LECTUR 10

THE 2-D HAAR WAVELET TRANSFORM.

1. Tensorproduct of functions

In 1-dimension we used step functions of the form

$$p = \sum_{j=0}^{2^n-1} s_j \chi_{\left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)}$$

to approximate "arbitrary" functions. Thus we made the following steps.

(1) Deside up on the resolution, and divide the interval $[0,1)$ into subintervalls $I_j = \left[\frac{j}{2^n}, \frac{j+1}{2^n}\right)$.

(2) Take the orthogonal projection $f \to p = \sum_{j=0}^{2^n-1} s_j \chi_{I_j}$

(3) Replace p by the vector $[s_0, \dots, s_{2^n-1})$

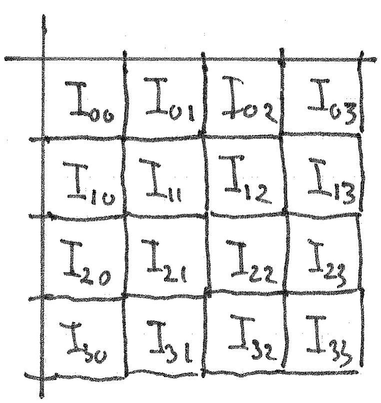
(4) Apply the fast Haar wavelet transform to the sequence $[s_0, \dots, s_{2^n-1})$.

In 2-D we have now two directions, x and y. We will therefore have to divede the square

$$I = [0,1) \times [0,1) = \{(x,y) \in \mathbb{R}^2 : 0 \leq x < 1,\ 0 \leq y < 1\}$$

into smaller squares

$$I_{ij}^m = \left\{(x,y) : \frac{i}{2^n} \leq x < \frac{i+1}{2^n},\ \frac{j}{2^n} \leq y < \frac{j+1}{2^n}\right\}$$



Then we approximate functions of two variables by functions that are constants on each of the squares I_{ij}, i.e.,

$$f \to p = \sum s_{ij} \chi_{I_{ij}}$$

We start by discussing special functions of two variables.

<u>Definition</u> Let $p, q : [0,1) \to \mathbb{R}$. Then $p \otimes q$ is the function on $[0,1) \times [0,1)$ given by

$$p \otimes q(x,y) = p(x)\, q(y)$$

<u>Example</u> Let $A, B \subseteq [0,1)$. Then $\chi_A \otimes \chi_B = \chi_{A \times B}$

<u>Proof.</u> We have $\chi_A \otimes \chi_B(x,y) = \chi_A(x)\chi_B(y)$. Thus $\chi_A \otimes \chi_B(x,y) = 1$ if and only if $x \in A$ and $y \in B$. But that is the same as saying that $(x,y) \in A \times B$.

<u>Example</u> Let

$$\varphi_{ij} = \chi_{[\frac{j}{2^i}, \frac{j+1}{2^i})}$$

and

$$\psi_{i,j} = \chi_{[\frac{j}{2^{i+1}}, \frac{j+1}{2^{i+1}})} - \chi_{[\frac{j+2}{2^{i+1}}, \frac{j+3}{2^{i+1}})}.$$

Recall that φ_{ij} is used to express the averages, and ψ_{ij} is used to express the details. We have

$$\varphi_{ni} \otimes \varphi_{nj} = \varphi_{I^n_{ij}}.$$

Can be used to expers the "average values" on I^n_{ij} and

$$\varphi_{nj} \otimes \psi_{nj}, \quad \psi_{ni} \otimes \varphi_{nj}, \quad \psi_{ni} \otimes \psi_{nj}$$

can be used to express details going from one column to the next, one row to the nex, and going diagonally.

Let f be a step function in $[0,1)\times[0,1)$. In stead of associating to f an array, we have to usa a matrix. Thus if

$$f(x,y) = \sum_{i,j=1}^{2^n-1} s_{ij}\, \chi_{I^n_{ij}}$$

Then f corresponds to the matrix

$$f \longleftrightarrow \begin{bmatrix} s_{00} & s_{01} & s_{02} & \cdots & s_{0,2^n-1} \\ s_{10} & s_{11} & s_{12} & \cdots & s_{1,2^n-1} \\ \vdots & & & & \\ s_{2^n-1,0} & s_{2^n-1,1} & & & s_{2^n-1,2^n-1} \end{bmatrix}$$

Notice, that the first indice (the on corresponding to the x-variable) indicates the row and the second indices corresponds to the columns.

<u>Example</u> 2×2-matrix.

$$\chi_{[0,\frac{1}{2})} \otimes \chi_{[0,1)} \longleftrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

To see this write

$\chi_{[0,1/2)}$ \\ $\chi_{[0,1)}$	1	1
1	$1\cdot 1$	$1\cdot 1$
0	$0\cdot 1$	$0\cdot 1$

multiply

<u>Example</u> Find the 2×2 matrix corresponding to the function

$$\chi_{[0,\frac{1}{2})} \otimes \left(\chi_{[0,\frac{1}{2})} - \chi_{[\frac{1}{2},1)}\right) + 2\chi_{[\frac{1}{2},1)} \otimes \chi_{[0,1)}.$$

Solution

$$\begin{array}{c|c|c} x_{[0,\frac{1}{2})} \backslash x_{[0,\frac{1}{2})} - x_{[\frac{1}{2},1)} & 1 & -1 \\ \hline 1 & 1 & -1 \\ \hline 0 & 0 & 0 \end{array} \quad + 2 \cdot \quad \begin{array}{c|c|c} x_{[\frac{1}{2},1)} \backslash x_{[0,1)} & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \end{array} \quad = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

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THE 2-D HAAR WAVELET TRANSFORM

In two dimension we apply the Haar wavelet transfor on the column <u>and</u> on the rows.

<u>Example</u> Suppose that the function f is represented by the 2×2 matrix

$$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix}$$

Then each row represent a one dimensional vector of length 2: $[4,2]$ and $[0,6]$. Applying the 1-D wavelet transform to this vector gives $[4,2] \to [3,1]$ and $[0,6] \to [3,-3]$.

Thus

$$\begin{pmatrix} 4 & 2 \\ 0 & 6 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 3 & -3 \end{pmatrix}$$

We now apply the 1-D transform to the rows $\begin{bmatrix} 3 \\ 3 \end{bmatrix} \to \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix} \to \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. This results in the new matrix

$$\begin{pmatrix} 3 & 1 \\ 3 & -3 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & -3 \end{pmatrix}.$$

Notice that the element 3 represent the average value of all the values:

$$\frac{4+2+0+6}{4} = \frac{12}{4} = 3.$$

To understand the other matrix elements let us do this for a general matrix

$$\begin{bmatrix} s_{00} & s_{01} \\ s_{10} & s_{11} \end{bmatrix} \xrightarrow{1^{th}\text{-step}} \begin{bmatrix} \frac{s_{00}+s_{01}}{2} & \frac{s_{00}-s_{01}}{2} \\ \frac{s_{10}+s_{11}}{2} & \frac{s_{10}-s_{11}}{2} \end{bmatrix}$$

We apply the 1-D transform to each column and get

$$\xrightarrow{2^{th}\text{-step}} \begin{bmatrix} \frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} + \frac{s_{10}+s_{11}}{2}\right] & \frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} + \frac{s_{10}-s_{11}}{2}\right] \\ \frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} - \frac{s_{10}+s_{11}}{2}\right] & \frac{1}{2}\left[\frac{s_{00}-s_{01}}{2} - \frac{s_{10}-s_{11}}{2}\right] \end{bmatrix}$$

- The numer $\frac{1}{2}\left[\frac{s_{00}+s_{01}}{2} + \frac{s_{10}+s_{11}}{2}\right] = \frac{s_{00}+s_{01}+s_{10}+s_{11}}{4}$

  Is the average value.

- The numbe at place $(0,1)$ is

$$\underbrace{\frac{1}{2}\Bigg[\underbrace{\frac{s_{00}-s_{01}}{2}}_{\substack{\text{detail}\\ \text{moving from}\\ (0,0)\to(0,1)}} + \underbrace{\frac{s_{10}-s_{11}}{2}}_{\substack{\text{detail}\\ \text{moving from}\\ (1,0)\to(1,1)}}\Bigg]}_{\text{average chang moving from first column to the second column.}}$$

- The number located at $(1,0)$ is

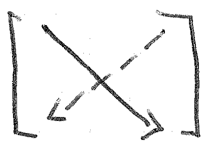
$$\frac{1}{2}\left[\frac{S_{00}+S_{01}}{2} - \frac{S_{10}+S_{11}}{2}\right] = \frac{1}{2}\left[\frac{S_{00}-S_{10}}{2} + \frac{S_{01}-S_{11}}{2}\right]$$

is the average change going from row 1 to row 2 (or the details for the average)

- Finally the last number at place $(1,1)$ is

$$\frac{1}{2}\left[\frac{S_{00}-S_{01}}{2} - \frac{S_{10}-S_{11}}{2}\right] = \frac{1}{2}\left[\frac{S_{00}+S_{11}}{2} - \frac{S_{01}+S_{10}}{2}\right]$$

represents the average changes moving along the diagonals



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The method for bigger matrices is the same:

$$\begin{pmatrix} 5 & 7 & -1 & -5 \\ 1 & -1 & -3 & -3 \\ 7 & 3 & 2 & 2 \\ 3 & -1 & 0 & 0 \end{pmatrix} \xrightarrow[\text{rows}]{\text{1-D on}} \begin{pmatrix} 6 & -3 & -1 & 2 \\ 0 & -3 & 1 & 0 \\ 5 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{Columns}]{\text{1-D on}} \begin{pmatrix} 3 & -3 & 0 & 1 \\ 3 & 1 & 2 & 0 \\ 3 & 0 & -1 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix}$$

Now notice that the 2×2-matrix $\begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix}$ located in the upper left corner consist of <u>average</u> values. We apply the 2-D wavelet transform again to this matrix and get

$$\begin{pmatrix} 3 & -3 \\ 3 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}.$$

The final answer is then

THE INVERSE 2-D WAVELET TRANSFORM

Given a 2x2-matrix or a 4x4-matrix knowing that this matrix is a result of a 2-D Haar wavelet transform, how can we reconstruct the original matrix?

Let us start with a 2×2-matrix. Recall that the 2-D Haar wavelet transform consist of 2-steps:

1^{th}-step $\left(\begin{matrix}\longrightarrow\\\longrightarrow\end{matrix}\right)$ 1-D Haar wavelet transform on the rows

2^{th}-step $(\downarrow\ \downarrow)$ 1-D Haar wavelet transform on the colums.

Therefore, to invert this we need to:

<u>1^{th}-step:</u> Apply the inverse 1-D transform to the columns.

<u>2^{th}-step:</u> Apply the inverse 1-D transform to the rows.

<u>Example</u> Given the matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$. Find the initial matrix.

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \xrightarrow[\substack{\text{1-D inverse}\\\text{transform on}\\\text{columns}}]{} \begin{pmatrix} 2-1 & 1+0 \\ 2-(-1) & 1-0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix}$$

$$\xrightarrow[\substack{\text{1-D inverse}\\\text{on rows}}]{} \begin{pmatrix} 1+1 & 1-1 \\ 3+1 & 3-1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 2 \end{pmatrix}.$$

Now look at 4×4-matrices. Recall, that the last step was to apply the 2-D transform on the upper left corner (2×2) matrix. Here is an overview of the steps.

<u>1the-step</u> 1-D on the rows

<u>2th-step</u> 1-D on the columns

<u>3th-step</u> Take the 2×2 matrix in the left upper corner and apply step 1 and two to this matrix keeping all other numbers the same.

For the inverse we need to invert each of those steps. Thus.

<u>1th-step</u> Pick the 2×2-matrix in the left upper corner. Apply the inverse 1-D transform on the columns.

<u>2th-step</u>: Apply the 1-D inverse transform on the rows in the 2×2-matrix from step 1.

<u>3th-step</u> We have now a new 4×4-matrix. Apply the inverse 1-D transform to the columns.

<u>4th-step</u>. Apply the <u>inverse</u> 1-D transform to the rows.

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<u>Example</u>: Given the matrix $\begin{pmatrix} 2 & 1 & 0 & 3 \\ 1 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix}$, find the initial matrix.

<u>Solution</u> We start by the 2×2-matrix $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ and apply the inverse transform to this matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow[\text{columns}]{} \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow[\text{rows}]{} \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}.$$

We have now the new matrix

$$\begin{pmatrix} 4 & 2 & 0 & 3 \\ 2 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix}$$

and have to apply the inverse row/column transform to this matrix.

$$\begin{pmatrix} 4 & 2 & 0 & 3 \\ 2 & 0 & 2 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & 3 \end{pmatrix} \xrightarrow[\text{Columns}]{\text{inverse on}} \begin{pmatrix} 5 & 4 & -1 & 5 \\ 3 & 0 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 2 & 1 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow[\text{rows}]{\text{inverse on}} \begin{pmatrix} 4 & 6 & 9 & -1 \\ 4 & 2 & 1 & -1 \\ 5 & -1 & 1 & -3 \\ 3 & 1 & -3 & 5 \end{pmatrix}.$$

So the final answer is

$$\begin{pmatrix} 4 & 6 & 9 & -1 \\ 4 & 2 & 1 & -1 \\ 5 & -1 & 1 & -3 \\ 3 & 1 & -3 & 5 \end{pmatrix}$$

LECTURE 11

COMPLEX NUMBERS AND COMPLEX VECTOR SPACES

Up to now we have only considered vector spaces over the field of real numbers. For the Fourier transform we need vector spaces where we can multiply vectors by complex numbers.

If $x \in \mathbb{R}$, then we know that $x^2 \geq 0$ and $x^2 = 0$ only if $x = 0$. Therefore there is no <u>real</u> number x such that $x^2 = -1$, or : There is no <u>real</u> solution to the equation

$$x^2 + 1 = 0.$$

More generally let us look at the equation

$$x^2 + bx + c = 0$$

By completing the square we get

$$x = -\frac{b}{2} \pm \frac{1}{2}\sqrt{b^2 - 4c}. \qquad (*)$$

There are now 3 possibilities:

(1) $b^2 - 4c > 0$. Then equation $(*)$ gives two solutions

$$x = -\frac{b}{2} + \frac{1}{2}\sqrt{b^2-4c}$$

and

$$x = -\frac{b}{2} - \frac{1}{2}\sqrt{b^2-4c}$$

(2) $b^2 - 4c = 0$. Then we have one solution

$$x = -\frac{b}{2}$$

(3) $b^2 - 4c < 0$. Then there is <u>no</u> real solution to the quadratic equation $x^2 + bx + c = 0$.

We introduce now a new numbe $i = \sqrt{-1}$ such that

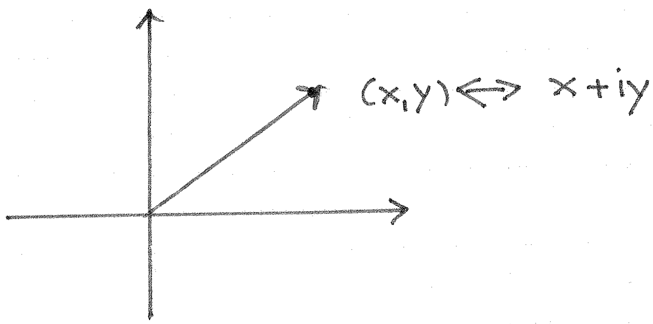
$$i^2 = -1$$

The <u>complex numbers</u> are all expressions of the form

$$z = x + yi \quad , \quad x, y \in \mathbb{R}.$$

The set of complex numbers is denoted by $\mathbb{C}$.

We say that $x = \mathrm{Re}(z)$ is the real part of z and $y = \mathrm{Im}(z)$ is the imaginar part of z. Recall that the set of real numbers can be thought of as a <u>line</u>, the <u>real line</u>. To picture the set of complex numbers we use the plane. A vector $\vec{V} = (x, y)$ corresponds to the complex number $z = x + iy$.



The addition of two complex numbers $z = x + iy$ and $w = s + it$ then corresponds to the addition of the corresponding vectors. Thus

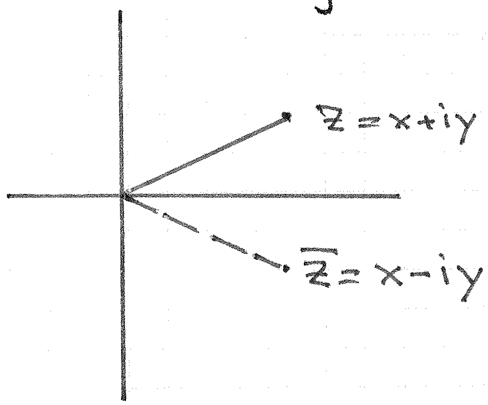
$$(x + iy) + (s + it) = [x + s] + i[y + t].$$

To find out what the product of z and w is, we just act as if all the ruls that we are so familiar with, still holds and use that $i^2 = -1$. Thus

$$\begin{aligned}(x+iy)\cdot(s+it) &= x\cdot s + x\cdot it + iy\cdot s + (iy)it \\ &= x\cdot s + i^2 yt + i(xt + ys) \\ &= (xs - yt) + i(xt + ys)\end{aligned}$$

Before we use this to find the inverse - or reciprocal - of $z = x+iy$, we need to introduce the complex conjugate:

$$\overline{x+iy} = x - iy.$$



Thus, complex conjugation corresponds to a reflection around the x-axis.

Now multiply z by $\bar{z}$:

$$z \cdot \bar{z} = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2.$$

Thus $\sqrt{z \cdot \bar{z}} =: |z|$ is the length of the vector (x, y).
It is therefore natural to call $|z| = \sqrt{z \cdot \bar{z}}$ the length, or absolute value, of the complex number z.

<u>Lemma</u> Let $z = x+iy$ be a complex number with $|z| \neq 0$. Then

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x-iy}{x^2+y^2}.$$

<u>Proof.</u> We have $z \cdot \frac{x-iy}{x^2+y^2} = \frac{(x+iy)(x-iy)}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$ ▨

<u>Examples</u>

(1) $(2+3i) + (5-2i) = 7+i$

(2) $(2-3i) \cdot (1+i) = (2+3) + (-3+2)i = 5 - i.$

(3) $\frac{1}{2+i} = \frac{2-i}{5} = \frac{2}{5} - \frac{1}{5} i.$

(4) $\frac{2+3i}{1+5i} = \frac{(2+3i)(1-5i)}{26} = \frac{(2+15)+(3-10)i}{26}$

$= \frac{17-7i}{26}.$

THE COMPLEX EXPONENTIAL FUNCTION

The idea behind the Fourier transform is to represent a function (or signal) in the frequency domain using the complex exponential function.

Definition Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers. We say that z_n converges to the complex number w if for all $\varepsilon > 0$ there exists a $N \in \mathbb{N}$ such that

$$|z_n - w| < \varepsilon$$

for all $n \geq N$. We write $z_n \to w$ or $\lim_{n\to\infty} z_n = w$, if z_n converges to w.

Let $z \in \mathbb{C}$ and define $z_n = \sum_{k=0}^{n} \frac{z^k}{k!}$. Then it can be shown that the sequence $\{z_n\}$ converges. We denote the limit by

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{z^k}{k!}.$$

It can be shown, that

$$e^{z+w} = e^z e^w.$$

In particular $\frac{1}{e^z} = e^{-z}$.

Theorem Let $z = x + iy \in \mathbb{C}$. Then

$$e^z = e^x(\cos(y) + i\sin(y))$$

(The Euler formula).

This was proved in class.

Examples

(1) $e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$

(2) $e^{\pi i/2} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

(3) $e^{2+\frac{\pi i}{4}} = e^2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$$= \frac{e^2}{\sqrt{2}}(1+i)$$

Lemma Let $z = x+iy$. Then

$$\overline{e^z} = e^{\bar{z}}$$

Proof. $\overline{e^z} = \overline{e^x(\cos y + i\sin y)}$

$$= e^x(\cos y - i\sin y)$$

$$= e^{x-iy} = e^{\bar{z}}.$$

COMPLEX VALUED FUNCTIONS

Let $F: I \to \mathbb{C}$, $I \subseteq \mathbb{R}$ an interval, be a function. Then we can write

$$F(t) = f(t) + ig(t)$$

where $f, g: I \to \mathbb{R}$. The function F is continuous if and only if f and g are both continuous. In that case we have

$$\lim_{t \to t_0} F(t) = \lim_{t \to t_0} f(t) + \left(\lim_{t \to t_0} g(t)\right) i$$

$$= f(t_0) + g(t_0) i$$

$$= F(t_0).$$

We integrade and differentiate F by integrating $f(t)$ and $g(t)$ (differentiating $f(t)$ and $g(t)$). Thus

$$\int_a^b F(t)dt = \int_a^b f(t)dt + \left(\int_a^b g(t)dt\right) i$$

$$\frac{dF}{dt}(t) = \frac{df}{dt}(t) + \frac{dg}{dt}(t)\, i .$$

<u>Ex</u> 1) $\int_0^1 2t + 3t^2 i \, dt = t^2\Big]_0^1 + t^3\Big]_0^1 i$

$$= 1 + i .$$

2) $\int_0^{2\pi} e^{(1+i)t} dt = \int_0^{2\pi} e^t \cos t + e^t \sin t \, dt$

We have

$$\int_0^{2\pi} e^t \cos t \, dt = \cos t \, e^t\Big]_0^{2\pi} + \int_0^{2\pi} \sin t \, e^t dt$$

$$= e^{2\pi} - 1 + \sin t e^t\Big]_0^{2\pi} - \int_0^{2\pi} \cos t e^t dt$$

$$= e^{2\pi} - 1 - \int_0^{2\pi} e^t \cos t dt$$

Thus $\int_0^{2\pi} e^t \cos t \, dt = (e^{2\pi} - 1)/2$

Similarly

$$\int_0^{2\pi} \sin t \, e^t dt = \sin t \, e^t\Big]_0^{2\pi} - \int_0^{2\pi} \cos t \, e^t dt$$

$$= -\frac{1}{2}(e^{2\pi} - 1).$$

Thus $\int_0^{2\pi} e^{(1+i)t} dt = \frac{1}{2}(e^{2\pi} - 1)[1 - i]$.

But we can also have used the rule

$$\int e^{at} dt = \frac{1}{a} e^{at} + C$$

where a is any complex number, $a \neq 0$. Then :

$$\int_0^{2\pi} e^{(1+i)t}\,dt = \frac{1}{1+i}\, e^{(1+i)t}\Big]_0^{2\pi}$$

$$= \frac{1}{1+i}\left[e^{2\pi + i2\pi} - e^0\right]$$

$$= \frac{1}{1+i}\left[e^{2\pi} - 1\right]$$

Further more

$$\frac{1}{1+i} = \frac{1}{2}(1-i)$$

Thus

$$\int_0^{2\pi} e^{(1+i)t}\,dt = \frac{e^{2\pi}-1}{2}(1-i) \quad \blacksquare$$

<u>Example</u> Evaluate the integral $\int_0^1 (t+2it^2)(t^2-3it)\,dt$.

<u>Solution</u> : First we have to carry out the multiplication

$$(t+2it^2)(t^2-3it) = t^3 + 6t^3 - 2it^4 - 3it^2$$

$$= 7t^3 - (2t^4 + 3t^2)\,i$$

Thus

$$\int_0^1 (t+2it^2)\cdot(t^2-3it)\,dt = \int_0^1 7t^3\,dt - \left(\int_0^1 2t^4 + 3t^2\,dt\right) i$$

$$= \frac{7}{4}t^4\Big]_0^1 - \left[\frac{2}{5}t^5 + t^3\right]_0^1 i$$

$$= \underline{\underline{\frac{7}{4} - \frac{7}{5}\,i}}$$

COMPLEX VECTOR SPACES

The axioms for complex vector spaces are the same as those for real vector spaces except the scalars are now complex numbers.

Examples Let

$$\mathbb{C}^n = \{(z_1, \dots, z_n) : z_1, \dots, z_n \in \mathbb{C}\}.$$

The addition is given by

$$u = (z_1, \dots, z_n), \; v = (w_1, \dots, w_n)$$

$$u + v = (z_1 + w_1, z_2 + w_2, \dots, z_n + w_n)$$

The scalar multiplication is

$$\lambda u = (\lambda z_1, \dots, \lambda z_n).$$

Example Let $I \subseteq \mathbb{R}$ be an interval. Let V be the space of functions $f : I \to \mathbb{C}$. The addition and scalar multiplication is given by

$$(f+g)(t) = f(t) + g(t)$$

$$[\lambda f](t) = \lambda f(t). \qquad \blacksquare$$

The big change is by the definition of inner product. Let V be a complex vector space. A map $(\cdot, \cdot)$ from $V \times V$ into the field of complex numbers is called an inner product if

(1) $(u, u) \geq 0$ for all $u \in V$

(2) $(u, u) = 0$ if and only if $u = 0$

(3) For fixed $v \in V$, the map $u \mapsto (u, v)$ is linear, i.e. $(\lambda u + \mu w, v) = \lambda (u, v) + \mu (w, v)$ for all $\lambda, \mu \in \mathbb{C}$ and all $u, w \in V$.

(4) For all $u, v \in V$ we have $(u, v) = \overline{(v, u)}$.

<u>Lemma</u> Let $(\cdot,\cdot)$ be an inner product on the complex vector space V. Then

$$(v, \lambda u + \mu w) = \overline{\lambda}\,(v,u) + \overline{\mu}\,(v,w)$$

for all $\lambda, \mu \in \mathbb{C}$ and all $v, u, w \in V$.

<u>Proof.</u> We have

$$\begin{aligned}
(v, \lambda u + \mu w) &= \overline{(\lambda u + \mu w, v)} && \text{(by rule 4)} \\
&= \overline{[\lambda (u,v) + \mu (w,v)]} && \text{(by (3))} \\
&= \overline{\lambda}\,\overline{(u,v)} + \overline{\mu}\,\overline{(w,v)} \\
&= \overline{\lambda}\,(v,u) + \overline{\mu}\,(v,w) && \text{(by (4))}. \ \blacksquare
\end{aligned}$$

<u>Examples</u>

(1) Let $V = \mathbb{C}^n$. Define for $u = (z_1, \dots, z_n)$ and $v = (w_1, \dots, w_n)$

$$(u, v) = z_1 \overline{w_1} + \cdots + z_n \overline{w_n}.$$

Then

$$\begin{aligned}
(u,u) &= z_1 \overline{z_1} + \cdots + z_n \overline{z_n} \\
&= |z_1|^2 + \cdots + |z_n|^2 \geq 0
\end{aligned}$$

If $(u,u) = 0$, then we must have $|z_1| = |z_2| = \cdots = |z_n| = 0$, so $u = 0$. Let $w = (t_1, \dots, t_n) \in V$ and $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}
(u + w, v) &= (z_1 + t_1)\overline{w_1} + \cdots + (z_n + t_n)\overline{w_n} \\
&= (z_1 \overline{w_1} + \cdots + z_n \overline{w_n}) + (t_1 + \cdots + t_n)\overline{w_n} \\
&= (u, v) + (w, v)
\end{aligned}$$

Similarly

$$\begin{aligned}
(\lambda u, v) &= (\lambda z_1)\overline{w_1} + \cdots + (\lambda z_n)\overline{w_n} \\
&= \lambda (z_1 \overline{w_1}) + \cdots + \lambda (z_n \overline{w_n}) \\
&= \lambda (u, v)
\end{aligned}$$

Finally

$$(u,v) = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$
$$= \overline{\overline{z_1}w_1 + \cdots + \overline{z_n}w_n}$$
$$= \overline{(v,u)} \quad \boxtimes$$

<u>Example</u> Let V be the space of piecewise continuous functions $f:[0,1]\to\mathbb{C}$. Define

$$(f,g) = \int_0^1 f(t)\overline{g(t)}\,dt. \quad \boxtimes$$

<u>Definition</u> Let V be a complex vector space with inner product $(\cdot,\cdot)$. Then the norm (or length) of a vector $u\in V$ is defined by

$$\|u\| := \sqrt{(u,u)}$$

<u>Remark</u> Notice that $\|u\|=0$ if and only if $u=0$ and that $\|\lambda u\| = |\lambda|\cdot\|u\|$.

<u>Example</u>

(1) $V=\mathbb{C}^2$ and $u=(1,i)$. Then

$$\|u\|^2 = 1 + i\cdot\overline{i} = 1 + i(-i) = 1+1 = 2.$$

Notice, that without the complex conjugation we would get $1+i^2 = 1-1=0$, wich is not possible.

(2) $V=\mathbb{C}^2$, $u=(1+i,\ 2+3i)$:

$$\|u\|^2 = (1+i)(1-i) + (2+3i)(2-3i)$$
$$= 1+1+4+9 = 15$$

or

$$\|u\| = \sqrt{15}.$$

(3) Let V = space of piesewise continuous functions on $[0,1]$.
Let $a \in \mathbb{R}$ and

$$f(t) = e^{ait} = \cos(at) + i\sin(at).$$

Then
$$f(t)\overline{f(t)} = |f(t)|^2 = (\cos(at) + i\sin(at))(\cos(at) - i\sin(at))$$
$$= (\cos(at))^2 + (\sin(at))^2$$
$$= 1.$$

Hence $\|f\| = \sqrt{\int_0^1 |f(t)|^2 dt} = \sqrt{\int_0^1 1\, dt} = \underline{\underline{1}}$

(4) Let V be as in the last example and

$$f(t) = t + it^2$$

Then
$$|f(t)|^2 = (t + it^2)(t - it^2)$$
$$= t^2 + t^4.$$

Hence
$$\|f\| = \sqrt{\int_0^1 t^2 + t^4 dt} = \sqrt{\frac{1}{3} + \frac{1}{5}} = \underline{\underline{\sqrt{\frac{8}{15}}}}$$

(5) Let V be as in the last two examples and

$$f(t) = t^2 + 1 + 2i(t-3).$$

Then
$$|f(t)|^2 = (1+t^2)^2 + 4(t-3)^2$$
$$= 1 + 2t^2 + t^4 + 4t^2 - 24t + 36$$
$$= t^4 + 6t^2 - 24t + 37$$

Hence
$$\|f\|^2 = \int_0^1 t^4 + 6t^2 - 24t + 37\, dt$$
$$= \frac{1}{5} + 2 - 12 + 37$$
$$= 27 + \frac{1}{5} = \underline{\underline{\frac{136}{5}}}$$

Exercises, Complex numbers, complex vector spaces, and complex valued functions

1) Evaluate the following complex numbers:

1. $(2+3i)+(4-2i) = 6+i$
2. $(3-2i)\cdot(2+5i) = 16+11i$
3. $(2+3i)\cdot(2-3i) = 4+9=13$
4. $\frac{4-i}{3+2i} = \frac{10-11i}{13}$
5. $\frac{3+2i}{3-i} = \frac{7+9i}{10}$
6. $e^{\pi i} = -1$
7. $e^{2\pi i} = 1$

2) Evaluate the following integrals:

1. $\int_0^1 3t + 2it^2 dt = \frac{3}{2} + \frac{2}{3}i$
2. $\int_0^{1/2} e^{2\pi i t}\ dt = i/\pi$
3. $\int_0^1 (2t+it^2)\cdot(2+3it)\ dt = \frac{5}{4} + \frac{8}{3}i$

3) Evaluate the following inner products:

1. $((1+i, 2-i),(1-i,3)) = (1+i)(1+i) + (2-i)\cdot 3 = 6-i$
2. $((i, 1+i, 3),(i, 2-i, 2)) = i\cdot(-i) + (1+i)(2+i) + 6 = 8+3i$
3. $((2, 3+i, i),(2-i, 2, 1+i)) = 2\cdot(2+i) + (3+i)\cdot 2 + i(1-i)$
4. $((2+i, 1+2i, 1),(2-i, 1-2i, 1)) = 11+5i$

4) Find the norm of the following vectors:

1. $\|(1,i)\| = \sqrt{1+1} = \sqrt{2}$
2. $\|(1+i, 1-i)\| = \sqrt{2+2} = 2$
3. $\|(2-i, 2, 3+i)\| = \sqrt{5+4+10} = \sqrt{19}$

Quiz 11-18-2003.

1) Multiply the following complex numbers

a) $(2+3i)\cdot(1+i) = 2\cdot 1 + 2\cdot i + 3i\cdot 1 + 3i\cdot i$

$$= 2-3+(2+3)i$$

$$= -1+5i$$

b) $\dfrac{2-i}{3+i} = \dfrac{2-i}{3+i}\cdot\dfrac{3-i}{3-i} = \dfrac{(6-1)-3i-2i}{10} = \dfrac{5-5i}{10} = \dfrac{1-i}{2}$

2) Find the length of the following complex numbers:

(a) $z = 2+3i$: $|z| = \sqrt{4+9} = \underline{\underline{\sqrt{13}}}$

(b) $z = 1-3i$: $|z| = \sqrt{1+9} = \underline{\underline{\sqrt{10}}}$

(c) $z = \dfrac{1}{2+i}$, $|z| = \dfrac{1}{\sqrt{4+1}} = \underline{\underline{\dfrac{1}{\sqrt{5}}}}$

Quiz, 11-20-2003

1) Find the length of the vector $(1+i, 1+2i, i)$.

Solution: Recall that the length (or norm) of the vector $(z_1, \dots, z_n) \in \mathbb{C}^n$ is given by

$$\|(z_1, \dots, z_n)\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

Here the length square of a complex number $z = x+iy$ is

$$|z|^2 = x^2 + y^2.$$

Thus $\|(1+i, 1+2i, i)\| = \sqrt{|1+i|^2 + |1+2i|^2 + |i|^2}$

$$= \sqrt{2+5+1} = \sqrt{8}$$

$$= 2\sqrt{2}$$

2) Evaluate the integral $\int_0^1 (2t+i)\cdot(3t+2it^2)\,dt$.

Solution: The first step is to evaluate the function, that we are integrating

$$(2t+i)(3t+2it^2) = 6t^2 + 4it^3 + 3it - 2t^2$$

$$= 4t^2 + (4t^3+3t)i$$

Thus $\int_0^1 (2t+i)\cdot(3t+2it^2)\,dt = 4\int_0^1 t^2\,dt + \left(\int_0^1 4t^3+3t\,dt\right)i$

$$= \frac{4}{3} + \left(1 + \frac{3}{2}\right)i$$

$$= \frac{4}{3} + \frac{5}{2}i$$

3) Find the norm of the functions $t \rightsquigarrow e^{\pi i t}$ and $f(t) = t + it^2$.

<u>Solution</u> Let me first recall the following: On the vector space of continuous function on $[0,1]$, the inner product is

$$(f, g) = \int_0^1 f(t)\overline{g(t)}\,dt$$

and the norm is

$$\|f\| = \sqrt{\int_0^1 |f(t)|^2 dt}.$$

We have
$$|e^{\pi i t}| = |\cos(\pi t) + i\sin(\pi t)|$$
$$= \sqrt{\cos^2(\pi t) + \sin^2(\pi t)} = 1$$

Therefore
$$\|e^{\pi i t}\| = \sqrt{\int_0^1 1\,dt} = \underline{\underline{1}}$$

We have $|t + it^2|^2 = t^2 + t^4$. Hence

$$\|t + it^2\| = \sqrt{\int_0^1 t^2 + t^4\,dt}$$
$$= \sqrt{\frac{1}{3} + \frac{1}{5}} = \underline{\underline{\sqrt{\frac{8}{15}}}}$$