| \\ &= |\int (f(xy) - f(x))\, dm(x)| \\ &\le \int_{UVU} |f(xy) - f(x)|\, dm(x) \\ &\le \epsilon I(f).\end{aligned}$$

But $I(\rho(y)f) = \Delta(y^{-1})I(f)$. Consequently, $|\Delta(y^{-1}) - 1| \le \epsilon$ for $y \in W$. So Δ is continuous at e.

Finally we show (e). Define $J(f) = \int f(x^{-1})\Delta(x^{-1})\, dm(x)$ for $f \in C_c(G)$. Clearly J is positive. We show J is left invariant. Indeed, by (f),

$$\begin{aligned}J(\lambda(g)f) &= \int f(g^{-1}x^{-1})\Delta(x^{-1})\, dm(x) \\ &= \Delta(g)^{-1} \int f(g^{-1}(xg^{-1})^{-1})\Delta((xg^{-1})^{-1})\, dm(x) \\ &= \Delta(g)^{-1} \int f(x^{-1})\Delta(x^{-1})\Delta(g)\, dm(x) \\ &= J(f).\end{aligned}$$

Thus by uniqueness of left Haar integrals, there is a $c > 0$ with $J = cI$. Hence

$$\int f(x^{-1})\Delta(x^{-1})\, dm(x) = c \int f(x)\, dm(x)$$

for $f \in C_c(G)$. To see $c = 1$, note

$$\begin{aligned}\int f(x)\, dm(x) &= \frac{1}{c} \int f(x^{-1})\Delta(x^{-1})\, dm(x) \\ &= \frac{1}{c^2} \int f(x)\Delta(x)\Delta(x^{-1})\, dm(x) \\ &= \frac{1}{c^2} \int f(x)\, dm(x).\end{aligned}$$

Thus $c^2 \int f(x)\, dm(x) = \int f(x)\, dm(x)$. So $c = 1$.

□

The function Δ in Proposition 6.2.2 is called the modular function for the group G. If Δ is identically one, the group G is said to be **unimodular**. Thus a left Haar measure on G is right invariant if and only if G is unimodular.

Lemma 6.2.3. *Let $K \subseteq G$ be compact, then $\Delta_G|_K = 1$. In particular, if G is compact, then G is unimodular.*

Proof. As Δ is continuous, it follows that $\Delta(K)$ is a compact subgroup of $\mathbb{R}^+$ and hence equal to $\{1\}$. □

Example 1 ($GL(n,\mathbb{R})$). Recall that $\mathrm{GL}(n,\mathbb{R})$ can be viewed as an open dense subset of $M(n,\mathbb{R})$, and $M(n,\mathbb{R})$ can be naturally identified with $\mathbb{R}^{n^2}$ by stacking the n column vectors of $n\times n$ matrices into a column vector of length n^2. Define a Radon measure μ on $\mathrm{GL}(n,\mathbb{R})$ by

$$\begin{aligned}\int_{\mathrm{GL}(n,\mathbb{R})} f(X)\,d\mu(X) &:= \int_{\mathrm{GL}(n,\mathbb{R}))} f([x_{i,j}])|\det([x_{i,j}])|^{-n} dx_{1,1}\cdots dx_{1,n}\cdots dx_{2,1}\cdots dx_{n,n} \\ &= \int_{\mathrm{GL}(n,\mathbb{R})} f(X)\,|\det(X)|^{-n} d\lambda(X)\end{aligned}$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{n^2}$. Let $C, X \in \mathrm{GL}(n,\mathbb{R})$ and denote by $\mathbf{x}_1,\ldots,\mathbf{x}_n$ the column vectors of X. Then the matrix CX is given by

$$CX = (C\mathbf{x}_1,\ldots,C\mathbf{x}_n)$$

Hence left multiplication by C on $\mathrm{GL}(n,\mathbb{R})$ corresponds after stacking column vectors to the linear transformation on $\mathbb{R}^{n^2}$ having $n^2 \times n^2$ matrix

$$L_C = \begin{pmatrix} C & & 0 \\ & \ddots & \\ 0 & & C \end{pmatrix}.$$

This transformation has determinant $\det(C)^n$. It follows using Theorem **??** that

$$\begin{aligned}\int f(CX)\,d\mu(X) &= \int f(CX)\,|\det(X)|^{-n}\ d\lambda(X) \\ &= |\det C|^n \int f(CX)\,|\det(CX)|^{-n}\ d\lambda(X) \\ &= \int f(X)\,|\det(X)|^{-n}\ d\lambda(X)\end{aligned}$$

Hence μ is a left Haar measure.

6.3 Strongly quasi-invariant measures on G/H

In this section G denotes a locally compact Hausdorff topological group and H a closed subgroup of G. By Lemma **??**, the homogeneous space G/H with quotient topology is Hausdorff and the mapping

$$(g, xH) \mapsto gxH$$

is a continuous action of G on G/H. Moreover, since the mapping $\kappa : G \to G/H$ is an open mapping, the space G/H is locally compact. Our aim is to study quasi invariant measure on the homogeneous space $X = G/H$.

Lemma 6.3.1. *Let $K \subseteq X$ be compact. Then there exists a compact set $L \subseteq G$ such that $\kappa(L) = K$.*

Proof. For each $x \in \kappa^{-1}(K)$ let $U_x \in \mathcal{N}(x)$ be compact. Then $\kappa(U_x)$ is a compact neighborhood of xH. Hence, there exists finitely many $U_j = U_{x_j}$, $j = 1, \ldots, n$, such that $K \subseteq \cup_{j=1}^{n} \kappa(U_j)$. Let

$$L = \left(\bigcup_{j=1}^{n} U_j \right) \cap \kappa^{-1}(K)\,.$$

As $\kappa^{-1}(K)$ is closed, it follows that L is compact, and by construction we have $\kappa(L) = K$. □

Lemma 6.3.2. *Use dm_H to denote a left Haar measure on H. The mapping $f \mapsto f_H$ defined by $f_H(xH) = \int f(xh)\,dh$ maps $C_c(G)$ onto $C_c(G/H)$.*

Proof. Suppose $f \in C_c(G)$. To see f_H is continuous, let $\epsilon > 0$. Choose a compact neighborhood N of e. By left uniform continuity of f, we choose a neighborhood N' of e contained in N such that

$$|f(ny) - f(y)| \leq \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\mathrm{supp} f)} \text{ for all } y \in G \text{ for } n \in N'.$$

Let $n \in N'$. Then $f(nxh) = 0$ and $f(xh) = 0$ for $h \notin H \cap x^{-1}N^{-1}\mathrm{supp} f$. Hence

$$\begin{aligned}|f_H(nxH) - f_H(xH)| &\leq \int_H |f(nxh) - f(xh)|\,dh \\ &\leq \int_{H\cap x^{-1}N^{-1}\mathrm{supp} f} \frac{\epsilon}{m(H \cap x^{-1}N^{-1}\mathrm{supp}\, f)}\,dh \\ &= \epsilon.\end{aligned}$$

So f_H is continuous.

Moreover, if $\kappa : G \to G/H$ is the mapping $g \mapsto gH$, we have $\mathrm{supp}\,(f_H) \subseteq \kappa(\mathrm{supp}\,(f))$. Hence $f_H \in C_c(G/H)$ for $f \in C_c(G)$.

Now suppose $F \in C_c(G/H)$. Let K be the support of F. Let $L \subseteq G$ be compact such that $\kappa(L) =$. By Lemma **??**, there exists $\varphi \in C_c(G)$, $0 \leq \varphi \leq 1$ such that $\varphi = 1$ on K. Then $\varphi_H > 0$ on K. Define $f(x) = \frac{\varphi(x)}{\varphi_H(xH)} F(xH)$ on K

and $f(x) = 0$ outside K. Then f is continuous, see Exercise **??**. Finally

$$\begin{aligned} f_H(xH) &= \int_H f(xh)\,dh \\ &= \int \frac{\varphi(x)}{\varphi_H(xH)} F(xH)\,dh \\ &= F(xH) \end{aligned}$$

for $x \in G$. □

Definition 6.3.3. *Let H be a closed subgroup of G. A continuous function $\rho : G \to \mathbb{R}^+$ is called a **rho-function** if*

$$\rho(xh) = \rho(x)\frac{\Delta_H(h)}{\Delta_G(h)} \tag{6.3.1}$$

for all $x \in G$ and $h \in H$.

Note, that if ρ is a rho-function, then, for a fixed a, the function $x \mapsto \rho(ax)/\rho(x)$ is H-right invariant and hence

$$G \times G/H \ni (a, xH) \mapsto \frac{\rho(ax)}{\rho(x)} \in \mathbb{C} \tag{6.3.2}$$

is well defined and continuous.

Theorem 6.3.4 (Strongly quasi-invariant measures). *Assume G is locally compact and Hausdorff and H is a closed subgroup. Let μ be a positive Radon measure on X. Then μ is strongly quasi-invariant if and only if there exists a rho-function ρ such that*

$$I(f) = \int_X f_H(x)\,d\mu(x) = \int_G f(x)\rho(x)\,dm_G(x)$$

for all $f \in C_c(G)$. In this case the Radon-Nikodym derivative is given by

$$\mu_X(a, x) = \frac{\rho(ax)}{\rho(x)}\,. \tag{6.3.3}$$

Proof. Assume that the rho-function ρ is given. We first show that I is well defined. Indeed, suppose $f_H = 0$. Choose $g \in C_c(G)$ with $g_H = 1$ on the compact set $\kappa(\operatorname{supp} f)$. Suppose $g(x)f(xh) \neq 0$. Then $x \in \operatorname{supp}(f)$ and $xh \in \operatorname{supp}(f)$. Hence $(x, h) \in \operatorname{supp}(g) \times (\operatorname{supp}(f)\operatorname{supp}(g)^{-1}$ which is compact. Hence

$$G \times H \ni (x, h) \mapsto F(x, h) = \rho(x)g(x)f(xh) \in \mathbb{C}$$

is in $C_c(G \times H)$. It follows that F is integrable on $G \times H$ and vanishes outside a set of finite measure. Thus we are allowed to use Fubini's Theorem in the following argument:

$$\begin{aligned}
0 &= \int\int \rho(x)g(x)f(xh)\,dm_H(h)\,dm_G(x) \\
&= \int\int \rho(x)g(x)f(xh)\,dm_G(x)\,dm_H(h) \\
&= \int\int \rho(xh^{-1})g(xh^{-1})f(x)\Delta_G(h^{-1})\,dm_G(x)\,dm_H(h) \\
&= \int\int \Delta_G(h)\Delta_H(h^{-1})\rho(xh)g(xh)f(x)\,dm_H(h)\,dm_G(x) \\
&= \int\int \rho(x)g(xh)f(x)\,dm_H(h)\,dm_G(x) \\
&= \int \rho(x)f(x)g_H(xH)\,dm_G(x) \\
&= \int \rho(x)f(x)\,dm_G(x).
\end{aligned}$$

So I is well defined. Let $a \in G$. Then

$$\begin{aligned}
\int f_H(axH)\,d\mu(xH) &= \int f(ax)\rho(x)\,dm_G(x) \\
&= \int f(x)\rho(a^{-1}x)\,dm_G(x) \\
&= \int f(x)\frac{\rho(a^{-1}x)}{\rho(x)}\,\rho(x)\,dm_G(x) \\
&= \int \frac{\rho(a^{-1}x)}{\rho(x)} f_H(xH)\,d\mu(xH).
\end{aligned}$$

In particular we get that $\mu_X(a,x) = \frac{\rho(ax)}{\rho(x)}$.

Assume now that μ is a strongly quasi-invariant measure on X with Radon-Nikodym derivative μ_X. Define $\rho(a) = \mu_X(a, eH)$. Then ρ is well defined because μ_X is continuous. We need to show that (6.3.1) holds. From the cocycle relation in Lemma 6.1.8 it follows for $a \in G$ and $h \in H$, that $\rho(ah) = \mu_X(a, hH)\mu_X(h, eH) = \rho(a)\rho(h)$. Define $I : C_c(G) \to \mathbb{C}$ by

$$J(f) = \int_X \int_H f(gh)\rho(gh)^{-1}\,dm_H(h)d\mu(gH) = \int_X (f/\rho)_H(x)\,d\mu(x)\,.$$

Let $a \in G$, then, by using the cocycle relation, we get:

$$\begin{aligned} J(\lambda(a)f) &= \int_X (f/\lambda(a^{-1})\rho)_H(a^{-1}x)\, d\mu(x) \\ &= \int_X (f/\lambda(a^{-1})\rho)_H(x)\mu(a,x)\, d\mu(x) \\ &= \int_X \int_H f(gh)\rho(agh)^{-1}\mu(a,gH)\, dm_H(h)d\mu(gH) \\ &= \int_X \int_H f(xh)\rho(gh)^{-1}\mu(a,gH)^{-1}\mu(a,gH)\, dm_H(h)d\mu(aH) \\ &= J(f) \end{aligned}$$

Hence, there exists a $c > 0$ such that

$$J(f) = \int_X \int_H f(gh)\rho(gh)^{-1}\, dm_H(h)d\mu(gH) = c\int_G f(g)\, dm_G(g)\,.$$

Replacing f by $f\rho$ it follows that

$$\int_X f_H(x)\, d\mu(x) = c\int_G f(g)\rho(g)\, dg$$

for all $f \in C_c(G)$. From this we get

$$\begin{aligned} c\int_G f(gh)\rho(g)\, dm_G(g) &= c\Delta_G(h)^{-1}\int_G f(g)\rho(gh^{-1})\, dm_G(g) \\ &= c\Delta_G(h)^{-1}\rho(h^{-1})\int_G f(g)\rho(g)\, dm_G(g) \\ &= \Delta_G(h)^{-1}\rho(h^{-1})\int_X f_H(x)\, d\mu(x)\,. \end{aligned}$$

On the other hand

$$\begin{aligned} c\int_G f(gh)\rho(g)\, dm_G(g) &= c\int_G (\rho(h)f)(g)\rho(g)\, dm_G(g) \\ &= c\int_G (\rho(h)f)(g)\rho(g)\, dm_G(g) \\ &= \int_X \int_H f(gkh)\, dm_H(k)d\mu(gH) \\ &= \Delta_H(h^{-1})\int_X f_H(x)\, d\mu(x)\,. \end{aligned}$$

By taking $f \in C_c(G)$ such that $\int_X f_H(x)\, d\mu(x) \neq 0$, and replacing h by h^{-1} it follows that

$$\rho(h) = \Delta_H(h)/\Delta_G(h)$$

as was to be shown. □

Corollary 6.3.5. *There exists an invariant measure on G/H if and only if $\Delta_H = \Delta_G|H$.*

Corollary 6.3.6. *If $K \subseteq G$ is a compact subgroup, then there exists a invariant measure on G/K.*

Proof. This follows from Lemma 6.2.3 and Corollary 6.3.5. □

Theorem 6.3.7 (Relatively invariant measures). *Let μ be a positive Radon measure on X. Then μ is strongly quasi-invariant if and only if there exists a homomorphism $\rho : G \to \mathbb{R}^+$ such that $\rho(h) = \Delta_H(h)/\Delta_G(h)$ for all $h \in H$ and such that*

$$\int_X f_H(x)\, d\mu(x) = \int_G f(x)\rho(x)\, dm_G(x) \tag{6.3.4}$$

for all $f \in C_c(G)$.

Proof. If ρ is as in the theorem, then ρ is rho-function. By Theorem 6.3.4 there exists a quasi invariant measure such that (6.3.4) holds. By (6.3.3) it follows that $\mu_X(a,x) = \rho(a)$ is independent of x, and hence the measure is relatively invariant.

Assume now that μ is relatively invariant and define ρ by

$$I(\lambda(a)f) = \int_X f(ax)\, d\mu(x) = \rho(a) \int_X f(x)\, d\mu(x)\,.$$

As $\lambda(ab)f = \lambda(a)[\lambda(b)f]$ it follows that ρ is a homomorphism. Let $f \in C_c(X)$ be such that $I(f) = 1$. then $\rho(a) = I(\lambda(a)f)$. Let $\epsilon > 0$ be given. We can assume that $\epsilon < 1$. As f is uniformly continuous, there exists a $V \in \mathcal{N}(e)$ such that $|f(ax) - f(x)| < 1$. Let $U \in \mathcal{N}(e)$, $U \subseteq V$ be such that

$$|f(ax) - f(x)| < \frac{\epsilon}{\mu(V^{-1}\mathrm{supp}(f))}\,.$$

Then for $a \in U$ we get

$$|\rho(a) - 1| \le \int_X |f(ax) - f(x)|\, d\mu \le \epsilon\,.$$

□

Recall a Hausdorff space X is **paracompact** if every open covering has an open locally finite refinement. Examples include both metrizable spaces and compact Hausdorff spaces. Since second countable locally compact Hausdorff spaces are metrizable, homogeneous spaces G/H are paracompact if G is a second countable locally compact Hausdorff group and H is a closed subgroup. However, for groups more is true. Indeed, every locally compact Hausdorff group G is paracompact and so are their quotients G/H for closed subgroups H; see Exercises **??.??** and **??.??**.

Lemma 6.3.8. *Let G be a locally compact Hausdorff group with closed subgroup H. Then there is a positive continuous function $\phi(x)$ with $\phi(xh) = \phi(x)\frac{\Delta_H(h)}{\Delta_G(h)}$ for all $x \in G$ and $h \in H$.*

Proof. We use G/H is paracompact. Since G/H is locally compact, we can find a locally finite cover $\mathcal{U}$ of G/H consisting of open sets U with each $\bar{U}$ compact. Now consider the collection of all open sets V with $\bar{V} \subseteq U$ for some $U \in \mathcal{U}$. This is an open cover of G/H. Hence it has a locally finite refinement $\mathcal{V}$ of open sets covering G/H.

For each open set U in $\mathcal{U}$, set $W_U = \bigcup\{V \in \mathcal{V} \mid \bar{V} \subseteq U\}$. The sets W_U for $U \in \mathcal{V}$ form an open cover for G/H. Since $W_U \subseteq U$, $\bar{W}_U \subseteq \bar{U}$. Thus each $\bar{W}_U$ is compact. We finally note $\bar{W}_U \subseteq U$. Indeed, let $x \in \bar{W}_U$. Choose a neighborhood N_x of x that meets only finitely many V in $\mathcal{V}$. In particular, $\{V \in \mathcal{V} \mid \bar{V} \subseteq U, N_x \cap V \neq \emptyset\}$ consists of finitely many sets $V_1, V_2, \ldots, V_n$. This implies $x \in \overline{V_1 \cup V_2 \cup \cdots \cup V_n} = \cup_{k=1}^n \bar{V}_k \subseteq U$. So $\bar{W}_U \subseteq U$.

Now by Lemma **??**, one can find for each $U \in \mathcal{U}$ a continuous function F_U of compact support inside U and satisfying $0 \leq F_U \leq 1$ and $F_U = 1$ on W_U. By Lemma 6.3.2 and its proof, there are nonnegative $f_U \in C_c(G)$ such that

$$F_U(xH) = \int f_U(xh)\,dh$$

for all xH. Define $f = \sum_{U \in \mathcal{U}} f_U$. Note if $x \in G$, there is an open set N in G/H with $xH \in N$ and N meets only finitely many U. Since F_U has compact support in U, this implies f_U is zero on $\kappa^{-1}(N)$ for all but finitely many U. Thus f is defined, nonnegative, and continuous. Moreover, for each x, $f_U(xh) > 0$ for some U and h; and the set of h with $f_U(xh) > 0$ is precompact.

Now set $\delta(h) = \frac{\Delta_H(h)}{\Delta_G(h)}$. Define $\phi(x) = \int_H f(xh)\,\delta(h^{-1})\,dh$. Note ϕ is continuous for

$$\int_H f(yh)\delta(h^{-1})\,dh = \sum_{U \cap N \neq \emptyset} \int f_U(yh)\delta(h^{-1})\,dh \text{ when } y \in N.$$

Moreover, $\phi(xh') = \int f(xh'h)\delta(h^{-1})\,dh = \int \psi(xh)\delta(h^{-1}h')\,dh = \delta(h')\phi(x)$. $\square$