§2 Subspaces.

In most applications we will be working with a subset W of a set V which we know is a vectorspace.

Q: Do we have to test all the axioms to find out if W is a vector space?

The answer is no!

Theorem: Let $W \neq \emptyset$ be a subset of the vector space V. Then W, with the same addition and scalar multiplication as V, is a vector space if and only if:

- $u + v \in W$ for all $u, v \in W$ (or $W + W \subseteq W$)
- $r \cdot u \in W$ for all $r \in \mathbb{R}$ and all $u \in W$ (or $\mathbb{R}W \subseteq W$)

In this case we say that W is a subspace of V.

Proof: Assume that $W + W \subseteq W$ and $\mathbb{R} W \subseteq W$.

To show that W is a vector space we have to show that all the 10 axioms hold for W.

But that follows because the axioms hold for V:

A1] $u + v = v + u$

vectors in V

same addition as in V

commutativity holds in V.

A4] Take any vector $u \in W$. Then by assumption $0 \cdot u = \vec{0} \in W$. Hence $\vec{0} \in W$.

A5] Similarly, if $u \in W$, then $-u = (-1) \cdot u \in W$.

all the other axioms follows in the same way $\boxtimes$

Usually the situation is, that we have given a vector space V and a a subset of vector that satisfy some conditions

$$W = \{ v \in V : \text{ some conditions on } v \}$$

V ↑ vector space

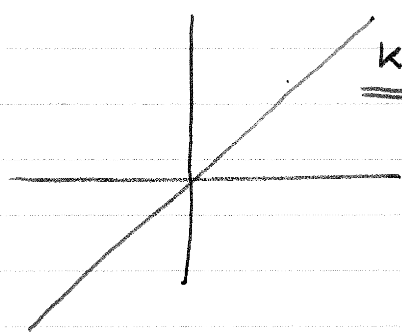
We will then have to show that

$$\begin{matrix} v, w \in W \\ r \in \mathbb{R} \end{matrix} \qquad \left. \begin{matrix} v + w \\ r \cdot v \end{matrix} \right\} \text{ satisfy the same conditions.}$$

<u>Example</u> $V = \mathbb{R}^2$,

$$W = \{ (x, y) \mid y = kx \} \quad (k \text{ given})$$

$= \text{line through } (0,0) \text{ with slope } k.$



$kx - y = 0$

Let $u = (x_1, y_1)$, $v = (x_2, y_2) \in W$
Then $y_1 = kx_1$ and $y_2 = kx_2$

$$\begin{aligned} u + v &= (x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2, kx_1 + kx_2) \\ &= (x_1 + x_2, k(x_1 + x_2)) \end{aligned}$$

And the same for $ru = (rx_1, krx_1)$.

- So what are the subspaces of $\mathbb{R}^2$?

(1) $\{0\}$

(2) Lines, but only those that contain $(0,0)$. Why?

(3) $\mathbb{R}^3$.

- What are the subspaces of $\mathbb{R}^3$.

(1) $\{0\}$ and $\mathbb{R}^3$

(2) Planes: A plain $W \subseteq \mathbb{R}^3$ is given by a normal vector (a, b, c) and the distence from $(0,0,0)$ or

$$(*) \quad W = \{(x,y,z) : \underbrace{ax + by + cz = p}_{\text{condition on } (x,y,z)}\}$$

<u>First test</u>: If W is a subspace, then $\vec{0} \in W$.

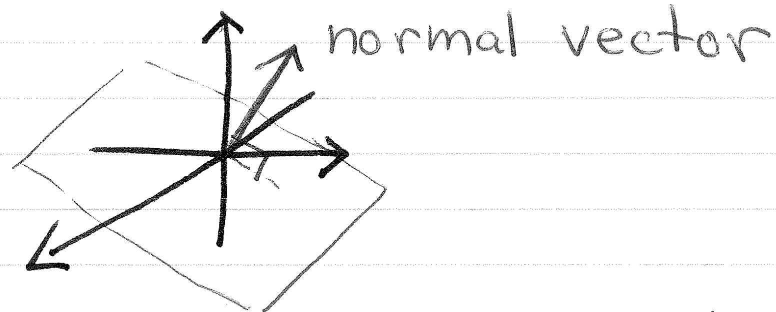
<u>Thus</u>: If $\vec{0} \notin W$, then W is not a subspace!

If we apply this to $(*)$, then

$$p = a\cdot 0 + b\cdot 0 + c\cdot 0 = 0$$

or $\boxed{p = 0}$

But we can NOT conclude from the fact that $\sigma \in W$, that W is a subspace.



But a plane through $(0,0,0)$ is always a subspace.

Proof! $ax_1 + by_1 + cz_1 = 0$

$ax_2 + by_2 + cz_2 = 0$

Then $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2)$

$$= \underbrace{(ax_1 + by_1 + cz_1)}_{\sigma} + \underbrace{(ax_2 + by_2 + cz_2)}_{\sigma} = 0$$

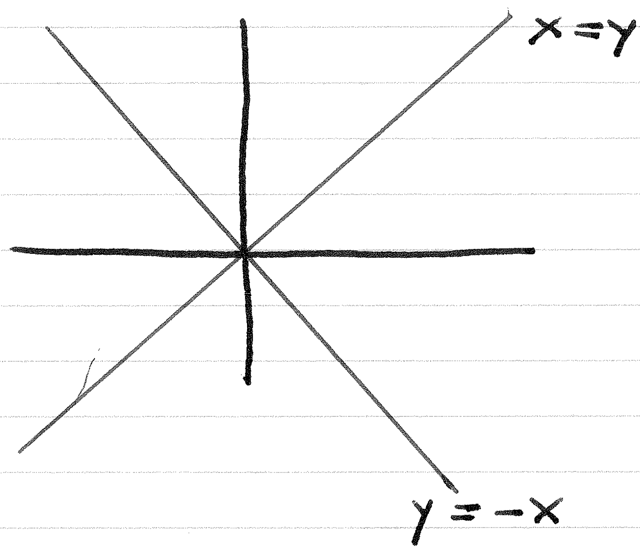
and $a(rx_1) + b(ry_1) + c(rz_1) = r[ax_1 + by_1 + cy_2]$

$= 0.$

(3) lines containing zero = Intersection of two planes.

An EXAMPLE of a subset in $\mathbb{R}^2$ that is not a subspace:

$$W = \{(x,y) \mid x^2 - y^2 = 0\}$$



We have $(1,1), (1,-1) \in W$ but

$$(1,1) + (1,-1) = (2,0) \notin W$$

Notice that $(0,0) \in W$ and W is closed under multiplication by scalars.

<u>Exercises</u> Which of the following subsets of $\mathbb{R}^n$ is a subspace and which is not (give the arguments):

1) $W = \{(x,y) \in \mathbb{R}^2 : \ xy = 0\}$

2) $W = \{(x,y,z) \in \mathbb{R}^3 : 3x + 2y^2 + z = 0\}$

3) $W = \{(x,y,z) \in \mathbb{R}^3 : 2x + 3y - z = 0\}$

4) The set of all vectors (x_1, x_2, x_3) satisfying $2x_3 = x_1 - 10x_2$

5) All the vectors in $\mathbb{R}^4$ satisfying the system of linear equations

$$\begin{aligned} 2x_1 + 3x_2 + 5x_4 &= 0 \\ x_1 + x_2 - 3x_3 &= 0 \end{aligned}$$

6) The set of points (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ satisfying

$$x_1 + 2x_2 + 3x_3 + x_4 = -1.$$