Solution to the homework due 3-31-06

5.5-5: (a) The sequence $\sum_{k=1}^{\infty} e^{-kx}$ converges uniformly on $[1, \infty)$. For that note that on this interval we have

$$e^{-kx} \leq e^{-k} = (1/e)^k$$

and the series $\sum_{k=1}^{\infty}(1/e)^k$ converges. The claim follows then by the Weierstrass M-test.
(b) $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$ converges uniformly on $\mathbb{R}$ because

$$\left|\frac{\sin(kx)}{k^3}\right| \leq \frac{1}{k^3}$$

and the series $\sum_{k=1}^{\infty} 1/k^3$ converges. The claim follows then by the Weierstrass M-test.
(c) The series $\sum_{k=1}^{\infty} \sin^k(x)$ converges uniformly on $[0, \pi/4]$ because on this interval $|\sin^k(x)| \leq (1/\sqrt{2})^k$ and the series $\sum_{k=1}^{\infty}(1/\sqrt{2})^k$ converges.
(d) No, the series $\sum_{k=1}^{\infty} \tan^k x$ does not even converge at $x = \pi/4$.

5.5-6: Let $f(x) = \sum_{k=1}^{\infty} \sin(kx)/k^3$ on $\mathbb{R}$. Show that $f \in C^1(\mathbb{R})$ and find an expression for $f'(x)$ in terms of an infinite series.

Solution: We use Theorem 5.5.1 with $f_k(x) = \sin(kx)/k^3$. Then

$$f_k'(x) = \frac{\cos(kx)}{k^2}$$

and, as the series $\sum_{k=1}^{\infty} k^{-2}$ converges, $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly (and absolutely) on $\mathbb{R}$. As the series for $f(x)$ converges for all points, it follows that

$$f'(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$$

by Theorem 5.5.1-c and f' is continuous by 5.5.1-a.

5.6.-2 (Look for the statement in the book, p. 140)

a) We have

$$\frac{1}{t} = \frac{1}{1-(1-t)} = \sum_{k=0}^{\infty}(1-t)^k = \sum_{k=0}^{\infty}(-1)^k(t-1)^k$$

as long as $|t-1| < 1$ of $t \in (0, 2)$. If $t = 2$, then we have

$$\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + \ldots$$

which does not converge (note that $s_n = 1$ if n is even and $s_n = 0$ if n is odd.). For $t = 0$ we have

$$\sum_{k=0}^{\infty} 1 = 1 + 1 + \ldots$$

which does not converge.

(b) We have

$$\begin{aligned}
\log x &= \int_1^x \frac{1}{t}\, dt \\
&= \int_1^x \sum_{k=0}^{\infty} (-1)^k (t-1)^k \, dt \\
&= \sum_{k=0}^{\infty} (-1)^k \int_1^x (t-1)^k \, dt \\
&= \sum_{k=0}^{\infty} (-1)^k \frac{1}{k+1}\, x^{k+1} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\, x^k \,.
\end{aligned}$$

Here we need the uniform convergence of the series

$$\sum_{k=0}^{\infty} (-1)^k (t-1)^k$$

on any closed subinterval $[\alpha, \beta] \subset (0, 2)$ (see Theorem 5.6.1) to be able to apply Theorem 5.5.1-b.