Solutions to the homework, due Oct. 25

14.7 – 10: Find the local maximum and minimum values and saddle point(s) of the function

$$f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$$

Solution:

Step 1: To find the critical points we solve $\nabla f(x,y) = \vec{0}$. Thus
$f_x(x,y) = 6x^2 + 10x + y^2 = 0 = 2xy + 2y = f_y(x,y)$.
In particular

$$y(x+1) = 0$$

Thus $x = -1$ or $y = 0$. Inserting $x = -1$ in the first equation gives $6 - 10 + y^2 = 0$ or $y^2 = 4$, $y = \pm 2$
Inserting $y = 0$ into the first equation gives $2x(3x+5) = 0$ or $x = 0$, $x = -5/3$. The critical points are:
$(0,0)$, $(-1, 2)$, $(-1, -2)$, $(-\frac{5}{3}, 0)$.

Step 2: Now we use the 2th derivatives test.
$f_{xx}(x,y) = 12x + 10 = 2(6x+5)$
negative at $(-1, \pm 2)$, $(-\frac{5}{3}, 0)$, positive at $(0,0)$
$f_{yy}(x,y) = 2(x+1)$
$f_{xy}(x,y) = 2y$

$$D = 4\left[(6x+5)(x+1) - y^2\right] \begin{cases} > 0 & (0,0) \text{ and } (-\frac{5}{3}, 0) \\ < 0 & (-1, \pm 2) \end{cases}$$

Thus $f(-\frac{5}{3}, 0) = \frac{125}{27}$ is a local maximum.
$f(0,0) = 0$ is a local minimum and $(-1, \pm 2)$ are saddle points.

14.7-38: Find the point on the plane $x-y+z=4$ that is closest to the point $(1,2,3)$.

Solution: The point is $\frac{1}{3}(5,4,11)$.

First solution method: The distance of (x,y,z) to $(1,2,3)$ is given by $\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$. This function takes it minimal value if and only if $(x-1)^2+(y-2)^2+(z-3)^2$ takes its minimal value. We therefore use (given that $z=4-x+y$

$$f(x,y)=(x-1)^2+(y-2)^2+(1-x+y)^2.$$

$$f_x(x,y) = 2(x-1)+2(1-x+y)=2(2x-y-2) = 0$$
$$f_y(x,y) = 2(y-2+1-x+y)=2(-x+2y-1) = 0.$$

We get the two equations:

$$2x-y=2$$
$$-x+2y=1$$

Multiplying the first equation by 2 and add, gives

$$3x=5 \quad \text{or} \quad x=5/3$$

Inserting that into the first equation gives

$$y=2x-2=\frac{10}{3}-2=4/3$$

Using that $z=4-x+y$ gives

$$z=4-5/3+4/3=11/3$$

2^{th} - method: We can also use the Lagrange Multipliers method from 14.8. Here

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

$$g(x,y,z) = x - y + z = 4$$

Thus:

$$2(x-1) = \lambda$$
$$2(y-2) = -\lambda$$
$$2(z-3) = \lambda$$

or: $y = -\frac{\lambda}{2} + 2 = -x+1+2 = -x+3$

$z = \frac{\lambda}{2} + 3 = x - 1 + 3 = x + 2.$

Inserting this into the equation for the plane gives

$x - y + z = x + x - 3 + x + 2 = 3x - 1 = 4$ or $x = 5/3$

$y = -x + 3 = -\frac{5}{3} + 3 = 4/3$ and $z = x + 2 = \frac{5}{3} + 2 = 11/3$

14.8-8 Use the Lagrange multipliers to find the maximum and minimum values of $f(x,y,z) = 8x - 4z$ subject to $g(x,y,z) = x^2 + 10y^2 + z^2 = 5$

Solution maximum $= 20$, minum $= -20$

The equation $\nabla f = \lambda \nabla g$ gives

$$8 = 2\lambda x$$
$$0 = 20\lambda y$$
$$-4 = 2\lambda z$$

It is clear, that $\lambda \neq 0$. Hence $y = 0$, $x = \frac{4}{\lambda}$, $z = -\frac{2}{\lambda} = -\frac{x}{2}$

Inserting this information into $g = 5$ gives:

$$(2z)^2 + 0^2 + z^2 = 5z^2 = 5 \quad \text{or} \quad z = \pm 1$$

We get the points $(2,0,1), (-2,0,-1)$. Inserting into f gives

$$f(2,0,1) = 16+4 = 20$$
$$f(-2,0,-1) = -16-4 = -20.$$

14.8-16: Find the maximum value and minimum value of $f(x,y,z) = 3x - y - 3z$ with the constrains $g(x,y,z) = x + y - z = 0$ and $h(x,y,z) = x^2 + 2z^2 = 1$.

Solution max $= \frac{12}{\sqrt{6}} = 2\sqrt{6}$, min $= -\frac{12}{\sqrt{6}} = -2\sqrt{6}$

There are real numbers λ and μ such that

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

This gives

$$3 = \lambda + 2\mu x$$
$$-1 = \lambda$$
$$-3 = -\lambda + 4\mu z$$

Inserting $\lambda = -1$ into the other two equations, gives

$$4 = 2\mu x \quad \text{or} \quad 2 = \mu x$$
$$-4 = 4\mu z \quad \text{or} \quad -1 = \mu z$$

As $\mu \neq 0$ we get $x = -2z$. Inserting this into $h = 1$ gives $4z^2 + 2z^2 = 1$ or $z = \pm 1/\sqrt{6}$, $x = \mp 2/\sqrt{6}$, $y = z - x = 3z = \pm 3/\sqrt{6}$. Inserting into f gives

$$f\left(-\frac{1}{\sqrt{6}}, \frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{6}{\sqrt{6}} - \frac{3}{\sqrt{6}} - \frac{3}{\sqrt{6}} = -\frac{12}{\sqrt{6}}$$

$$f\left(\frac{1}{\sqrt{6}}, \frac{-3}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = \frac{12}{\sqrt{6}}$$