1. Linear Groups

1.1. **Manifolds.** Lie groups are mathematical objects with **two** different structures:

(1) **Analytical structure:** Locally they look like Euclidean space $\mathbb{R}^n$, we can talk about differentiable functions, we can differentiate those functions, in short: **We can do analysis.**
(2) **Algebraic structure:** We have group structure, so that we can compose element in the group. We can let the group then act on spaces, functions etc. We can then try to understand how those objects behave under the action of the group.

Those two structures are related by the requirement, that the algebraic operations are **smooth**. One can formulate this in one definition:

Definition 1. *Assume that G is a group an a manifold. Then G is called a **Lie group** if the map*

$$G \times G \ni (a, b) \mapsto a^{-1}b \in G$$

is smooth.

Let us recall the definition of a smooth map. Let $U \neq \emptyset$ be an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $e_1, \ldots, e_n$ be the standard orthonormal basis for $\mathbb{R}^n$. I.e.

$$e_j = \underbrace{(0, \ldots, 0, 1, 0, \ldots, 0)}_{1\ at\ the\ j^{th}-place}{}^T$$

Let $F : U \to \mathbb{R}^m$. Define

$$D_j F(x) = \frac{\partial F}{\partial x_j}(x) := \lim_{h \to 0} \frac{F(x + he_j) - F(x)}{h}$$

if the limit exists. Let $m = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ be a multi-index. Let

$$D^m F(x) = D_1^{m_1} \ldots D_n^{m_n} F(x)$$

be defined inductively in the obvious way. Then we say that F is smooth if all the partial derivatives $D^m F$, $m \in \mathbb{Z}_+^n$, exists on U and are continuous maps $D^m F : U \to \mathbb{R}^m$.

We can now define a manifold. Let M be a topological space with a countable basis for the topology. Let $p \in M$. **A chart** around p is a pair $(U, \mathbf{x})$ where $U \subset M$ is an open neighborhood of p and $\mathbf{x} : U \to V \subset \mathbb{R}^n$, is a homeomorphism, where V is an open subset of $\mathbb{R}^n$. Thus $\mathbf{x}$ is continuous, bijective, and $\mathbf{x}^{-1} : V \to U$ is also continuous. An **atlas** for M is a collection $\mathcal{A} = \{(U, \mathbf{x}_U) \mid U \subset M \text{ open}\}$ such that

$$M = \bigcup_{(U, \mathbf{x}_U) \in \mathcal{A}} U$$

and $(U, \mathbf{x}_U)$ is a chart (around any point $p \in U$). We say that the atlas is continuous, smooth, etc. if all the coordinate changes

$$\mathbf{x}_V \circ \mathbf{x}_U^{-1} : \mathbf{x}(U \cap V) \to \mathbb{R}^n$$

are continuous, smooth, etc. Notice that $\mathbf{x}(U \cap V)$ is an open subset of $\mathbb{R}^n$ because $\mathbf{x}_U : U \to \mathbf{x}(U)$ is a homeomorphism and $U \cap V$ is an open subset of U.

Definition 2. *A set M is a n-dimensional manifold if M is a topological space with a countable basis for the topology and there exists a **maximal** smooth atlas for M.*

If M is a m-dimensional manifold, then we write M^m to indicate the dimension. We will also use the notation $\mathcal{A}_M$ for a maximal atlas for M without comments. If $(U, \mathbf{x})$ is a chart, then we will write

$$\mathbf{x}(p) = (x_1(p), \ldots, x_m(p))$$

and call the functions $x_j : U \to \mathbb{R}$ the coordinates. If M is a manifold and $N \subset M$ open and non-empty. Then N is a manifold with an atlas

$$\mathcal{A}_N = \{(U \cap N, \mathbf{x}_U|_{U\cap N}) \mid (U, \mathbf{x}) \in \mathcal{A}_M,\, U \cap N \neq \emptyset\}$$

If M^m and N^n are manifolds, then the product

$$M^m \times N^n = \{(p, q) \mid p \in M, q \in N\}$$

is a $(m + n)$-dimensional manifold with atlas

$$\mathcal{A}_{M\times N} := \mathcal{A}_M \times \mathcal{A}_n = \{(U \times V, \mathbf{x} \times \mathbf{y}) \mid (U, \mathbf{x}) \in \mathcal{A}_M, (V, \mathbf{y}) \in \mathcal{A}_N\}\ .$$

The manifold $M \times N$ with this structure is called the **product manifold**.

Definition 3. *Let M^m and N^n be manifold with maximal atlas $\mathcal{A}_M$ and $\mathcal{A}_N$ respectively. Let $F : M \to N$ be a continuous function. Then F is called smooth if the maps*

$$\mathbf{y} \circ F \circ \mathbf{x}^{-1} : \mathbf{x}(U \cap F^{-1}(V)) \to \mathbb{R}^n$$

are smooth for all $(U, \mathbf{x}) \in \mathcal{A}_M$ and $(V, \mathbf{y}) \in \mathcal{A}_N$ such that $U \cap F^{-1}(V) \neq \emptyset$.

Example 1 (Vector spaces)**.** *Let E be a finite dimensional vector space over $\mathbb{R}$. Fix a basis $f_1, \ldots, f_k$ for E. Define a map $\mathbf{x}^{-1} : \mathbb{R}^k \to E$ by*

$$\mathbf{x}^{-1}(x_1, \ldots, x_k) = \sum_{j=1}^{k} x_j f_j$$

Then $\mathcal{A} = \{(E.\mathbf{x})\}$ is an atlas for E. It follows in particular that any open, non-empty subset of a finite dimensional vector space over $\mathbb{R}$ is a manifold.

Example 2 (The sphere S^n)**.** *Let $S^n = \{x \in \mathbb{R}^{n+1} \mid ||x|| = 1\}$. Then S^n is a manifold.*

1.2. **The General Linear Group** $\mathrm{GL}(n, \mathbb{R})$**.** Let $M = M(n, \mathbb{R})$ be the space of $n \times n$ matrices $X = (x_{ij})_{i,j=1}^{n}$. Then M is a n^2-dimensional vector space and hence a manifold. In this section we will discuss some examples of submanifold of M. First define

$$\det : M \to \mathbb{R}$$

to be the determinant function

$$\det(X) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{1\sigma(1)} \cdot \ldots \cdot x_{n\sigma(n)}$$

where S_n is the permutation group of n-elements. Then det is polynomial and hence smooth (in fact analytic). Thus

$$\mathrm{GL}(n, \mathbb{R}) = \{a \in M(n, \mathbb{R}) \mid \det(a) \neq 0\}$$

is an open subset of M and hence a n^2-dimensional manifold. Denote the coordinate functions $(a_{ij}) \mapsto a_{ij}$ by x_{ij}. The multiplication of two matrices $a = (a_{ij})$ and $b = (b_{ij})$ is given by

$$x_{ij}(ab) = \sum_{\mu=1}^{n} a_{i\mu} b_{\mu j}$$

and is a polynomial function in a and b. In particular

$$\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \ni (a, b) \mapsto ab \in \mathrm{GL}(n, \mathbb{R})$$

is smooth. Similarly it follows by Cramers rule that the inversion map $a \mapsto a^{-1}$ is a rational function in the coordinates of a:

$$x_{ij}(a^{-1}) = \frac{(-1)^{ij}}{\det(a)} \det A_{ji}$$

and hence smooth. Thus $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. As the determinant is continuous it follows that the group

$$\mathrm{SL}(n, \mathbb{R}) = \{a \in \mathrm{GL}(n, \mathbb{R}) \mid \det(a) = 1\}$$

is a closed subgroup. The group $\mathrm{GL}(n, \mathbb{R})$ is called **the general linear group** and the group $\mathrm{SL}(n, \mathbb{R})$ is the **special linear group**.

1.3. **Orthogonal group.** To be added: Bilinear form and orthogonal groups.

1.4. **The Heisenberg group.**

2. The Exponential Map

Let $F = \mathbb{R}$ or $= \mathbb{C}$ and let $X, Y \in M(n, \mathbb{F})$. Then we define the **inner product** $(X \mid Y)$ by

$$(X \mid Y) = \mathrm{Tr}(XY^*)$$

where $Y^* = \bar{Y}^T$. Then

$$(X \mid Y) = \sum_{i.j=1}^{n} x_{ij} \overline{y_{ij}}\,.$$

The corresponding norm is given by

$$||X|| = \sqrt{\sum_{i,j=1}^{n} |x_{ij}|^2}\,.$$

This is set up so that our canonical isomorphism $M(n, \mathbb{F}) \simeq \mathbb{F}^{n^2}$ is an unitary isomorphism. Using the norm $M(n, \mathbb{F})$ becomes a metric space. In particular the notation of **convergence** and **Cauchy sequences**. Notice that a sequence $(X_n) \in M(n, \mathbb{F})$ is convergent if and only if it is a Cauchy sequence. We will also need the **operator norm** $||\cdot||_{op}$ of a matrix X. This norm is denoted by

$$||X||_{op} = \sup_{||u||=1} ||Xu|| = \sup_{u \neq 0} \frac{||Xu||}{||u||}\,.$$

If X is symmetric, and hence diagonalizable, we have

$$||X||_{op} = \max |\lambda_j|$$

where $\lambda_1, \ldots, \lambda_n$ are the **eigenvalues** of X. Notice that there exists constants $A, B > 0$ such that

$$A\,||X|| \leq ||X||_{op} \leq B\,||X||\,.$$

Lemma 1. *Let $X, Y \in M(n, \mathbb{F})$. Then the following holds:*

(1) $||X^*|| = ||X||$ *and* $||X^*||_{op} = ||X||_{op}$
(2) $||XY||_{op} \leq ||X||_{op}\,||Y||_{op}$.

(Add few words on analytic functions and power series). Let $X \in M(n, \mathbb{F})$ and define

$$X_n = \sum_{k=0}^{n} \frac{X^k}{k!}\,.$$

Assume that $n < m$. Then

$$\begin{aligned} ||X_m - X_n||_{op} &= \left|\left| \sum_{k=n+1}^{m} \frac{X^k}{k!} \right|\right|_{op} \\ &\leq \sum_{k=n+1}^{m} \frac{||X||_{op}^k}{k!} \end{aligned}$$

As the series

$$e^{||X||} = \sum_{k=0}^{\infty} \frac{||X||_{op}^k}{k!}$$

converges it follows that for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $m > n \geq N$ we have

$$||X_n - X_m|| \leq \sum_{k=n+1}^{m} \frac{||X||_{op}^k}{k!} < \epsilon\,.$$

Hence $\{X_n\}_{n \in \mathbb{Z}_+}$ is a Cauchy sequence and

$$e^X := \lim_{N \to \infty} \sum_{k=0}^{N} \frac{X^k}{k!} =: \sum_{k=0}^{\infty} \frac{X^k}{k!}$$

exits. The map $X \mapsto e^X$ is called the (matrix) **exponential function**.

Example 3. *If $a_1, \ldots, a_n \in \mathbb{F}$ denote by $d(a_1, \ldots, a_n)$ the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. Thus*

$$d(a_1, \ldots, a_n) = \begin{pmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \vdots & 0 & a_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & a_n \end{pmatrix}$$

Assume that $X = d(a_1, \ldots, a_n)$. Then

$$e^X = d(e^{a_1}, \ldots, e^{a_n})\,.$$

Example 4. *A matrix X is called* ***upper triangular*** *if $x_{ij} = 0$ for all $i > j$. Assume that*

$$X = \begin{pmatrix} a_1 & & * \\ 0 & \ddots & \\ 0 & 0 & a_n \end{pmatrix} = d(a_1, \dots, a_n) + N$$

where N is upper triangular with zero on the main diagonal. Then e^X is upper triangular with diagonal entries $e^{a_1}, \dots, e^{a_n}$.

Example 5. *Let $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $Y = tX$. Then $Y^2 = -t^2 I$, and by induction*

$$Y^{2k} = (-1)^k t^{2k} I, \qquad Y^{2k+1} = (-1)^k t^{2k+1} X\,.$$

Hence

$$\begin{aligned} e^Y &= \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} X \\ &= \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}. \end{aligned}$$

Notice that this examples implies that the exponential map is ***not*** *injective.*

Let $X \in M(n, \mathbb{F})$ and consider the map

$$\gamma_X : \mathbb{R} \to \mathrm{GL}(n, \mathbb{F})$$

given by

$$\gamma_X(t) = e^{tX}\,.$$

Then γ_X is continuous, and in fact analytic. Furthermore by the above

$$\gamma_X(t+s) = \gamma_X(t)\gamma_X(t)\,.$$

By direct calculation one can show that γ_X is differentiable:

$$\left|\left| \frac{\gamma_X(t+h) - \gamma_X(t)}{h} \right|\right|_{op} = \left|\left| e^{tX} \frac{e^{hX} - I}{h} \right|\right|_{op}$$

$$\leq ||e^{tX}|| \left|\left| \frac{e^{hX} - I}{h} \right|\right|_{op} \tag{2.1}$$

$$\leq ||e^{tX}|| \sum_{k=0}^{\infty} \frac{|h|^k}{(k+1)!} ||X||_{op}^{k+1} \tag{2.2}$$

It follows that the limit $h \to 0$ exists and in fact

$$\lim_{h \to 0} \frac{\gamma_X(t+h) - \gamma_X(t)}{h} = X\gamma_X(t) = \gamma_X(t)X\,. \tag{2.3}$$

Lemma 2. *Let $I \subset \mathbb{R}$ be an open interval containing zero. Let $\gamma : I \to M(n, \mathbb{F})$ be differentiable and*

$$\frac{d\gamma(t)}{dt} = X\gamma(t)\,.$$

Then

$$\gamma(t) = \gamma(0)e^{tX}\,.$$

Proof. Let $F(t) = e^{-tX}\gamma(t)$. Then

$$\begin{aligned} F'(t) &= -e^{-tX}X\gamma(t) + e^{-tX}\gamma'(t) \\ &= e^{-tX}\left(-X + X\right)\gamma(t) \\ &= 0. \end{aligned}$$

Hence F is constant $= F(0) = \gamma(0)$ on I. Hence the claim. □

Definition 4. *A* ***one parameter subgroup of*** $\mathrm{GL}(n,\mathbb{F})$ *is a continuous map* $\gamma : \mathbb{R} \to \mathrm{GL}(n,\mathbb{F})$ *such that*

$$\gamma(s+t) = \gamma(s)\gamma(t).$$

Notice that if γ is a one parameter subgroup, then $\gamma(\mathbb{R}) \subset \mathrm{GL}(n,\mathbb{F})$ is a subgroup of $\mathrm{GL}(n,\mathbb{F})$.

Lemma 3. *Let $A \in M(n,\mathbb{F})$ and assume that $||A - I|| < 1$. Then $A \in \mathrm{GL}(n,\mathbb{F})$.*

Proof. Assume that $Au = 0$. Then $(A - I)u = -u$ and hence $||A - I||_{op} \geq 1$ a contradiction. □

Lemma 4. *Let γ be a one parameter subgroup. Then γ is differentiable and there exists a $X \in M(n,\mathbb{F})$ such that*

$$\gamma = \gamma_X$$

Proof. We have $\gamma(0) = I$ and hence

$$\frac{\gamma(t+h) - \gamma(t)}{h} = \gamma(t)\frac{\gamma_X(h) - I}{h}. \tag{2.4}$$

Thus we only have to show that γ is differentiable at 0. As $\gamma(0) = I$ and γ is continuous, there exists a $\epsilon > 0$ such that

$$\left|\left|\gamma(t)^{-1} - I\right|\right|_{op} < 1/2$$

for $|t| < \epsilon$. Choose $f \in C_c(\mathbb{R})$ such that $\mathrm{Supp}(f) \subset (-\epsilon, \epsilon)$, $f(t) \geq 0$, $\int f dt = 1$. Let

$$\tilde{\gamma}(t) = \int f(u)\gamma(t-u)\,du = \int f(t-u)\gamma(u)\,du.$$

In particular $\tilde{\gamma}$ is smooth. Furthermore

$$\begin{aligned} \tilde{\gamma}(t) &= \int f(u)\gamma(t-u)\,du \\ &= \gamma(t)\int f(u)\gamma(u)^{-1}\,du \\ &= \gamma(t)A \end{aligned}$$

where $A = \int f(u)\gamma(u)^{-1}\,du$. If we can show that A is invertible then

$$\gamma(t) = \tilde{\gamma}(t)A^{-1}$$

and it follows that γ is smooth. But if $x \in \mathbb{F}^n$ then

$$\begin{aligned} ||A - I||_{op} &= \left|\left|\int f(u)\gamma(u)^{-1}\, du - I\right|\right|_{op} \\ &\leq \int f(u)\, ||\gamma(u)^{-1} - I||\, du \\ &\leq \frac{1}{2}\int f(u)\, du \\ &= \frac{1}{2} \end{aligned}$$

hence $A \in \mathrm{GL}(n, \mathbb{F})$.

It follows now from (2.4) that

$$\frac{d\gamma}{dt}(t) = \left(\frac{d\gamma}{dt}(0)\right)\gamma(t)\,.$$

Let $X = \gamma'(0)$. Then Lemma 4 and the fact that $\gamma(0) = I$ implies that

$$\gamma(t) = e^{tX}.$$

□

The exponential function has the following properties:

Lemma 5. *Let $X, Y \in M(n, \mathbb{F})$. Then the following holds:*

(1) $e^{X+Y} = e^X e^Y$ *if* $XY = YX$.
(2) $e^{X^T} = \left(e^X\right)^T$.
(3) *Let* $a \in \mathrm{GL}(n, \mathbb{F})$ *then*

$$e^{aXa^{-1}} = ae^X a^{-1}.$$

(4) $\det(e^X) = e^{\mathrm{Tr}(X)}$. *In particular* $\det(e^X) \neq 0$ *so* $e^X \in \mathrm{GL}(n, \mathbb{F})$.

Proof. 1) Consider the curves $\gamma(t) = e^{t(X+Y)}$ and $\beta(t) = e^{tX}e^{tY}$. Then both satisfies the differential equation

$$F'(t) = (X + Y)F(t) \qquad F(0) = I\,.$$

Hence the claim.

2) **To be added.**

3) Let $N \in \mathbb{N}$. Then

$$a\left(\sum_{k=0}^{N} \frac{X^k}{k!}\right)a^{-1} = \sum_{k=0}^{N} \frac{(aXa^{-1})^k}{k!}$$

Then claim now follows by taking the limit $N \to \infty$. The

c) Choose $a \in \mathrm{GL}(n, \mathbb{F})$ orthogonal such that $aXa^T = aXa^{-1} = Y$ is upper triangular. Then $\mathrm{Tr}(aXa^{-1}) = \mathrm{Tr}(Y)$ and

$$\begin{aligned} \det(e^X) &= \det(e^{aXa^{-1}}) \\ &= \det(e^Y) \\ &= e^{\mathrm{Tr}(Y)} \\ &= e^{e^{\mathrm{Tr}(X)}}\,. \end{aligned}$$

□

Definition 5. *Let $U, V \subset \mathbb{R}^n$ be open and nonempty. A map $F : U \to V$ is a said to be a diffeomorphism if*

(1) *F is bijective;*
(2) *F and F^{-1} are smooth.*

Theorem 1 (Inverse Function Theorem)**.** *Let $U, V \subset \mathbb{R}^n$ be open and nonempty. Let $F : U \to V$ be smooth and $x \in U$. If $DF(x) : \mathbb{R}^n \to \mathbb{R}^n$ is regular then there exists $x \in U_1 \subset U$ open and $F(x) \in V_2 \subset V$ open such that $F : U_1 \to V_1$ is a diffeomorphism.*

Lemma 6. *Let $\mathfrak{l}$ and $\mathfrak{m}$ be subspaces of $M(n, \mathbb{R})$ such that $\mathfrak{l} \cap \mathfrak{m} = \{0\}$. Define $\phi : \mathfrak{l} \times \mathfrak{m} \to M(n, \mathbb{R})$ by*

$$\phi(X, Y) = e^X e^Y .$$

Then $D\phi(0, 0) : \mathfrak{l} \times \mathfrak{m} \to M(n, \mathbb{R})$ is given by

$$D\phi(0, 0)(X, Y) = X + Y .$$

In particular $D\phi(0, 0)$ is injective. If $\mathfrak{l} \oplus \mathfrak{m} = M(n, \mathbb{R})$ then $D\phi(0, 0)$ is an insomorphism. In that case there exists open sets $0 \in U \subset \mathfrak{l}$, $0 \in V \subset \mathfrak{m}$, and $e \in W \subset \mathrm{GL}(n, \mathbb{R})$ such that $\phi : U \times V \to W$ is a diffeomorphism.

Proof. Let $t \in \mathbb{R}$, $X \in \mathfrak{l}$, $Y \in \mathfrak{m}$. Then

$$\begin{aligned} D\phi(0, 0)(X, Y) &= \frac{d}{dt}\phi(tX, tY)|_{t=0} \\ &= \frac{d}{dt} e^{tX} e^{tY}|_{t=0} \\ &= \left(\frac{de^{tX}}{dt}\right)_{t=0} e^{0Y} + e^{0X}\left(\frac{de^{tY}}{dt}\right)_{t=0} \\ &= X + Y \end{aligned}$$

If $\mathfrak{l} \oplus \mathfrak{m} = M(n, \mathbb{R})$ then $D\phi(0, 0)$ is an isomorphism and the inverse function theorem implies that ϕ is a local diffeomorphism. □

Corollary 1. *There exists a open set $U \subset M(n, \mathbb{R})$, $0 \in U$, and an open set $e \in V \subset \mathrm{GL}(n, \mathbb{R})$ such that* $\exp : U \to V$ *is a diffeomorphism.*

We will not need the exact form of the map $(\exp|_U)^{-1} : V \to U$, but few remarks are at place. Define

$$\log(A) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(A - \mathrm{id})^k}{k} .$$

As

$$\left|\left|\sum_{k=1}^{\infty} (-1)^{k-1} \frac{(A - \mathrm{id})^k}{k}\right|\right|_{\mathrm{op}} \leq \sum_{k=1}^{\infty} \frac{||A - \mathrm{id}||_{\mathrm{op}}^k}{k} < \infty$$

if

$$||A - \mathrm{id}||_{\mathrm{op}} < 1$$

it follows that $\log(A)$ is well defined for $A \in B_{1,\mathrm{op}}(\mathrm{id}) = \left\{B \in M(n) \mid ||B - \mathrm{id}||_{\mathrm{op}} < 1\right\}$. By Lemma 3 it follows that $B_{1,\mathrm{op}}(\mathrm{id})$. As in the case of real numbers it can be shown that locally $\exp \circ \log = \mathrm{id}$ and $\log \circ \exp = \mathrm{id}$.

Lemma 7. *Let $X, Y \in M(n, \mathbb{R})$ and $t \in \mathbb{R}$. Then*

(1) *There exists a* $\delta > 0$ *such that for* $|t| < \delta$ *we have* $e^{tX}e^{tY} = exp\big(t(X+Y) + O(t^2)\big)$
(2) *There exists a* $\delta > 0$ *such that for* $|t| < \delta$ *we have* $e^{tX}e^{tY}e^{-tX}e^{-tY} = \exp(t^2\,[X,Y] + O(t^3))$

Proof. 1) For $Z \in M(n,\mathbb{R})$ and $t \in \mathbb{R}$ write

$$F(t,Z) := \sum_{k=2}^{\infty} \frac{t^{k-2}Z^k}{k!}$$

The series converges for all t and Z.. Furthermore

$$\begin{aligned} e^{tX}e^{tY} &= \big(I + tX + t^2F(t,X)\big)\,\big(I + tY + t^2F(t,Y)\big) \\ &= I + t(X+Y) + t^2(F(t,X) + F(t,Y) + t^2XY) \\ &\quad + t^3(F(t,X)Y + XF(t,Y)) + t^4F(t,X)F(t,Y) \\ &= I + t(X+Y) + t^2G(t,X,Y) \end{aligned}$$

where

$$G(t,X,Y) = F(t,X) + F(t,Y) + t^2XY + t\,(F(t,X)Y + XF(t,Y)) + t^2F(t,X)F(t,Y)$$

is continuous in t, X, and Y. In particular G is bounded on compact subset. Let U, V be such that $\exp : U \to V$ is a diffeomorphism. Choose $\delta > 0$ such that $e^{tX}e^{tY} \in V$ for all $t \in (-\delta,\delta)$. Then it follows that

$$\log\left(e^{tX}e^{tY}\right) = t(X+Y) + O(t^2)$$

which implies the claim.

2) The prove of (2) goes almost exactly as (2) by multiplying out on the left hand side. □

Theorem 2. *Let* $X, Y \in M(n,\mathbb{R})$ *then the following holds*

(1) $\exp(X+Y) = \lim_{k\to\infty}\left(\exp\left(\frac{1}{k}X\right)\exp\left(\frac{1}{k}Y\right)\right)^k$.
(2) $\exp([X,Y]) = \lim_{k\to\infty}\left(\exp\left(\frac{1}{k}X\right)\exp\left(\frac{1}{k}Y\right)\exp\left(-\frac{1}{k}X\right)\exp\left(-\frac{1}{k}Y\right)\right)^{k^2}$.

Proof. 1) We have according to Lemma 7 for k big:

$$\begin{aligned} \left(\exp\left(\frac{1}{k}X\right)\exp\left(\frac{1}{k}Y\right)\right)^k &= \left(\exp\left(\frac{1}{k}(X+Y) + O(\frac{1}{k})\right)\right)^k \\ &= \exp(X + Y + o(\frac{1}{k})) \end{aligned}$$

and hence

$$\lim_{k\to\infty}\left(\exp\left(\frac{1}{k}X\right)\exp\left(\frac{1}{k}Y\right)\right)^k = \exp(X+Y)\,.$$

2) This goes exactly the same way by using (2) in Lemma 7. □

Exerise:

1) Show that a group G which is also a manifold is a Lie group if and only if the maps

$$G \ni a \mapsto a^{-1} \in G$$

and

$$G \times G \ni (a,b) \mapsto ab \in G$$

are smooth.

2) Show that S^1 is a manifold.

3) Let $F : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the real linear isomorphism:

$$F((x_1 + iy_1, \ldots, x_n + iy_n)^T) = (x_1, \ldots, x_n, y_1, \ldots, y_n)^T .$$

If $T : \mathbb{C}^n \to \mathbb{C}^n$ is $\mathbb{C}$-linear, define $T_F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$T_F := F \circ T \circ F^{-1}$$

In this way we get a linear injective map

$$\Psi : M(n, \mathbb{C}) \to M(2n, \mathbb{R}).$$

(1) Find the matrix $J := \Psi(T)$ of the linear map $T(z) = iz$.
(2) Find the matrix $I_{n,n} := \Psi(\mathrm{conj})$ where $\mathrm{conj}(z) = \bar{z}$.
(3) Show that

$$\begin{aligned}\mathrm{Im}(\Psi) &= \{A \in M(2n, \mathbb{R}) \mid JA = AJ\} \\ &= \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid \alpha, \beta \in M(n, \mathbb{R}) \right\} \cap \mathrm{GL}(2n, \mathbb{R})\end{aligned}$$

4) Show that the group $\mathrm{O}(2, 1)$ has four connected components.

5) This is a generalization of problem (4): Show that $\mathrm{O}(p, q)$ has 4 connected components if $pq \neq 0$, but $\mathrm{O}(p)$ has two connected components.

3. The Lie algebra of a linear group

3.1. **Lie algebras.** Let $\mathfrak{g}$ be a vector space over $\mathbb{F}$ with a bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then $\mathfrak{g}$ (or $(\mathfrak{g}, [\cdot, \cdot])$ to be exact) is called a **Lie algebra** if the following holds:

(1) (Anticommutative) $\forall X, Y \in \mathfrak{g} : [X, Y] = -[Y, X]$.
(2) (Jacobi identity) For all $X, Y, Z \in \mathfrak{g}$ we have

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 .$$

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subspace. Then $\mathfrak{h}$ is a **Lie subalgebra** if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$. It is easy to see that $(\mathfrak{h}, [\cdot, \cdot])$ is a Lie algebra. The subspace $\mathfrak{h}$ is an **ideal** if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and all $Y \in \mathfrak{h}$. In this case $\mathfrak{h}$ is a Lie subalgebra.

Example 6. *Let A be an associative algebra over $\mathbb{F}$. Define for $a, b \in A$:*

$$[a, b] = ab - ba$$

then $(A, [\cdot, \cdot])$ is a Lie algebra. In particular $M(n, \mathbb{R})$ is a Lie algebra. The Lie algebra $M(n, \mathbb{F})$ is usually denoted by $\mathfrak{gl}(n, \mathbb{F})$.

Example 7. *Let A be a real symmetric matrix $n \times n$-matrix. Define*

$$\mathfrak{o}(A) = \left\{ X \in M(n, \mathbb{R}) \mid X^T A + AX = 0 \right\} .$$

Then $\mathfrak{o}(A)$ is a Lie algebra. The only thing that we need to show is that for $X, Y \in \mathfrak{o}(A)$ we have $[X, Y] \in \mathfrak{o}(A)$. For that we simply calculate

$$\begin{aligned}[X, Y]^T A + A[X, Y] &= (XY - YX)^T A + A(XY - YX) \\ &= \left(Y^T X^T A - X^T Y^T A\right) + AXY - AYX \\ &= Y^T AX - X^T AY + AXY - AYX \\ &= AYX - AXY + AXY - AYX \\ &= 0\end{aligned}$$

where we have used that $X^T A = AX$ and $Y^T A = YA$.

Example 8. *As* $\operatorname{Tr}(XY - YX) = 0$ *for all* X, Y *we have that*
$$\mathfrak{sl}(n,\mathbb{F}) = \{X \in \mathfrak{gl}(n,\mathbb{F}) \mid \operatorname{Tr}(X) = 0\}$$
is an ideal in $\mathfrak{gl}(n,\mathbb{F})$.

Theorem 3. *Let* $G \subset \mathrm{GL}(n,\mathbb{R})$ *be a linear group. Define*
$$\mathfrak{g} := \left\{X \in \mathfrak{gl}(n,\mathbb{R}) \mid e^{\mathbb{R}X} \subset G\right\} .$$
Then $\mathfrak{g}$ *is a Lie algebra.*

Proof. Let $\lambda \in \mathbb{R}$ and $X \in \mathfrak{g}$. Then $\mathbb{R}(\lambda X) \subset \mathbb{R}X$ and hence $\lambda X \in \mathfrak{g}$. Let $X, Y \in \mathfrak{g}$ and $t \in \mathbb{R}$.. Then
$$\exp\left(t(X+Y)\right) = \lim_{k\to\infty} \left(\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right)\right)^k .$$
By definition of $\mathfrak{g}$ it follows that
$$\exp\left(\frac{t}{k}X\right), \exp\left(\frac{t}{k}Y\right) \in G .$$
Hence, as G is a group, $\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right) \in G$. As G is a closed subgroup it finally follows that
$$\exp\left(t(X+Y)\right) = \lim_{k\to\infty} \left(\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right)\right)^k \in G$$
As t was arbitrary it follows that $X + Y \in \mathfrak{g}$. By the second part of Theorem 2 we have
$$\exp\left(t\,[X,Y]\right) = \lim_{k\to\infty} \left(\exp\left(\frac{t}{k}X\right)\exp\left(\frac{t}{k}Y\right)\exp\left(-\frac{t}{k}X\right)\exp\left(-\frac{t}{k}Y\right)\right)^{k^2} \in G .$$
□

Definition 6. *Let* $G \subset \mathrm{GL}(n,\mathbb{R})$ *be a closed group. Then the Lie algebra*
$$\mathfrak{g} = \left\{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \forall t \in \mathbb{R} : e^{tX} \in G\right\}$$
is called ***the Lie algebra of*** G.

Example 9 (The Lie algebra of $\mathrm{SL}(n,\mathbb{R})$)**.** *Recall that* $\mathrm{SL}(n,\mathbb{R})$ *is the group of elements* $g \in \mathrm{GL}(n,\mathbb{R})$ *such that* $\det(g) = 1$. *Then* X *is in the Lie algebra of* $\mathrm{SL}(,\mathbb{R})$ *if and only if*
$$\det(e^{tX}) = 1$$
for all $t \in \mathbb{R}$. *As*
$$\frac{d}{dt}\det(e^{tX})|_{t=0} = \operatorname{Tr}(X)$$
it follows that $Tr(X) = 0$. *On the other hand assume that* $\operatorname{Tr}(X) = 0$. *Then*
$$\begin{aligned}\frac{d}{dt}\det(e^{tX}) &= \left(\frac{d}{dt}\det(e^{tX})\right)_{t=0}\det(e^{tX})\\ &= \operatorname{Tr}(X)\det(e^{tX})\\ &= 0 .\end{aligned}$$
Hence $t \mapsto \det(e^{tX})$ *is constant. As* $e^{0X} = I$ *it follows that this constant in* 1, *or* $e^{tX} \in \mathrm{SL}(n,\mathbb{R})$ *for all* $t \in \mathbb{R}$. *Thus the Lie algebra of* $\mathrm{SL}(n,\mathbb{R}) = \mathfrak{sl}(n,\mathbb{R})$.

Example 10 (The orthogonal groups). *To be added*

Our next aim is to show that $\exp : \mathfrak{g} \to G$ is a local diffeomorphism. We will then use that to turn G into a Lie group.

Lemma 8. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{l} \subset \mathfrak{gl}(n, \mathbb{R})$ be a complementary subspace, i.e., $\mathfrak{gl}(n, \mathbb{R}) = \mathfrak{l} \oplus \mathfrak{g}$. Then there exists $0 \in V \subset \mathfrak{l}$, $0 \in U \subset \mathfrak{g}$, and $e \in W \subset \mathrm{GL}(n, \mathbb{R})$, open and such that*

1. *$\varphi : V \times U \to W$, $(X, Y) \mapsto e^X e^Y$ is a diffeomorphism,*
2. *Let $X \in V$ and $Y \in U$. Then $\varphi(X, Y) \in G$ if and only if $X = 0$.*

Proof. ByAssume that the Lemma does not hold. We can find $B_R(0) \subset \mathfrak{g}$ and $B_S(0) \subset \mathfrak{l}$ such that φ is a diffeomorphism. Assume that for all $n \in \mathbb{N}$, there □

Theorem 4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then there exists a zero neigborhood $V \subset \mathfrak{g}$, and a e-neighborhood $U \subset G$ such that*

$$\exp : V \to U$$

is a homeomorphism.

Exercise:

1. Let A be an associative algebra over $\mathbb{F}$. Show that A with the Lie product
$$[a, b] = ab - ba$$
is $(A, [\cdot, \cdot])$ a Lie algebra.

4. Homogeneous spaces

Definitions

Example 11. *Let $\mathrm{GL}(n, \mathbb{R})$ acts on $\mathbb{R}^n$ in the canonical way*

$$(A, x) \mapsto Ax\,.$$

As $A0 = 0$ it follows that $\{0\}$ is one orbit. Let $x \in \mathbb{R}^n$, $x \neq 0$. Extend x to a basis $\{x = x_1, x_2, \ldots, x_n\}$ of $\mathbb{R}^n$ and let A be the matrix with columns x_j, $A = [x_1, x_2, \ldots, x_n]$. Then $A \in \mathrm{GL}(n, \mathbb{R})$

$$Ae_1 = x\,.$$

Hence $\mathbb{R}^n \setminus \{0\}$ is another orbit. Suppose that $Ae_1 = e_1$. Then

$$A = [e_1, *]$$

where $*$ stands for arbitrary elements $x_2, \ldots, x_n \in \mathbb{R}^n \setminus \{0\}$ such that $e_1, x_2, \ldots, x_n$ is linearly independent. It follows that

$$H = \mathrm{GL}(n, \mathbb{R})^{e_1} = \left\{ \begin{pmatrix} 1 & x \\ 0 & A \end{pmatrix} \mid x^T \in \mathbb{R}^{n-1}, A \in \mathrm{GL}(n-1, \mathbb{R}) \right\}.$$

We will later in this section show that $\mathrm{GL}(n, \mathbb{R})/H$ is diffeomorphic to $\mathbb{R}^n \setminus \{0\}$.

Lemma 9. *Let $H \subset G$ be a closed subgroup. Then there exists a unique differentiable structure on $M = G/H$ such that the following holds:*

1. *The quotient map $\kappa : G \to G/H$, $a \mapsto aH$, is smooth*
2. *If N is a manifold and $f : G/H \to N$ is continuous, then f is smooth if and only if $f \circ \kappa : G \to N$ is smooth.*

Corollary 2. *The group G acts smoothly on G/H.*

Proof. We have to show that the map

$$G \times G/H \ni (a, bH) \to abH \in G/H$$

is smooth. But we can factorize this map as

$$(a, b) \mapsto ab \mapsto \kappa(ab)$$

which is smooth. □

We will need the **Baire Category Theorem** for the next Theorem.

Definition 7. *Let X be a topological space. A set $A \subset X$ is* ***nowhere dense*** *if its closure has no interior. Sets which are countable unions of nowhere dense sets are said to be of* ***first category****. The set A is of the* ***second category*** *if it is not of the first category.*

Theorem 5 (Baire Category Theorem)**.** *A complete metric space or a locally compact Hausdorff space is second category.*

Let us now fix some notations before we state and prove the next theorem. Let M be a G space. Denote the action of G on G/H by $\ell(a)$ and the action of G on M by $\tau(a)$. Thus

$$\ell(a)(bH) = (ab)H$$

and

$$\tau(a)m = a \cdot m\,.$$

Fix $p \in M$ and let $H = G^p$. Then we can define a map $\Phi_p : G/H \to M$ by

$$\Phi_p(aH) = a \cdot p\,.$$

Then Φ_p is well defined because $aH = bH$ implies that $b^{-1}a = h \in H$ and hence

$$a \cdot p = (bh) \cdot p = b \cdot (h \cdot p) = h \cdot p\,.$$

Furthermore $\Phi_p \circ \kappa : G \to M$ is given by

$$\Phi_p \circ \kappa(a) = a \cdot p \tag{4.1}$$

and hence smooth. Let $a, b \in G$. Then

$$\begin{aligned}\Phi_p(\ell(a)bH) &= \Phi_p(abH)\\ &= (ab) \cdot p\\ &= a \cdot (b \cdot p)\\ &= \tau(a)(\Phi_p(bH))\,.\end{aligned}$$

Thus Φ_p is a G-map. Let us now fix a subspace $\mathfrak{l} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}\,.$$

Then $\mathfrak{l} \simeq T_{eH}(G/H)$ where $X \in \mathfrak{l}$ is viewed as a tangent vector by

$$X_{eH}(f) = \frac{d}{dt} f(\exp tX \cdot H)|_{t=0}.$$

The differential $d\ell(a)_{eH} : T_{eH}(G/H) \to T_{aH}(G/H)$ is an isomorphism. Thus $T_{eH}(G/H) \simeq \mathfrak{l}$ where the isomorphism is now given by

$$\begin{aligned} X_{aH}(f) &= X_{eH}(f \circ \ell(a)) \\ &= \frac{d}{dt} f(a \exp(tX) \cdot H)|_{t=0} . \end{aligned}$$

If we differentiate the relation (4.1) we get

$$(d\Phi_p)_{aH} \circ (d\ell(a))_{eH} = (d\tau(a))_p \circ (d\Phi_p)_{eH} . \tag{4.2}$$

Identifying $T_{aH}(G/H)$ with $\mathfrak{l}$ as above this reads:

$$\begin{aligned} (d\Phi_p)_{aH}(X_{aH})f &= (d\Phi_p)_{aH}((d\ell(a))_{eH} X_{eH})f \\ &= \frac{d}{dt} f(a \exp tX \cdot p)|_{t=0} . \end{aligned}$$

We also see that $(d\Phi_p)_{aH}(X_{aH}) = 0$ if and only if $(d\Phi_p)_{eH}(X_{eH}) = 0$.

Lemma 10. *Let M be a G-space, $p \in M$, and $H = G^p$. Then the map*

$$G/H \ni aH \overset{\Phi_p}{\mapsto} ap \in M$$

is a G-diffeomorphism.

Proof. We only have to show that Φ is a diffeomorphism. If we can show that there exists an open set $eH \in U \subset G/H$ and an open set $p \in V \subset M$ such that $\Phi : U \to V$ is a diffeomorphism, then we are done. In fact let $q \in M$. Then there exists an $a \in G$ such that $a \cdot p = q$. We have $aH \in aU \subset G/H$ and $q \in a \cdot V \subset M$ and $\Phi^{-1} : aV \to aU$ is given by

$$m \mapsto a^{-1}m \mapsto (\Phi|_U)^{-1}(a^{-1}m) \mapsto a(\Phi|_U)^{-1}(a^{-1}m)$$

and all those maps are smooth. Choose a subspace $\mathfrak{l} \subset \mathfrak{g}$ as above, i.e., such that

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{h}.$$

We will now show that $(d\Phi)_{eH}$ is bijective.

Let $X \in \mathfrak{l}$ and assume that $(d\Phi)_{eH}(X) = 0$. Let $(V, \mathbf{y} = (y_1, \ldots, y_m))$ be coordinates around $p \in M$. As $\Phi_p \circ \kappa$ is continuous there exists a open set $e \in U \subset G$ such that $\Phi_p(UH) \subset V$. Then $(d\Phi)_{aH}(X_{aH}) = 0$ for all $a \in U$ and

$$\begin{aligned} (d\Phi)_{aH}(X)(y_j) &= X_{aH}(y_j \circ \Phi) \\ &= \frac{d}{dt} y_j(a \exp tX \cdot p)|_{t=0} \\ &= 0 \end{aligned}$$

Choose $\delta > 0$ such that $\exp(sX) \in U$ for all $s \in (-\delta, \delta)$. Then it follows that the maps

$$(-\delta, \delta) \mapsto y_j(\exp sX \cdot p)$$

are all constant equal to $y_j(p)$. It follows that $\exp(sX) \cdot p = p$ for all $s \in (-\delta, \delta)$. But then

$$\exp(n(-\delta, \delta)X) \subset H$$

for all $n \in \mathbb{N}$ which implies that $\exp(\mathbb{R}X) \subset \mathfrak{h}$ or $X \in \mathfrak{h}$. As $\mathfrak{l} \cap \mathfrak{h} = \{0\}$ we get $X = 0$ or $(d\Phi_p)_{aH}$ is injective for all $a \in G$.

We now show that Φ_p is an open map. Let $V \subset G/H$ be open and $U = \Phi_p(V)$. Let $q \in U$. Choose $aH \in V$ such that $\Phi_p(aH) = q$. Let W be a symmetric neighborhood around e such that $a\kappa(\bar{W}^2) \subset V$ and $\bar{W}$ is compact. Then - because G has a countable basis for the topology - there exists a finite or countable infinite set J and elements $a_j \in G$, $j \in J$, such that

$$G = \bigcup_{j \in J} a_j W.$$

As G acts transitively it follows that

$$M = \bigcup_{j \in J} a_k W \cdot p \subset \bigcup a_k \bar{W} \cdot p\,.$$

in particular $\cup a_k \bar{W} \cdot p = M$. As $a_k \bar{W}$ is compact and $\Phi_p \circ \kappa$ is continuous it follows that $a_k \overline{W} \cdot p$ is compact and hence closed. It follows that

$$a_k \bar{W} \cdot p = \overline{a_k W \cdot p}\,.$$

By the Baire Category Theorem there exists a $j_0 \in J$ such that

$$(a_k \bar{W} \cdot p)^o \neq \emptyset\,.$$

But then, as each $\tau(x)$, $x \in G$, is a diffeomorphism, $(\bar{W} \cdot p)^o \neq \emptyset$. Choose $x \in \bar{W}$ such that $x \cdot p \in (\bar{W} \cdot p)^o$, then - again because $\tau(x)^{-1}$ is a diffeomorphism -

$$p \in \left(\tau(x)^{-1} \bar{W} \cdot p\right)^o \subset \left(\bar{W}^2 \cdot p\right)^o$$

As $a\kappa(\bar{W}^2) \subset V$ it follows that

$$q = a \cdot p \in a(\bar{W}^2 \cdot p)^o \subset a\bar{W}^2 \cdot p \subset U.$$

Thus q is an inner point. As q was arbitrary it follows that U is open. Let $(U, \mathbf{x})$ be coordinates around $eH \in G/H$ and $(V, \mathbf{y})$ coordinates around $p \in M$. We may then assume that $\Phi_p(U) \subset V$. Then the map

$$\mathbf{y} \circ \Phi_p \circ \mathbf{x}^{-1} : \mathbf{x}(U) \to \mathbf{y}(V \cap \Phi_p(U))$$

is a homeomorphism. By the invariance of dimension it follows that

$$\dim \mathbf{x}(U) = \dim \mathbf{y}(V \cap \Phi_p(U))$$

or

$$\dim(G/H) = \dim(M)\,.$$

As $(d\Phi_p)_{aH}$ is injective, it follows from $\dim T_{aH}(G/H) = \dim T_{a \cdot p}(M)$ that $(d\Phi_p)_{aH}$ is surjective. By the inverse function theorem it follows that Φ_p is locally a diffeomorphism and hence globally a diffeomorphism. □

Example 12. *Let $k \in \mathbb{N}$, and $\mathbf{n} = (n_1, n_2, \ldots, n_k)$ with $n_1 < n_2 < \ldots < n_k < n$. A $\mathbf{n}$-flag is a array $(E_1, \ldots, E_k)$ of subspaces of $\mathbb{R}^n$ such that*

$$E_j \subset E_{j+1}$$

and

$$\dim E_j = n_j\,.$$

Let $M(\mathbf{n})$ be the set of all $\mathbf{n}$-flags. At the moment we do not put any topology on $M(\mathbf{n})$. Let $E = (E_1, \ldots, E_k) \in M(\mathbf{n})$ and $g \in \mathrm{GL}(n, \mathbb{R})$. Define

$$g \cdot E = (g(E_1), \ldots, g(E_k))\,.$$

It is clear that $g \cdot E \in M(\mathbf{n})$ and that this defines a $\mathrm{GL}(n, \mathbb{R})$-action on $M(\mathbf{n})$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and define E_j by

$$E_j = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_{n_j}$$

Then

$$E = (E_1, \ldots, E_k) \in M(\mathbf{n})\,.$$

Let $F = (F_1, \ldots, F_k) \in M(\mathbf{n})$. *Let* $f_1, \ldots, f_{n_1}$ *be an orthonormal basis for* F_1. *Extend it to an orthonormal basis* $f_1, \ldots, f_{n_2}$ *of* F_2. *In that way we construct an orthonormal basis* $f_1, \ldots, f_{n_j}$ *of* F_j *such that* $f_1, \ldots, f_{n_{j-1}}$ *is an orthonormal basis for* F_{j-1}. *We can assume (by replacing* f_1 *by* $-f_1$ *if necessary) that* $f_1, \ldots, f_n$ *has a positive orientation. Define* $g \in \mathrm{SO}(n, \mathbb{R})$ *by*

$$ge_j = f_j$$

Then

$$g \cdot E = F\,.$$

Thus $\mathrm{SO}(n, \mathbb{R})$ *(and then also* $\mathrm{GL}(n, \mathbb{R})$*) acts transitively on* $M(\mathbf{n})$. *We next determine the stabilizer of* E. *First we notice that* $g \in \mathrm{GL}(n, \mathbb{R})^E$ *if and only if* $g(E_j) = E_j$. *Thus if* $i \leq n_j$ *then*

$$g(e_i) = \sum_{r \leq n_j} g_{ri} e_r$$

or

$$g_{ri} = 0 \qquad \text{if} \qquad r > n_j\,.$$

It follows that g *has the block form*

$$g = \begin{pmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & A_k \end{pmatrix}$$

As every matrix of this form stabilizes E *it follows that*

$$\mathrm{GL}(n, \mathbb{R})^E = P(\mathbf{n}) = \left\{ \begin{pmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & A_k \end{pmatrix} \mid A_j \in \mathrm{GL}(n_j - n_{j-1}) \right\}$$

where we have put $n_0 = 0$. *In particular*

$$M(\mathbf{n}) = \mathrm{GL}(n, \mathbb{R})/P(\mathbf{n})\,.$$

Recall now that $\mathrm{SO}(n)$ *acts transitively on* $M(\mathbf{n})$. *Furthermore*

$$\begin{aligned} \mathrm{SO}(n) \cap P(\mathbf{n}) &= \left\{ \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & A_k \end{pmatrix} \mid \begin{array}{c} A_j \in 0(n_j - n_{j-1}) \\ \det A_1 \cdot \ldots \cdot \det A_k = 1 \end{array} \right\} \\ &= S(0(n_1) \times \ldots \times 0(n_k - n_{k-1}))\,. \end{aligned}$$

Thus $M(\mathbf{n})$ *is a compact connected manifold.*

Exerise:
1) Let $M = \{z \in \mathbb{C} \mid |z| \leq 1\}$. For

$$g \in G = \mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}$$

define

$$g \cdot z = \frac{\alpha z + \beta}{\bar{\alpha} z + \bar{\beta}} .$$

(1) Show that $g \cdot z \in M$ and that this defines a $\mathrm{SU}(1,1)$ action on M.
(2) Show that there are exactly two G-orbits, the boundary $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and the interior $B_1(0) = \{z \in \mathbb{C} \mid |z| < 1\}$.
(3) Determine the stabilizers of $0 \in B_1(0)$ and $1 \in \mathbb{T}$.

2) For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ and $z \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ define

$$g \cdot z = \frac{az + b}{cz + d} .$$

Show that this defines a transitive $\mathrm{SL}(2,\mathbb{R})$-action on $\mathbb{C}^+$ and that $\mathrm{SL}(2,\mathbb{R})^i = \mathrm{SO}(2)$.

3) Let $S = \left\{ \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \right\}$. Then S is diffeomorphic to $\mathbb{C}^+$.

4) Let G_1 and G_2 be Lie groups and let M_1 and M_2 be manifolds. A (G_1, G_2) map from M_1 to M_2 is a pair (Φ, ϕ) where $\Phi : M_1 \to M_2$ is smooth, $\phi : G_1 \to G_2$ is a Lie group homomorphism, and furthermore

$$\Phi(g \cdot m) = \phi(g) \cdot \Phi(m) .$$

Let the notation be as in problems 1 and 2. Define $\Phi : B_1(0) \to \mathbb{C}$ and $\phi : \mathrm{SU}(1,1) \to \mathrm{GL}(2,\mathbb{R})$ by

$$\Phi(z) = \frac{z + i}{iz + 1}$$

and

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} .$$

Show that (Φ, ϕ) is a $(\mathrm{SU}(1,1), \mathrm{SL}(2,\mathbb{R}))$-isomorphism of $B_1(0)$ onto $\mathbb{C}^+$.

5) Let $G = \mathrm{SL}(2,\mathbb{R})$ act on it Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ by conjugation $(a, X) \mapsto \mathrm{Ad}(a)X = aXa^{-1}$. Show that we have the following type of orbits:

a) The hyerbolic orbits $\mathcal{H}_t = \mathrm{Ad}(G)\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}$, $t > 0$.

b) Elliptic orbits $\mathcal{E}_t = \mathrm{Ad}(G)\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}$, $t \in \mathbb{R}^*$.

c) The nilpotent orbits $\mathcal{N}_+ = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \mid t > 0 \right\}$ and $\mathcal{N}_- = \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \mid t < 0 \right\}$

d) The trivial orbit $\{0\}$.

5. Vector bundles

In this section $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. We will always assume that vector spaces are over $\mathbb{F}$ and state if necessary if $\mathbb{F}$ is assume to be $\mathbb{R}$ or $\mathbb{C}$.

Definition 8. *Let M be a manifold. A vector bundle over M is a pair (E, π) where*

(1) *E is a manifold,*
(2) *$\pi : E \to M$ is a smooth surjective map,*
(3) *$\pi^{-1}(p) = E_p$ is a vector space,*

(4) *For each* $p \in M$ *there exists* $p \in U \subset M$ *open and a diffeomorphism*

$$f_U : E_U = \pi^{-1}(U) \to U \times \mathbb{F}^n$$

such that F_U *has the form*

$$f_U(u, v) = (u, T(u)v)\,, \qquad (u, v) \in E_u := \pi^{-1}(u)$$

with $T(u) : E_u \to \mathbb{F}^n$ *a linear isomorphism.*

We sometimes write $\pi : E \to M$ *or* $E \xrightarrow{\pi} M$ *for a vector bundle. If* $\dim E_p = 1$ *for all* $p \in M$ *then we say that* E *is a* ***line bundle*** *over* M*. The vector space* E_u *is called* ***the fiber*** *over* u*.*

Definition 9. *Let* $E \xrightarrow{\pi} M$ *and* $F \xrightarrow{\rho} M$ *be two vector bundles over* M*. A bundle map from* E *to* F *is a smooth map* $\phi : E \to F$ *such that* $\phi(E_p) \subset F_p$ *for all* $p \in M$ *and* $\phi_x = \phi|_{E_x} : E_x \to F_x$ *is linear. We say that* $E \xrightarrow{\pi} M$ *and* $F \xrightarrow{\rho} M$ *are isomorphic if there exists a bundle map* $\phi : E \to F$ *which is a diffeomorphism.*

Remark 1. *The maps* f_U *is called a* ***local trivialization*** *of the bundle.*

Example 13. *Let* $E = M \times \mathbb{F}^n$ *and with the product topology and* $\pi(m, v) = m$*. Then* E *is a vector bundle. Any vector bundle over* M *which is isomorphic to* $M \times \mathbb{F}^n$ *is called the* ***trivial bundle*** *over* M*.*

Example 14. *Let* $M = M^m$ *be a manifold. Then* $T(M)$ *is a vector bundle over* M*. Let* $(U, \mathbf{x})$ *be local coordinates around* $p \in M$*. Then* $(d\mathbf{x})_u : T_uM \to \mathbb{R}^m$ *is a linear isomorphism and* $u \mapsto (d\mathbf{x})_u$ *is smooth by the definition of the manifold structure on* TM*. Define*

$$f_U(u, v) := (u, (d\mathbf{x})_u^{-1}(v)) \in T_u(M)\,.$$

Example 15. *Let* $k \in N$*,* $k < n$*. Let* $M(k) = M((k))$ *be the compact space of* k*-dimensional subspaces of* $\mathbb{R}^n$*. Define a vector bundle over* $M(k)$ *by*

$$E = \{(V, V) \in M(k) \times M(k) \mid V \in M(k)\}$$

with projection $(V, x) \mapsto V$ *where* x *is any point in* V*.*

We will now discuss *how to construct vector bundles.* The idea is to glue together trivial vector bundles over open subsets of M. To motivate the construction let $U, V \subset M$ be open such that $U \cap V \neq \emptyset$ and such that we have local trivialization of the vector bundle $E \to M$, denoted by

$$f_U : E_U \to U \times \mathbb{F}^n\,, \quad f_U(u, v) = (u, T_U(u)v)$$

and

$$f_V : E_V \to V \times \mathbb{F}^n\,, \quad f_V(u, w) = (u, T_V(u)w)\,.$$

Then we can define a map $g_{V,U} : U \cap V \to \mathrm{GL}(n, \mathbb{F})$ by

$$g_{V,U}(u)x = \mathrm{pr}_2(f_V \circ f_U^{-1}(u, x))$$

where pr_2 is the projection onto the second factor

$$V \times \mathbb{F}^n \ni (v, x) \mapsto x \in \mathbb{F}^n\,.$$

Thus in particular

$$(u, x) \mapsto g_{V,U}(u)x$$

is smooth and

$$g_{U,U}(u) = \mathrm{id}\,.$$

If W is a third open set in M such that $U \cap V \cap W \neq \emptyset$ then we have the following *cocycle relation*:

$$\begin{aligned} g_{V,W}(u)g_{W,U}(u)x &= \mathrm{pr}_2\left(f_V \circ f_W^{-1}(u, g_{W,U}(u)x)\right) \\ &= \mathrm{pr}_2\left(f_V \circ f_W^{-1}(u, \mathrm{pr}_2(f_W \circ f_U^{-1}(u,x))\right) \\ &= \mathrm{pr}_2((f_V \circ f_W^{-1}) \circ (f_W \circ f_U^{-1})(u,x)) \\ &= \mathrm{pr}_2(f_V \circ f_U^{-1}(u,x)) \\ &= g_{V,U}(u)x \end{aligned} \tag{5.1}$$

In particular we have

$$g_{V,U}(u)^{-1} = g_{U,V}(u)$$

Given a covering $\mathcal{W}$ and families of maps $\{g_{V,U}\}$ satisfying the cocycle relation (5.1) we call $\{g_{V,U}\}$ for a cocycle adapted to $\mathcal{W}$.

Example 16. *The cocyle for the tangent bundle.*

Let now $\mathcal{W}$ be an open covering of M such that each $U \in \mathcal{W}$ is a coordinate neighborhood. Furhtermore assume that $\{g_{U,V}\}$ is a cocycle such that $g_{V,U} : U \cap V \to \mathrm{GL}(n, \mathbb{F})$ is smooth. We would like to construct a vector bundle E over M such that the corresponding cocycles are exactly the ones that we started with. Let

$$\mathcal{X} := \{(p, x, U) \in M \times \mathbb{F}^n \times \mathcal{W} \mid p \in U\}.$$

We make $M \times \mathbb{F}^n \times \mathcal{W}$ into a topological space by considering the product topology with the discrete topology on $\mathcal{W}$. On $\mathcal{X}$ we concider then the relative topology. We say that $(p, x, U), (q, y, V) \in \mathcal{X}$ are equivalent, denoted by $(p, x, U) \backsim (q, y, V)$, if

(1) $p = q$;
(2) $U \cap V \neq \emptyset$;
(3) $g_{V,U}(u)x = y$.

It is then easy to see that the cocycle relation is exactly what we need to make $\simeq$ into an equivalence relation. Let

$$E := \mathcal{X}/\backsim$$

If $(p, x, U) \in \mathcal{X}$ then we denote the corresponding equivalence class by $[p, x, U]$. We make E into a topological space by requiring the canonical quotient map $(p, x, U) \mapsto [p, x, U]$ is continuous. Notice that for fixed p and U the map

$$\mathbb{F}^n \ni x \mapsto [p, x, U] \in E$$

is injective because $g_{U,U}(p) = \mathrm{id}$. Define $\pi : E \to M$ by

$$\pi([p, x, U]) = p.$$

Then π is well defined because $[p, x, U] = [q, y, V]$ implies that $p = q$.

The next step is then to define the vector space structure on $E_p = \pi^{-1}(p)$, $p \in M$. Let $v, w \in E_p$ and $\lambda \in \mathbb{F}$. Then there exists a $U \in \mathcal{W}$ and $x, y \in \mathbb{F}^n$ (only depending on U) such that $v = [p, x, U], w = [p, y, U]$. Define

$$\lambda v + w = [p, \lambda x + y, U].$$

We have to show that this is well defined. Notice that our only choice was that of U. So assume that we have $v = [p, x_1, V]$ and $w = [p, y_1, V]$. Then by the definition of the equivalence relation we have

$$x_1 = g_{V,U}(p)x$$
$$y_1 = g_{V,U}(p)y$$

and hence

$$\lambda x_1 + y_1 = g_{V,U}(p)(\lambda x + y)\,.$$

It follows that

$$[p, \lambda x + y, U] = [p, \lambda x_1 + y_1, V]\,.$$

The final step before we state our theorem is to make E into a manifold such that $\pi : E \to M$ is smooth. Let $U \in \mathcal{W}$. Let $\mathbf{x}_U : U \to \mathbf{x}_U(U)$ be the corresponding coordinate map. Define $g_U : U \times \mathbb{F}^n \to \pi^{-1}(U)$ by

$$g_U(p,x) := [p,x,U]\,.$$

Then by the definition of the topology on E it follows that g_U is a homeomorphism. Let $f_U : \pi^{-1}(U) \to U \times \mathbb{F}^n$ be the inverse. We will now make E into a manifold such that the maps f_U are smooth. For that let $w \in E$. Then there exists $U \in \mathcal{W}$ and $p \in U$ such that $w = [p,x,U]$. Define a map $\mathbf{y}_U \colon \pi^{-1}(U) \to \mathbf{x}(U) \times \mathbb{F}^n$ by

$$\mathbf{y}_U(e) := (\mathbf{x}_U(\mathrm{pr}_1(f_U(e))), \mathrm{pr}_2(f_U(e)))\,.$$

Thus

$$\mathbf{y}_U([q,x,U]) = (\mathbf{x}_U(q), x)\,.$$

Let $V \in \mathcal{W}$ be such that $U \cap V \neq \emptyset$. Then we have coordinates $\mathbf{x}_V$ and $\mathbf{y}_V$. But

$$\mathbf{y}_V \circ \mathbf{y}_U^{-1}(u,x) = (\mathbf{x}_V^{-1} \circ \mathbf{x}_U(u), g_{V,U}(\mathbf{x}_U^{-1}(u))x)$$

which is obviously smooth. We have now proven the following Theorem:

Theorem 6. *Let $\mathcal{W}$ be an open covering of M consisting of coordinate neighborhoods. Assume that we have a smooth cocycle $\{g_{V,U}\}_{(V,U)\in\mathcal{W}\times\mathcal{W}, V\cap U\neq\emptyset}$. Define E and $\{f_U\}_{U\in\mathcal{W}}$ as above. Then $E \xrightarrow{\pi} M$ is a vector bundle with fiber isomorphic to $\mathbb{F}^n$, local trivialization maps F_U, $U \in \mathcal{W}$, and cocycles given by the functions $g_{V,U}$.*

The following Theorem decribes if two vector bundles are isomorphic in terms of their cocycles.

Theorem 7. *Let $E \xrightarrow{\pi} M$ and $F \xrightarrow{\rho} M$ be two vector bundles with $\mathrm{GL}(n,\mathbb{F})$-valued $\mathcal{W}$-cocycles $\{g_{U,V}\}$ and $\{h_{U,V}\}$ respectively. Then E and F are isomorphic if and only if for each U there exists a smooth map $s_U : U \to \mathrm{GL}(n,\mathbb{F})$ such that for each pair U and V with $U \cap V \neq \emptyset$ we have*

$$g_{V,U}(p) = s_V(p)h_{V,U}(p)s_U(p)^{-1}\,, \quad p \in U \cap V\,. \tag{5.2}$$

Proof. Assume first that $\phi : E \to F$ is an isomorphism. Assume that $\{g_{V,U}\}$ corresponds to the local trivialization $\{e_U\}$ and that $\{h_{U,V}\}$ corresponds to the local trivialization $\{f_U\}$. Define $s_U : U \to \mathrm{GL}(n,\mathbb{F})$ by

$$s_U(p)x = \mathrm{pr}_2 \circ e_U \circ \phi^{-1} \circ f_U^{-1}(p,x)\,.$$

Then

$$\begin{aligned} s_V(p)h_{V,U}(p)s_U(p)^{-1}x &= s_V(p)h_{V,U}(p)\mathrm{pr}_2(f_U(\phi(e_U^{-1}(p,x))))) \\ &= s_V(p)(\mathrm{pr}_2(f_V \circ f_U^{-1} \circ f_U \circ \phi(e_U^{-1}(p,x)))) \\ &= s_V(p)(\mathrm{pr}_2(f_V \circ \phi \circ e_U^{-1}(p,x))) \\ &= \mathrm{pr}_2(e_V \circ \phi^{-1} \circ f_V^{-1} \circ f_V \circ \phi \circ e_U^{-1}(p,x)) \\ &= \mathrm{pr}_2(e_V(e_U^{-1}(p,x)) \\ &= g_{V,U}(p)x \end{aligned}$$

Assume now that (5.2) holds. Then $E = \mathcal{X}/\sim_E$ and $F = \mathcal{X}/\sim_F$ where $(p,x,U) \sim_E (q,y,V)$ if and only if

(1) $p = q$;
(2) $U \cap V \neq \emptyset$;
(3) $g_{V,U}(p)x = y$.

Similarly for $\sim_F$ with $g_{V,U}$ replaced by $h_{V,U}$. We denote the corresponding equivalence classes by $[\ldots]_E$ respectively $[\ldots]_F$. Define a map $\Phi : \mathcal{X} \to \mathcal{X}$ by

$$\Phi((p,x,U)) := (p, s_U(p)^{-1}x, U)\,.$$

Assume that $(p,x,U) \sim_E (p,y,V)$. Then

$$\begin{aligned} h_{V.U}(p)s_U(p)^{-1}x &= s_V(p)^{-1}g_{V,U}(p)x \\ &= s_V(p)^{-1}y\,. \end{aligned}$$

Hence $\Phi((p,x,U)) \sim_F \Phi((p,x,U))$ and we can define $\phi : E \to F$ by

$$\phi([p,x,U]_E) := [p, s_U(p)^{-1}x, U]_F.$$

We need to show that ϕ is a linear isomorphism.

a) ϕ is linear because $s_U(p)^{-1}$ is linear.

b) Assume that $\phi([p,x,U]_F) = 0_{F_p} = [p,0,U]_F$. As $s_U(p)^{-1}$ is an isomorphism it follows that $x = 0$ and hence $[p,x,U]_E = 0_{E_p}$. Thus ϕ is injective.

c) Let $[p,y,U] \in F_p$. Let $x = s_U(p)y$. Then

$$\phi\left([p,x,U]\right) = [p,y,U]$$

and it follows that ϕ is surjective. □

Definition 10. *Let $E \xrightarrow{\pi} M$ be a vector bundle over M. A **section** of E is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. We denote the set of section by $\Gamma(M,E)$.*

Remark 2. *Let $\lambda \in \mathbb{F}$ and $s_1, s_2 \in \Gamma(M,E)$. Then we can make $\Gamma(M,E)$ into a vector space be defining*

$$\left(\lambda s_1 + s_2\right)(p) := \lambda s_1(p) + s_2(p)\,.$$

Example 17. *Let $E = TM$ be the tangent bundle of M. Then a section of E is just a vector field on M.*

Lemma 11. *Let $\mathcal{W} = \{U_i\}_{i\in I}$ be a open covering of M with cocycles $\{g_{V,U}\}$. Let $s_U : U \to \mathbb{F}^n$ be a family of smooth functions. Then $\{s_U\}_{U\in\mathcal{W}}$ defines a section of E if and only if*

$$s_V(p) = g_{V,U}(p)s_U(p) \tag{5.3}$$

for all $U, V \in \mathcal{W}$, and $p \in U \cap V$.

Proof. Assume first that (5.3) holds. Recall that $E \simeq \mathcal{X}/\sim$. As $s_V(p) = g_{V,U}(p)s_U(p)$ it follows that $(p, s_U(p), U) \sim (p, s_V(p), V)$ and we can define $s : M \to E$ by

$$s(p) = [p, s_U(p), U] .$$

Then $s : M \to E$ is a smooth section.

Assume now that $s : M \to E$ is a smooth section. Let $f_U : E_U \to U \times \mathbb{F}^n$ be a local trivialization. Define $\{s_U\}_U$ by

$$s_U(p) := \mathrm{pr}_2 f_U(s(p)) , \qquad p \in U .$$

Then $s_U : U \to \mathbb{F}^n$ is smooth. Suppose that $p \in V \cap U$. Then

$$\begin{aligned} s_V(p) &= \mathrm{pr}_2 f_V(s(p)) \\ &= \mathrm{pr}_2 f_V \circ f_U^{-1}(f_U(s(p))) \\ &= \mathrm{pr}_2 (f_V \circ f_U^{-1})(p, s_U(p)) \\ &= g_{V,U}(p) s_U(p) . \end{aligned}$$

□

6. Constructions with vector bundles

6.1. The dual bundle. Let V be a vector space over $\mathbb{F}$. Denote by $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ the space of continuous linear maps from V to $\mathbb{F}$. If $\dim V < \infty$ then every linear form is automatically continuous. If V is finite dimensional then $V^* \simeq V$. This works actually for any Hilbert space V but in the rest of this section we will assume that V is finite dimensional. Let $(\cdot \mid \cdot)$ be a inner product on V. For $u \in V$ define $f_u \in V^*$ by

$$f_u(x) = (x \mid u) .$$

Then $V \ni u \mapsto f_u \in V^*$ is a cojugate linear isomorphism. In particular let $f \in V^*$. Then there exists a unique $u \in V$ such that

$$f(x) = (x \mid u) \qquad \forall x \in V .$$

If $\{v_1, \ldots, v_n\}$ is a basis for V then the dual basis $\{f_1, \ldots, f_n\}$ for V^* is determined by

$$< f_i, v_i >= \delta_{i,j} .$$

If $F : V \to W$ is a linear map between two finite dimensional vector space, then we define a linear map $F^T : W^* \to V^*$ by

$$< F^T(f), x >=< f, F(x) > .$$

Assume that $W = V$. Let $\{v_1, \ldots, v_n\}$ be a basis and $\{f_1, \ldots, f_n\}$ the dual basis. Then F corresponds to a matrix $A = \{a_{ij}\}$ determined by

$$F(e_j) = \sum_{i=1}^{n} a_{ij} e_i .$$

Similarly F^T corresponds to a matrix $B = \{b_{ij}\}$ in the basis $\{f_1, \ldots, f_n\}$:

$$F^T(f_j) = \sum_{i=1}^{n} b_{ij} f_i$$

Thus

$$a_{ij} =< f_i, F(e_j) >=< F^T(f_i), e_j >= b_{ji}$$

Thus

$$B = A^T .$$

Notice that $A \mapsto A^T$ is an anti-homomorphism in the sense that $(AB)^T = B^T A^T$. Let now $E \xrightarrow{\pi} M$ be a vector bundle with local trivialization $\{f_U\}_{U \in \mathcal{W}}$ and corresponding cocycle $\{g_{V,U}\}$. Then as a set E is the disjoint union:

$$E = \bigcup_{p \in M} E_p$$

Define as a set:

$$E^* = \bigcup_{p \in M} E_p^* .$$

We can then make E^* into a vector bundle with cocycle $h_{V,U}(p) = g_{U,V}(p)^T$. The cocycle relation holds because

$$\begin{aligned} h_{V,W}(p) h_{W,U}(p) &= g_{W,V}(p)^T g_{U,W}(p)^T \\ &= (g_{U,W}(p) g_{W,V}(p))^T \\ &= g_{U,V}(p)^T \\ &= h_{V,U}(p) . \end{aligned}$$

Assume that $E = TM$. The bundle $TM^* = T^*M$ is called **the cotangential bundle of** M. If $f : M \to \mathbb{R}$ is smooth, then we define a section $df : M \to T^*M$ by

$$df(p) X_p = X_p(f) .$$

We also write df_p for $df(p)$. Let $\{U\}_{U \in \mathcal{W}}$ be an open covering of M such that $(U, \mathbf{x}_U)$ are coordinates. For $p \in U$ let

$$\left. \frac{\partial}{\partial x_j} \right|_p = (d\mathbf{x}_U(p))^{-1}(e_j)$$

be the standard basis of $T_p(M)$. Write $\mathbf{x}_U = (x_1, \ldots, x_n)$. Then $x_j : U \to \mathbb{R}$ is smooth and $dx_j : U \to T^*M_U$ is well defined. We have

$$\begin{aligned} dx_j \left(\left. \frac{\partial}{\partial x_i} \right|_p \right) &= \left. \frac{\partial}{\partial x_i} \right|_p (x_j) \\ &= \left. \frac{d}{dt} x_j(\mathbf{x}^{-1}(\mathbf{x}(p) + t e_i)) \right|_{t=0} \\ &= \left. \frac{d}{dt} \mathrm{pr}_j(\mathbf{x}(p) + t e_i) \right|_{t=0} \\ &= \left. \frac{d}{dt} (x_j(p) + t \mathrm{pr}_j(e_i)) \right|_{t=0} \\ &= \delta_{ij} . \end{aligned}$$

Hence $\{dx_1(p),\ldots,dx_n(p)\}$ is the dual base to $\left\{\left.\frac{\partial}{\partial x_1}\right|_p,\ldots,\left.\frac{\partial}{\partial x_n}\right|_p\right\}$. A section of $T^*(M)$ is called **a one-form** on M. Let ω be a one-form. Then locally

$$\omega_U(p)=\sum_{j=1}^n a_i(p)dx_i(p)\,.$$

Whe have

$$p\mapsto a_i(p)=\omega_p\left(\left.\frac{\partial}{\partial x_i}\right|_p\right)$$

and hence $a_i\in C^\infty(U)$. If $(V,\mathbf{y})$ is an other system of coordinates around p. Then

$$dy_j(p)=\sum\left(\frac{\partial y_j}{\partial x_i}\right)_p dx_i(p)$$

and hence

$$\begin{aligned}\omega_V(p)&=\sum_{j=1}^n b_j(p)dy_j(p)\\&=\sum_{j=1}^n b_j(p)\sum_{i=1}^n\left(\frac{\partial y_j}{\partial x_i}\right)_p dx_i(p)\\&=\sum_{i=1}^n\left(\sum_{j=1}^n b_j(p)\left(\frac{\partial y_j}{\partial x_i}\right)_p\right)dx_i(p)\end{aligned}$$

Hence

$$a_i(p)=\sum_{j=1}^n b_j(p)\left(\frac{\partial y_j}{\partial x_i}\right)_p$$

or

$$\begin{pmatrix}a_1(p)\\\vdots\\a_n(p)\end{pmatrix}=\left(\frac{\partial y_j}{\partial x_i}(p)\right)_{i,j=1}^n\begin{pmatrix}b_1(p)\\\vdots\\b_n(p)\end{pmatrix}$$

which shows that the cocycle is smooth. Hence T^*M is a smooth vector bundle over M.

Let V_j, $j=1,\ldots,N$, and W be finite dimensional vector spaces over $\mathbb{F}$. A map $\beta:V_1\times\ldots\times V_N\to W$ is called **multilinear** (or **N-linear**) if for all $j\in\{1,\ldots,N\}$ and fixed $v_i\in V_i$, $i\neq j$ the map

$$V_j\ni x\mapsto\beta(v_1,\ldots,v_{j-1},x,v_{j+1},\ldots,v_N)\in W$$

is linear. Let $M_k(V)$ be the space of k-linear mapst into $\mathbb{F}$. For $k=0$ let $M_k(V)=\mathbb{F}$. If $k=1$ then $M_k(V)=V^*$. If $N=2$ then we say that β is **bilinear**. Let S_k be the permutation group of k-elements. For $\beta\in M_k(V)$ and $\sigma\in S_k$ define

$$\sigma\cdot\beta(v_1,\ldots,v_k):=\beta(v_{\sigma(1)},\ldots,v_{\sigma(k)})\,.$$

This defines an action of S_k on $M_k(V)$.

Definition 11. *The k-linear form $\beta\in M_k(V)$ is said to be **symmetric** if for all $\sigma\in S_k$:*

$$\sigma\cdot\beta=\beta$$

and ***alternating*** *if*

$$\sigma \cdot \beta = \operatorname{sign}(\sigma)\, \beta\,.$$

We denote by $S_k(V)$ the space of symmetric k-forms on V and by $A_k(V)$ the space of alternating k-forms.

Let $\alpha \in A_r(V)$ and $\beta \in A_s(V)$. Define $\alpha \wedge \beta \in A_{r+s}(V)$ by

$$\alpha \wedge \beta(v_1, \dots, v_{r+s}) := \frac{1}{r!s!} \sum_{\sigma \in S_{r+s}} \operatorname{sign}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(r)}) \beta(v_{\sigma(r+1)}, \dots, v_{\sigma(r+s)})$$

Then $A(V) = \oplus_{r=0}^{\infty} A_r(V)$ is an algebra. We notice the following properties:

(1) Let $\alpha \in A_r(V)$ and $\beta \in A_s(V)$. Then

$$\alpha \wedge \beta = (-1)^{rs} \beta \wedge \alpha\,.$$

In particular if r is odd and $\alpha \in A_r(V)$ then $\alpha \wedge \alpha = 0$.

(2) $A_k(V) = \{0\}$ for $k > \dim(V)$.

(3) Let $n = \dim(V)$. Let $f_1, \dots, f_n$ be a basis of V^*. Then

$$\{f_{i_1} \wedge f_{i_2} \wedge \dots \wedge f_{i_k} \mid i_j \in \mathbb{N},\ i_1 < i_2 < \dots < i_n\}$$

is a basis for $A_k(V)$. Thus

$$\dim(A_k(V)) = \binom{n}{k} = \frac{n!}{k!(n-k)!}\,.$$

In particular $\dim A_n = 1$. Define $\det \in A_n(V)$ by

$$\det(x_1, \dots, x_n) := \det\left(f_i(x_j)\right)$$

then $A_n = \mathbb{F}\det$.

(4) Let $A = \oplus_{j \in \mathbb{S}} A_j$ be an algebra, where $\mathbb{S} = \mathbb{Z}_{\geq 0}$ or $\mathbb{S} = \mathbb{Z}/p\mathbb{Z}$. Then A is graded if $A_j A_k \subset A_{j+k}$. Then $A_k(V)$ is a graded algebra.

(5) Let $F : V \to W$ be linear. Define $\wedge^r F : A_k(W) \to A_k(V)$ by

$$\wedge^r F(\alpha)(x_1, \dots, x_r) := \alpha(F(x_1), \dots, F(x_r))\,.$$

Then $\wedge^r F$ is linear. If $G : W \to U$ then

$$\wedge^r (G \circ F) = \wedge^r F \circ \wedge^r G\,.$$

In particular, if F is an isomorphism, then $\wedge^r F$ is an isomorphism.

(6) If $r = \dim V$ then $\wedge^r F(\omega) = \det(F)\omega$ for all $\omega \in A_r(V)$.

We apply this now to the tangent bundle $TM \overset{\pi}{\to} M$. Define

$$A_r(T(M)) = \bigcup_{p \in M} A_r(T_p(M))\,.$$

For each $p \in M$ and $(U, \mathbf{x})$ coordinates around p we have a basis

$$dx_{i_1} \wedge \dots \wedge dx_{i_r}$$

for $A_r(T_p(M))$. If $(V, \mathbf{y})$ is another set of coordinats then, as we have seen

$$dx_j(p) = \sum \left(\frac{\partial x_j}{\partial y_i}\right)_p dy_i(p)\,.$$

Let us first assuming that $r = 2$. Then

$$dy_i \wedge dy_j = -dy_j \wedge dy_i$$

and hence

$$\begin{aligned} dx_{i_1} \wedge dx_{i_2} &= \left(\sum_i \left(\frac{\partial x_{i_1}}{\partial y_i}\right) dy_i\right) \wedge \left(\sum_i \left(\frac{\partial x_{i_2}}{\partial y_i}\right) dy_i\right) \\ &= \sum_{i<j} \left(\frac{\partial x_{i_1}}{\partial y_i}\frac{\partial x_{i_2}}{\partial y_j} - \frac{\partial x_{i_1}}{\partial y_j}\frac{\partial x_{i_2}}{\partial y_i}\right) dy_i \wedge dy_j \\ &= \sum_{i<j} \det\left(\frac{\partial x_{i_a}}{\partial y_b}\right)^2_{a=1,b=i,j} dy_i \wedge dy_j \end{aligned}$$

Similarly we get for general r:

$$\begin{aligned} dx_{i_1} \wedge \ldots \wedge dx_{i_r} &= \left(\sum_i \left(\frac{\partial x_{i_1}}{\partial y_i}\right) dy_i\right) \wedge \ldots \wedge \left(\sum_i \left(\frac{\partial x_{i_r}}{\partial y_i}\right) dy_i\right) \\ &= \sum_{1\leq j_1<\ldots<j_r} \det\left(\frac{\partial x_{i_a}}{\partial y_{i_b}}\right)^r_{a,b=1} dy_{i_1} \wedge \ldots \wedge dy_{i_r}\,. \end{aligned}$$

This gives us the cocycle for for the vector bundle $A_r(T(M))$. Notice that $A_0(TM) = C^\infty(M)$ and $A_1(T(M)) = T^*(M)$.

Definition 12. *We set $\Omega_k(M) = \Gamma(A_k(T(M)))$. The elements ω of Ω_k are called k-differential forms on M. If $m = \dim(M)$, then a volume form on M is an element $\omega \in \Omega_m(M)$ such that $\omega(p) \neq 0$ for all $p \in M$.*

Lemma 12. *Let M, N be manifolds. For $f \in C^\infty(M, N)$, $r \in \mathbb{Z}_{\geq 0}$ and $\omega \in \Omega_r(N)$. Then*

$$(f^*\omega)_p(v_1, \ldots, v_r) := \omega_{f(p)}\left((df)_p(v_1), \ldots, (df)_p(v_r)\right)$$

is in $\Omega_r(M)$ and $f^* : \Omega_r(N) \to \Omega_r(M)$ is contravariant, i.e., $(f \circ g)^* = g^* \circ f^*$. If $r = \dim(M) = \dim(N)$, $\omega \in \Omega_r(N)$, and $\eta \in \Omega_r(M)$, then there exists a smooth function $c : M \to \mathbb{R}$ such that

$$f^*\omega = c\eta\,.$$

Definition 13. *An atlas $\mathcal{A}$ for the manifold M is **oriented** if*

$$\det\left(\frac{\partial x_{Ui}}{\partial x_{Vj}}\right) > 0$$

for all $(U, \mathbf{x}), (V, \mathbf{y}) \in \mathcal{A}$, $U \cap V \neq \emptyset$. The manifold M is said to be orientable if there exists an oriented altas for M.

Lemma 13. *The following is equivalent:*

(1) *M is orientable;*
(2) *There exists a volume form on M.*

From now on we assume that there exists a volume form $\omega \in \Omega_m(M)$. Let $\mathcal{A} = \{(U, \mathbf{x})\}_J$ be an oriented atlas for M. We will now define the symbol

$$\int_M f\omega$$

for any $f \in C_c(M)$. Let $p \in M$. If $(U, \mathbf{x})$ are coordinates around p. Then

$$\omega_U = f_U\, dx_1 \wedge \ldots \wedge dx_n$$

for some $f_U \in C^\infty(U)$. As ω is a volume form $\mathrm{sign}(f_U(q))$ is constant on U. We can assume that $f_U(q) > 0$ for all $q \in U$. Let $(V, \mathbf{y})$ be another set of coordinates. Then

$$\begin{aligned} \omega_U &= f_U\, dx_1 \wedge \ldots \wedge dx_n \\ &= f_U \det\left(\frac{\partial x_i}{\partial y_j}\right) dy_1 \wedge \ldots \wedge dy_n \\ &= \omega_V\,. \end{aligned} \tag{6.1}$$

and $\det\left(\frac{\partial x_i}{\partial y_j}\right) > 0$. Hence $f_V = f_U \det\left(\frac{\partial x_i}{\partial y_j}\right)$ is also positive. Assume that $f \in C_c(M)$ andt $\mathrm{Supp}(f) \subset U$. Define

$$\int f\,\omega := \int_{\mathbb{R}^n} f \circ \mathbf{x}^{-1}(x_1, \ldots, x_n) f_U \circ \mathbf{x}^{-1}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n\,.$$

Recall the following *transformation rule* for integrals of functions on $\mathbb{R}^n$:

Lemma 14. *Let $U, V \subset \mathbb{R}^n$ be open, $U, V \neq \emptyset$. Let $F : U \to V$ be a diffeomorphism. Let $f \in C_c(V)$ then*

$$\int_U f(F(x))\, |\det(DF(x))|\, dx_1 \ldots dx_n = \int_V f(x)\, dx_1 \ldots dx_n$$

and

$$\int_U f(F(x))\, dx_1 \ldots dx_n = \int_V f(x) \left|\det DF^{-1}(x)\right|\, dx_1 \ldots dx_n$$

Assume that $(V, \mathbf{y}) \in \mathcal{A}$ is such that $\mathrm{Supp}(f) \subset V$. Then by (6.1) and the the above Lemma

$$\begin{aligned} &\int_{\mathbf{x}(U\cap V)} f \circ \mathbf{x}^{-1}(x)\ f_U \circ \mathbf{x}^{-1}(x)\, dx \\ &\qquad = \int_{\mathbf{x}(U\cap V)} (f \circ \mathbf{y}^{-1})(\mathbf{y} \circ \mathbf{x}^{-1})(x)\, (f_U \circ \mathbf{y}^{-1}) \circ (\mathbf{y} \circ \mathbf{x}^{-1})\, dx \\ &\qquad = \int_{\mathbf{y}(U\cap V)} f \circ \mathbf{y}^{-1}(y) f_U(\mathbf{y}^{-1})(y) \left|\det\left(D(\mathbf{x} \circ \mathbf{y}^{-1}\right)(y)\right|\, dy \\ &\qquad = \int_{\mathbf{y}(U\cap V)} f \circ \mathbf{y}^{-1}(y) f_V(\mathbf{y}^{-1})(y)\ dy \end{aligned}$$

as

$$\det\left(\frac{\partial(\mathbf{x} \circ \mathbf{y}^{-1})_i}{\partial y_j}\right) = \det\left(\frac{\partial x_j}{\partial y_i}\right)\,.$$

To generalize this to arbitary $f \in C_c(M)$ some preparation is needed. We start with a well known lemma from calculus:

Lemma 15. *The function*

$$\varphi(t) = \begin{cases} e^{-1/t} & 0 < t \\ 0 & \text{otherwise} \end{cases}$$

is smooth on $\mathbb{R}$ and $0 \le \varphi(t) \le 1$ for all $t \in \mathbb{R}$.

Lemma 16. *Let $0 < r < R$. Then there exists a smooth function $\psi \in C_c(\mathbb{R}^n)$ with the following properties:*

(1) $0 \le \psi(x) \le 1$;

(2) $\psi(x) = 1$ *for all* $x \in \overline{B_r(0)} = \{x \in \mathbb{R}^n \mid ||x|| \le r\}$;
(3) $\mathrm{Supp}(\psi) \subset B_R(0)$.

Proof. Choose δ such that $r < \delta < R$. Define $g \in C^\infty(\mathbb{R})$ by

$$g(t) := \varphi(\delta^2 - t)\varphi(t - r^2).$$

Then $0 \le g(t) \le 1$, and $g(t) = 0$ for $t > \delta^2$ and $t < r^2$. Let

$$G(x) := \frac{\int_x^\infty g(t)\,dt}{\int_{-\infty}^\infty g(t)\,dt}.$$

Then G is well defined as $g \ne 0$. Furthermore $G(x) = 0$ if $x > \delta^2$ and $G(x) = 1$ if $x \le r^2$. Finally we define

$$\psi(x) := G(||x||^2).$$

□

Lemma 17. *Let M be a manifold. Then there exists a sequcen $\{K_j\}$, finite or countable infinite, of compact subsets of M such that $K_j^o \ne \emptyset$, $M = \cup_j K_j$ and $K_j \subset K_j^o$.*

Proof. Let $p \in M$ then there exists an open set U such that $p \in U$ and $\overline{U}$ compact. Let $K_1 := \overline{U}$. Assume that we have defined $K_1 \subset K_2^o \subset K_2 \subset \ldots \subset K_j^o \subset K_j$ such that $K_1, \ldots, K_j$ are compact. Then for each $q \in K_j$ chose $U(q)$ open, $q \in U(q)$, $\overline{U(q)}$ compact. Then as K_j is compact there are $q_1, \ldots, q_k \in K_j$ such that $K_j \subset U(q_1) \cup \ldots \cup U(q_k)$. Let

$$K_{j+1} := \overline{U(q_1) \cup \ldots \cup U(q_k)}.$$

Then $K_{j+!}$ is compact and $K_j \subset K_j^o$. As M is second countable it follows that we are done after finite or countable finite number of steps. □

Lemma 18. *Let M^m be a manifold. Then there exists an altals $\mathcal{A} = \{(U_j, \mathbf{x}_j)\}_{j \in \mathbb{J}}$, $\mathbb{J}$ finite or countable, such that:*

(1) $M = \cup U_j$,
(2) *If $x \in M$ then there are only finitely many $j \in \mathbb{J}$ such that $x \in U_j$.*
(3) $\mathbf{x}_j(U_j) = B_2(0)$;
(4) $\left\{\mathbf{x}_j^{-1}(B_1(0))\right\}_{j \in \mathbb{J}}$ *is a covering of M.*

*We call any atlas of this form a **good** atlas.*

Proof. Let $K_1 \subset \ldots \subset K_j \subset K_{j+1}^o \subset \ldots$ be a sequence of compact subsets as in Lemma 17. Let $p \in K_1$ and let $(U, \mathbf{y})$ be local coordinates around p. By replacing U by $U \cap K_2^o$ we can assume that $U \subset K_2^o$ Let $V = \mathbf{y}(U) \subset \mathbb{R}^m$. Then there exits a $R > 0$ such that $B_R(\mathbf{y}(p)) \subset V$. Let $\mathbf{z}(u) := \frac{1}{2R}(\mathbf{y}(\mathbf{u}) - \mathbf{y}(\mathbf{p}))$, $U_1 = U \cap \mathbf{z}^{-1}(B_2(0))$, and $\mathbf{x}_1 = \mathbf{z}_1|_U$. Then $(U_1, \mathbf{x}_1)$ gives local coordinates around p such that 3 holds. Repeting this construction for all $p \in K_1$ using that K_1 is compact, we can find coordinates $(U_1, \mathbf{x}_1)$, [redacted] , $(U_r, \mathbf{x}_1)$ such that 3 holds and $\left\{\mathbf{x}_j^{-1}(B_1(0))\right\}_{j=1}^r$ covers K_1. Let us set $K_j = \emptyset$ if $j \le 0$. Suppose that for $j \ge 2$ we have found $\{(U_i, \mathbf{x}_i)\}_{i=1}^{s_j}$ such that 3 holds, $\cup U_j \subset K_{j+1}^o$, and $K_j \subset \cup \mathbf{x}_i^{-1}(B_1(0))$. Notice that $K_{j+1} \setminus K_j^o$ is compact and $(K_{j+1} \setminus K_j^o) \cap K_{j-1} = \emptyset$. By the above construction we can find local coordinates $(U_i, \mathbf{x}_i)_{i=s_j+1}^{s}$ such that 3 holds,

$$\cup_{i=s_j+1}^{s_{j+1}} U_i \subset K_{j+2}^o \setminus K_{j-1},$$

and

$$K_{j+1} \setminus K_j^o \subset \cup_{i=s_j+1}^{s_{j+1}} U_i \,.$$

In this way we get a finite or countable infinite collection of local coordinates $\{(U_j, \mathbf{x}_j)\}$ such that 1, 3, and 4 holds. Let $x \in M$. Then there is a j such that $x \in K_j \setminus K_{j-1}^o$. By construction it follows that x can also be contained in U_k for $s_{j-1} \leq k \leq s_{j+1}$. □

Lemma 19 (Partition of unity)**.** *Let $\mathcal{A}$ be a good atlas. Then there exists $\psi_j \in C_c^\infty(M)$, $j \in \mathbb{J}$, such that*

(1) $0 \leq \psi_j \leq 1$;
(2) $\mathrm{Supp}(\psi_j) \subset U_j$
(3) $\sum_{j\in\mathbb{J}} \psi_j(x) = 1$ *for all* $x \in M$;

Notice that the sum in (3) is finite. We say that the collection $\{\psi_j\}_{j\in\mathbb{J}}$ is a ***partition of unity subordinate to the altals*** $\mathcal{A}$.

Proof. Let $j \in \mathbb{J}$. Let ϕ be the function ψ in Lemma 16 for $r = 1$ and $R = 2$. Define

$$\varphi_j(q) := \begin{cases} \phi(\mathbf{x}_j(q)) & q \in U_j \\ 0 & \text{otherwise} \end{cases} \,.$$

The function φ_j is smooth as $\phi|_{\mathbf{x}_1(U_1)\setminus\overline{B_\delta(0)}} = 0$. Furthermore $\varphi_j(q) = 1$ for $q \in \mathbf{x}_j^{-1}(B_1(0))$ and

$$\mathrm{Supp}\,(\varphi_j) \subset \overline{\mathbf{x}_j^{-1}(B_\delta(0))} \subset \mathbf{x}_j^{-1}(B_2(0)) \subset U_j \,.$$

Let $\Psi(x) := \sum_{j\in\mathbb{J}} \varphi_j(x)$. Then $\Psi \in C^\infty(M)$ – as $M = \cup \mathbf{x}_j^{-1}(B_1(0))$ – $\Psi(x) > 0$ for all x. Hence $1/\Psi$ is smooth. Define now

$$\psi_j := \frac{\varphi_j}{\Psi} \in C^\infty(M) \,.$$

Then $0 \leq \psi_j(x) \leq 1$ and $\sum \psi_j(x) = \frac{\Psi(x)}{\Psi(x)} = 1$. □

Let $\mathcal{A}$ be a good atlas for M and $\{\psi_j\}$ a partition of unity subordinate to $\mathcal{A}$. If $f \in C_c(M)$ then $\mathrm{Supp}(f\psi_j) \subset U_j$ is compact and hence

$$\int f\psi_j\,\omega$$

is well defined. Furthermore $\mathrm{Supp}(f) \subset \cup_{\text{finite}} U_j$. Define

$$\int f\omega := \sum \int (f\psi_j)\,\omega \,.$$

Assume that $\{V_j\}_j$ is another good atlas and $\{\varphi_j\}$ a subordinate partition of unity. Then $\mathrm{Supp}(f\varphi_i\psi_j) \subset U_j \cap V_i$ and hence

$$\begin{aligned}\sum_j \int (f\psi_j)\,\omega &= \sum_j\sum_i \int (f\psi_j)\varphi_i\,\omega \\ &= \sum_i\sum_j \int (f\varphi_i)\,\psi_j\,\omega \\ &= \sum_i \int (f\varphi_i)\,\omega\end{aligned}$$

and therefore $\int f\omega$ is well defined. Furthermore there exists a unique Radon measure μ_ω on M such that

$$\int f\omega = \int_M f\, d\mu_\omega .$$

We denote by $L^p(M, d\mu_\omega)$ the corresponding L^p-spaces.

7. Invariant integration

Let $F : M \to M$ be a diffeomorphism. Then $(F^*\omega)_p = \det(dF)_p\omega_p$. By the transformation formula for integrals in $\mathbb{R}^n$ we get:

Lemma 20. *Let $F : M \to M$ be a diffeomorphism. Then*

$$\int f\, F^*\omega = \int f \circ F^{-1}\, \omega$$

for all $f \in C_c(M)$.

Let G be a linear Lie group acting on M. Denote by $\tau(g)$ the corresponding diffeomorphism. Define an action of G on $\Omega_k(M)$ by

$$g \cdot \omega = \tau(g^{-1})^*\omega .$$

If ω is a volume form then $(g\cdot\omega)_p = c(g,p)\omega_p$ for some smooth function $c : G\times M \to \mathbb{R}^+$. In particular if ω is G-invariant then $c(g,p) = 1$ for all g and p and

$$\int f(g\cdot p)\, d\mu_\omega = \int f\, d\mu_\omega$$

for all f. In particular $\mu_\omega(g\cdot E) = \mu_\omega(E)$ for all measuralbe sets E. Thus μ_ω is **G-invariant**.

Lemma 21. *Let G be a Lie group. Then the following holds:*

(1) *Let $f_1, \dots, f_n \in \mathfrak{g}$ be a bais. Define $\omega_j(g) = f_j \circ d\lambda(g)^{-1}$. Then $\omega_j \in \Omega_1(G)^G$.*

(2) *The map*

$$\Omega_k(G)^G \ni \omega \mapsto \omega(e) \in A_k(\mathfrak{g})$$

is an isomorphism with inverse

$$A_k(\mathfrak{g}) \ni \eta \mapsto (g \mapsto \eta \circ d\lambda(g)^{-1}) \in \Omega_k(G)^G .$$

In particular $\{\omega_{i_1} \wedge \dots \wedge \omega_{i_k} \mid i_1 < \dots < i_k\}$ is a basis for $\Omega_k(G)^G$.

(3) $\dim\Omega_n(G)^G = 1$. *In particular there exists a volume form ω which is G-invariant.*

Proof. Let $f_1, \dots, f_n \in \mathfrak{g}^*$ be a basis. Denote by $\lambda(g) : G \to G$ the left multiplication $x \mapsto g\cdot x$. Recall that $d\lambda(g) : \mathfrak{g} \to T_g(G)$ is an isomorphism. Define $\omega_j \in \Omega_1(G)$ by

$$\omega_j(g) = f_j \circ d\lambda(g)^{-1} .$$

Then

$$\begin{aligned}
(\lambda(g)^*\omega_j)(x) &= \omega_j(gx) \circ d\lambda(g)_x \\
&= f_j \circ \big(d\lambda(x)^{-1} \circ d\lambda(g)^{-1} \circ d\lambda(g)\big) \\
&= f_j \circ d\lambda(x)^{-1} \\
&= \omega_j(x) .
\end{aligned}$$

It follows that that ω_j is G-invariant. Let $i_1 < \ldots < i_k$ and define

$$\omega = \omega_{i_1} \wedge \ldots \wedge \omega_{i_k} .$$

Then $\omega \in \Omega_k(G)$ is a G-invariant.

On the other hand let $\omega \in \Omega_k(G)$. Then $\omega(e) \in A_k(\mathfrak{g})$. Hence

$$\omega(e) = \sum_{i_1 < \ldots < i_k} a_{i_1,\ldots,i_k} f_1 \wedge \ldots \wedge f_{i_k}$$

for some $a_{i_1,\ldots,i_k} \in \mathbb{R}$. We have

$$\begin{aligned} \omega(g) \circ d\lambda(g) &= \left(g^{-1} \cdot \omega\right)_e \\ &= \omega(e) \end{aligned}$$

because ω is G-invariant. Thus

$$\begin{aligned} \omega(g) &= \omega(e) \circ d\lambda(g)^{-1} \\ &= \sum_{1_1 < \ldots < i_k} a_{i_1,\ldots,i_k} \omega_{i_1} \wedge \ldots \wedge \omega_{i_k} . \end{aligned}$$

This proves (1) and (2). (3) follows now directly. □

Fix a G-invariant measure μ_G on G corresponding to a left invariant volume form ω. It is unique op to multiplication my a positve scalar. Any such choice is called a (left)-Haar measure on G. We will often write write dx for $d\mu_G$. If G is compact, then the constant function $x \mapsto 1$ is compactly supported and hence integrable and its integral is $\mu_G(G) < \infty$. We will allways assume that $\mu_G(G) = 1$ if G is compact. The Haar measure is in general **not** invariant under right multiplication $\rho(g)x = xg$. The group G is called **unimodular** if $\rho(g)^*\omega = \omega$. We notice the following:

$$\begin{aligned} \int f(xg)\, d\mu &= \int f(g^{-1}xg)\, d\mu \\ &= \int f \circ \left(\mathrm{Int}(g^{-1})\right)\, \omega \\ &= \int f \left(\mathrm{Int}(g)^*\omega\right) . \end{aligned}$$

Thus

$$\rho(g)^*\omega = \mathrm{Int}(g)^*\omega.$$

As $\rho(g)^*\omega$ is G-invariant it follows that

$$\rho(g)^*\omega = c(g)\omega .$$

As $d\mathrm{Int}(g)_e = \mathrm{Ad}(g)$ it follows that

$$c(g) = \det\left(\mathrm{Ad}(g)\right) .$$

Thus we have proved that:

Theorem 8. *Let G be a linear Lie group with left-invariant volume form ω. Then*

$$\rho(g)^*\omega = \det\left(\mathrm{Ad}(g)\right)\omega$$

for all $g \in G$. In particular G is unimodular if and only if $\det \mathrm{Ad}(g) = 1$ *for all $g \in G$.*

Lemma 22. *Let G be a compact Lie group. Then G is unimodular.*

Proof. The map $G \ni g \mapsto \det \mathrm{Ad}(g) \in \mathbb{R}^+$ is a continuous homomorphims. Hence $\det \mathrm{Ad}(G)$ is a compact subgroup (under multiplication) of $\mathbb{R}^+$. If there exists a $g \in G$ such that $c = \det \mathrm{Ad}(g) \neq 1$. Then we can assume that $\det \mathrm{Ad}(g) > 1$ (otherwise replace g by g^{-1}). But then

$$c^n = \det \mathrm{Ad}(g^n) \to \infty$$

contradicting the compactness of $\det \mathrm{Ad}(G)$. □

Let $M = G/H$ where H is compact. Then if $f \in C_c(M)$ the function $f \circ \kappa$, $\kappa : G \to G/H$ the canonical map, is compactly supported and we can define

$$\int_M f \, d\mu_M = \int f \circ \kappa(x) \, dx$$

Then $d\mu_M$ is G-invariant. It follows that:

Theorem 9. *Let $M = G/H$ be a homogeneous G-space with H compact. Then there exists an unique (up to scalar) G-invariant measure on M.*

Proof. We have seen that

$$f \mapsto \int f \circ \kappa(x) \, d\mu_M(x)$$

defines an invariant measure on M. Suppose that σ is a G-invariant measure on M. Fix a Haar-measure dh on H such that $\int dh = 1$. Let $f \in C_c(G)$. Then, as H is compact,

$$f^\diamond(x) = \int f(xh) \, dh$$

is continuous with compact support, and right H-invariant. Define $g : G/H \to \mathbb{C}$ by

$$g(xH) = f^\diamond(x) .$$

Then $g \in C_c(G/H)$ and hence $\int_M g(m) \, d\sigma(m)$ is well defined. We can therefore define a Radon measure η on G by

$$\begin{aligned} \int f(x) \, d\eta(x) &= \int g(xH) d\sigma(x) \\ &= \int_{G/H} \int_H f(xh) \, dh \, d\sigma(xH) . \end{aligned}$$

By the G-invariance of μ_M it follows that η is G-invariant. Thus there exists a $c > 0$ such that

$$\eta = c d\mu_G .$$

But then $d\sigma = c d\mu_M$. □

Lemma 23. *Assume that G is a connected Lie group such that*

$$[\mathfrak{g}, \mathfrak{g}] = \{[X, Y] \mid X, Y \in \mathfrak{g}\} = \mathfrak{g} .$$

Then G is unimodular.

Proof. Let $X \in \mathfrak{g}$. Then

$$\det \mathrm{Ad}(e^X) = e^{\mathrm{Tr}(\mathrm{ad}(X))} .$$

Choose $X_j, Y_j \in \mathfrak{g}$ such that

$$X = \sum [X_j, Y_j] .$$

Then

$$\begin{aligned}\operatorname{ad}(X) &= \sum_j \operatorname{ad}[X_j, Y_j] \\ &= \sum [\operatorname{ad}(X_j), \operatorname{ad}(Y_j)] \\ &= \sum_j \left(\operatorname{ad}(X_j)\operatorname{ad}(Y_j) - \operatorname{ad}(Y_j)\operatorname{ad}(X_j)\right) .\end{aligned}$$

As

$$\operatorname{Tr}\left(\operatorname{ad}(X_j)\operatorname{ad}(Y_j) - \operatorname{ad}(Y_j)\operatorname{ad}(X_j)\right) = 0$$

it follows that

$$\operatorname{Tr}(\operatorname{ad}(X)) = 0\,.$$

Hence

$$\det\left(\operatorname{Ad}(e^X)\right) = 1\,.$$

As $\exp : \mathfrak{g} \to G$ is a local diffeomorphism around 0 it follows that there exists an open neighborhood $U \ni e$ in G such that $\det \operatorname{Ad}(g) = 1$ for all $g \in U$. As $G = \cup U^n$ it finally follows that $\det \operatorname{Ad}(g) = 1$ for all $g \in G$. □

Example 18. *Here is an example of a linear group that is not unimodular. Let*

$$G = \left\{ x(a,x) = \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \mid a > 0,\, x \in \mathbb{R} \right\} .$$

For $f : G \to \mathbb{C}$ *write* $f(a,b) = f(x(a,b))$*. Then if* $f \in C_c(G)$*:*

$$\int f\, d\mu = \int_{-\infty}^{\infty} f(x(a,x))\, \frac{dadx}{a^2}\,.$$

Then

$$\int f\left(\begin{pmatrix} b & y \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\right)\frac{dadx}{a^2} = \int f\left(ab, bx + a^{-1}y\right)\frac{dadx}{a^2}\,.$$

Let

$$u = ba \qquad \text{and} \qquad v = bx + a^{-1}y$$

Then

$$du = bda \qquad \text{and} \qquad dv = bdx$$

Hence

$$\frac{dadb}{a^2} = \frac{dudv}{u^2}$$

and hence

$$\int f\left(\begin{pmatrix} b & y \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\right)\frac{dadx}{a^2} = \int f(a,x)\,\frac{dadx}{a^2}\,.$$

It follows that μ *is left-invariant. As*

$$\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} b & y \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} ab & ay + b^{-1}x \\ 0 & (ab)^{-1} \end{pmatrix}$$

it follows that

$$\int f\left(\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} b & y \\ 0 & b^{-1} \end{pmatrix}\right)\frac{dadx}{a^2} = \int f(ab, ay + b^{-1}x)\,\frac{dadx}{a^2}\,.$$

Let $u = ab$, and $v = ay + b^{-1}x$. Then $du = bda$ and $dv = b^{-1}dx$. Thus

$$\int f(ab, ay + b^{-1}x)\,\frac{dadx}{a^2} = b^2 \int f(u,v)\,\frac{dudv}{u^2}\,.$$

We can also see this by using Theorem 8. Choose a basis

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

for $\mathfrak{g}$. Let $g = x(a,x) \in G$. Then

$$\begin{aligned} \mathrm{Ad}(g)X &= \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix} \\ &= X \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ad}(g)Y &= \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix} \\ &= a^2 Y\,. \end{aligned}$$

It follows that

$$\det \mathrm{Ad}(x(a,x)) = a^2\,.$$

Exercises:

(1) Show that $\sigma \cdot \beta(v_1,\dots,v_k) := \beta(v_{\sigma(1)},\dots,v_{\sigma(k)})$ defines an action of S_k on $M_k(V)$.

(2) Define a map $\mathrm{pr} : M_k(V) \to M_k(V)$ by

$$\mathrm{pr}(\beta)(v_1,\dots,v_k) := \frac{1}{k!}\sum_{\sigma\in S_k} \mathrm{sign}(\sigma)\beta(v_{\sigma(1)},\dots,v_{\sigma(k)})\,.$$

Show that pr is a projection onto $A_k(V)$.

(3) Let $\varphi \in M_k$. Show that φ is alternating if and only if $v_i = v_j$, $j \neq i$, implies $\varphi(v_1,\dots,v_k) = 0$.

(4) Let M^m be a manifold. Define a map $d : \Omega_k(M) \to \Omega_{k+1}(M)$ by

$$\begin{aligned} d &\sum_{1\le i_1<\ldots<i_k\le m} f_{i_1,\dots,i_k}\,dx_{i_1}\wedge\ldots\wedge dx_{i_k} \\ &:= \sum_{1\le i_1<\ldots<i_k\le m} df_{i_1,\dots,i_k} \sum_{1\le i_1<\ldots<i_k\le m} f_{i_1,\dots,i_k}\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}\,. \end{aligned}$$

(a) If $X_1,\dots,X_{k+1} \in \mathcal{X}(M)$ then

$$\begin{aligned} d\omega(X_1,\dots,X_{k+1}) &= \sum_{j=1}^{k+1}(-1)^{j+1}X_j\left(\omega(X_1,\dots,\hat{X}_j,\dots,X_{k+1})\right) \\ &\quad + \sum_{i<j}(-1)^{i+j}\omega\left([X_i,X_j],X_1,\dots,\hat{X}_i,\dots\hat{X}_j,\dots,X_{k+1}\right) \end{aligned}$$

where $\hat{X}_n$ indicates that the element X_n should not be counted.

(b) Show that d is well defined.

(c) Show that $d^2 = 0$. In particular $\mathrm{im}(d) \subset \ker(d)$.

(d) If $\omega \in \Omega_r$ and $\eta \in \Omega_s$ then $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^r\omega\wedge d\eta$.

(5) Show that the function

$$\varphi(t) = \begin{cases} e^{-1/t} & 0 < t \\ 0 & \text{otherwise} \end{cases}$$

is smooth on $\mathbb{R}$.

8. Representations

Let $(\mathbf{H}, (\cdot \mid \cdot))$ be a Hilbert space over $\mathbb{F}$. Denote by $\mathrm{GL}(\mathbf{H})$ the group of continuous, invertible linear maps from to $\mathbf{H}$.

Definition 14. *Let G be a Lie group and $(\mathbf{H}, (\cdot \mid \cdot))$ a Hilbert space. A representation of G in $\mathbf{H}$ is a homomorphism $\rho : G \to \mathrm{GL}(\mathbf{H})$ such that the map*

$$G \times \mathbf{H} \ni (a, v) \mapsto \rho(a)v \in \mathbf{H}$$

is continuous. The representation ρ is unitary if $\rho(g)$ is unitary for all $g \in G$.

Remark 3. *If ρ is unitary, then $\rho(g)^* = \rho(g^{-1})$.*

Most of the time we will only consider complex Hilbert spaces, but there are some very natural representations in real Hilbert spaces. The most natural beeing the canonical embedding $G \subset \mathrm{GL}(n, \mathbb{R})$ for a linear group G. We will call representations in a real Hilbert space **real representations** and use the notion representations for representations in a complex Hilbert space. We write $(\rho, \mathbf{H})$ for ρ a representation of G in the Hilbert space $\mathbf{H}$. We list here some standard definitions for representations. From now on $(\rho, \mathbf{H})$, $(\pi, \mathbf{K})$ etc. will denote representations of G. We say that $\dim \mathbf{H}$ is the dimension of ρ, denoted by $d(\rho)$.

(1) Let $(\rho, \mathbf{H})$ and $(\pi, \mathbf{K})$ be two representations of G. An intertwining operator is a continuous linear operator $T : \mathbf{H} \to \mathbf{K}$ such that

$$T\rho(g) = \pi(g)T$$

for all $g \in G$. We denote by $I(\rho, \pi)$ the space of intertwining operators. If $\pi = \rho$, then we set $I(\rho) = I(\rho, \rho)$.

Lemma 24. *Let ρ be a unitary representation of G. Then $I(\rho)^* = I(\rho)$.*

Proof. Let $T \in I(\rho)$, $u, v \in \mathbf{H}$ and $g \in G$. Then

$$\begin{aligned} (u \mid T^*\rho(g)v) &= (Tu \mid \rho(g)v) \\ &= (\rho(g^{-1})Tu \mid v) \\ &= (T\rho(g^{-1})u \mid v) \\ &= (\rho(g^{-1})u \mid T^*v) \\ &= (u \mid \rho(g)T^*v) . \end{aligned}$$

As this holds for all u and v it follows that

$$T^*\rho(g) = \rho(g)T^*$$

and hence $T^* \in I(\rho)$. As $T = T^{**}$ it follows that $I(\rho)^* = I(\rho)$. □

(2) The representations ρ and π are said to be **equivalent** if $I(\rho, \pi)$ contains an isomorphis. If ρ and π are unitary and $I(\rho, \pi)$ contains an unitary isomorphism, then ρ and π are said to be **unitary equivalent**. We say that ρ is a **subrepresentations** of π if $I(\rho, \pi)$ contains an injective operator.

(3) A subspace $\mathbf{L} \subset \mathbf{H}$ is **invariant** if $\rho(g)\mathbf{L} \subset \mathbf{L}$ for all $g \in G$. Let $\mathbf{L} \subset \mathbf{H}$ be an invariant subspace. Then $(\rho_{\mathbf{L}}, \mathbf{L})$ defined by $\rho_{\mathbf{L}}(g) := \rho(g)|_{\mathbf{L}}$ for all $g \in G$, is a representation of G in $\mathbf{L}$. We say that ρ is **irreducible** if the only invariant subspaces of $\mathbf{H}$ are $\mathbf{H}$ and $\{0\}$.

(4) Let $\mathbf{L} \subset \mathbf{H}$ be invariant. Then we can define a representation of G in $\mathbf{H}/\mathbf{L}$ by
$$\rho_{\mathbf{H}/\mathbf{L}}(g)(u + \mathbf{L}) := \rho(g)u + \mathbf{L}\,.$$
The representation $(\pi, \mathbf{K})$ is a quotient of ρ if π is equivalent to $\rho_{\mathbf{H}/\mathbf{L}}$ for some invariant subspace $\mathbf{L} \subset \mathbf{H}$.

Example 19. *Let $G = \mathrm{SO}(n)$ and let $\mathbb{C}^k[\mathbb{R}^n]$ be the space of polynomials of degree $\leq k$. Let $\mathbb{C}_k[\mathbb{R}^n]$ be the subspace of polynomials of degree k. This is exactly the space of polynomials that are homogeneous of degree k i.e.,*
$$p(\lambda \mathbf{x}) = \lambda^k p(\mathbf{x})\,.$$
Define a representation ρ of G on $\mathbb{C}^k[\mathbb{R}^n]$ by
$$[\rho_k(g)p](x_1, \ldots, x_n) := p(g^{-1}x_1, \ldots, g^{-1}x_n)\,.$$
This representation is not irreducible because $\mathbb{C}_k[\mathbb{R}^n]$ is invariant. Thus the restriction of ρ_k to the space of polynomial that are homogeneous of degree n is a representation of G. But even this representation is not irreducible.

Lemma 25. *The multiplication operator $M : \mathbb{C}^{k-2}[\mathbb{R}^n] \to \mathbb{C}^k[\mathbb{R}^n]$, $p(\mathbf{x}) \mapsto (x_1^2 + \ldots + x_n^2)p(x) = ||x||^2\, p(x)$ is an intertwining operator. In particular it follows that $\mathrm{Im}(M)$ is an invariant subspace. If $n \geq 2$ then $\mathrm{Im}(M) \neq \mathbb{C}^k[\mathbb{R}^n]$.*

Proof. We have
$$\begin{aligned}(M\rho_{k-2}(g)p)(x) &= ||x||^2 \rho_{k-2}(g)p(x) \\ &= ||x||^2 p(g^{-1}x) \\ &= ||g^{-1}x||^2 p(g^{-1}x) \\ &= \rho_k(g)(Mp)(x)\end{aligned}$$
where we have used that $||g^{-1}x|| = ||x||$ because g, and hence also g^{-1}, is orthogonal. That $\mathrm{Im}(M)$ is invariant follows now from lemma **??**. If $n \geq 2$ then $x_1^k \notin \mathrm{Im}(M)$ and hence $\mathrm{Im}(M) \neq \mathbb{C}^k[\mathbb{R}^n]$. □

(1) Let $(\rho_j, \mathbf{H}_j)$, $j \in \mathbb{J}$, be a collection of unitary representations, where $\mathbb{J}$ is a finite or countable index set. Let
$$\bigoplus_{j \in \mathbb{J}} \mathbf{H}_j := \left\{ (u_j)_{j \in \mathbb{J}} \mid u_j \in \mathbf{H}_j\,,\ \sum_{j \in \mathbb{J}} ||u_j||^2 < \infty \right\}$$
with inner product
$$((u_j) \mid (v_j)) := \sum_{j \in \mathbb{J}} (u_j \mid v_j)_{\mathbf{H}_j}$$
Then $\bigoplus_{j \in \mathbb{J}} \mathbf{H}_j$ is a Hilbert space. Define $\oplus_j \rho_j : G \to \mathrm{GL}\left(\bigoplus_{j \in \mathbb{J}} \mathbf{H}_j\right)$ by
$$(\oplus_j \rho_j)\,(g)\,(u_j) := (\rho_j(g)u_j)\ .$$
Then $\oplus_j \rho_j$ is an unitary representation. The representation $\oplus_j \rho_j$ is called the **direct sum of** $(\rho_j)_{j \in \mathbb{J}}$.

Lemma 26. *Let $(\rho, \mathbf{H})$ be unitary and $\mathbf{L} \subset \mathbf{H}$ be invariant $\{0\} \neq \mathbf{L} \neq \mathbf{H}$. Then*

$$\mathbf{L}^{\perp} = \{u \in \mathbf{H} \mid \forall v \in \mathbf{L} : (u \mid v) = 0\}$$

is G-invariant and $\mathbf{H} = \mathbf{L} \oplus \mathbf{L}^{\perp}$.

Proof. Let $v \in \mathbf{L}$, $u \in \mathbf{L}^{\perp}$, and $g \in G$. Then

$$(v \mid \rho(g)u) = (\rho(g^{-1})v \mid u) = 0$$

and hence $\rho(g)u \in \mathbf{L}^{\perp}$. □

This lemma is not valid in general if we do not assume that ρ is unitary. Define 2-dimensional representation ρ of $\mathbb{R}$ by

$$\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Then $\mathbf{L} = \mathbb{C}e_1$ is invarinat. We have $\mathbf{L}^{\perp} = \mathbb{C}e_2$ but

$$\rho(t)e_2 = te_1 + e_2 \notin \mathbf{L}^{\perp}.$$

Assume that we have some invariant subspace $\mathbf{K}$ such that $\mathbb{C}^2 = \mathbf{L} \oplus \mathbf{K}$. Let $u \in \mathbf{K}$, $u \neq 0$. Then $u = ae_1 + be_2$, with $b \neq 0$. Furhtermore

$$\rho(t)u = (a + tb)e_1 + be_2$$

Let $b = -a/b$. Then it follows that $e_2 \in \mathbf{L}$. But $\dim \mathbf{L} = 1$ and hence $\mathbf{L} = \mathbb{C}e_2$ contradicting the fact that $\mathbb{C}e_2$ is not invariant.

(2) It is too much to ask for that every unitary representation is a direct sum of irreducible representations. Let $(\rho, \mathbf{H})$ be a representation. A vector $u \in \mathbf{H}$ is called **cyclic** if $\rho(G)u$ spanns $\mathbf{H}$.

Theorem 10. *Let ρ be a unitary representations. Then ρ is unitary equivalent to a direct sum of cyclic representations.*

The following theorem is also true for infinite dimensional representations, but the prove requires the spectral theorem for selfadjoint operators in a Hilbert space, so we only formulate it for finite dimensional representations.

Theorem 11 (Schur's Lemma)**.** *Let ρ be a finite dimensional irreducible unitary representations of G. Then $I(\rho) = I(\rho, \rho) = \mathbb{C}\mathrm{id}$.*

Proof. Let $T \in I(\rho)$. Then

$$\begin{aligned} T &= \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*) \\ &= \frac{1}{2}(T + T^*) + \frac{1}{2i}(i(T - T^*)) \end{aligned}$$

As the operators $T + T^*$ and $i(T - T^*)$ are both symmetric it is enough to show that every symmetric intertwining operator is of the form $\lambda\mathrm{id}$ for some $\lambda \in \mathbb{C}$. So we can assume that T is symmetric. But then there exists an eigenvalue $\lambda \in \mathbb{C}$. Let

$$\mathbf{H}(\lambda) := \{u \in \mathbf{H} \mid T(u) = \lambda u\} \neq \{0\}.$$

Let $g \in G$ and $u \in \mathbf{H}(\lambda)$ then

$$T(\rho(g)u) = \rho(g)T(u) = \lambda\rho(g)u.$$

Hence $\mathbf{H}(\lambda)$ is G-invariant. As ρ is irreducible it follows that $\mathbf{H} = \mathbf{H}(\lambda)$. □

Theorem 12. *Let ρ be a unitary representation of G. Then ρ is unitary equivalent to a direct sum of cyclic representations.*

Proof. Let $u \in \mathbf{H}$ be non-zero. Let

$$\mathbf{H}(u) = [\rho(G)u] := \overline{\left\{ \sum_{\text{finite}} \lambda_j \rho(g_j) u \mid \lambda_j \in \mathbb{C},\ g_j \in G \right\}} .$$

Then $\rho|\mathbf{H}(u)$ is a cyclic sub-representation of ρ. If $\mathbf{H}(u) = \mathbf{H}$ then we are done. If $\mathbf{H}(u) \neq \mathbf{H}$ let $\mathcal{S}$ be the collection of all sets $\mathcal{U} = \{\mathbf{H}_i\}_{i \in I}$ where $\mathbf{H}_i$ is a cyclic sub-representation of $\mathbf{H}$ and $\mathbf{H}_i \perp \mathbf{H}_j$ if $i \neq j$. Then $\{\mathbf{H}(u)\} \in \mathcal{S}$ so $\mathcal{S} \neq \emptyset$. For $\mathcal{U}, \mathcal{W} \in \mathcal{S}$ let $\mathcal{U} \leq \mathcal{W}$ if $\mathcal{U} \subset \mathcal{W}$. Let C be a chain in $\mathcal{S}$. Then

$$\mathcal{V} := \bigcup_{\mathcal{U} \in C} \mathcal{U} \in \mathcal{S}$$

and $\mathcal{U} \leq \mathcal{V}$ for all $\mathcal{U} \in C$. It follows that $\mathcal{S}$ has a maximal element. Let $\mathcal{U} \in \mathcal{S}$ be maximal. Let

$$\mathbf{W} := \bigoplus_{\mathbf{U} \in \mathcal{U}} \mathbf{U} .$$

Then each $\mathbf{U}$ is cyclic. We claim that $\mathbf{W} = \mathbf{H}$. If $\mathbf{H} \neq \mathbf{W}$. Then $\mathbf{W}^{\perp} \neq \{0\}$ is G-invariant. Let $u \in \mathbf{W}^{\perp}$, $u \neq 0$, and let $\mathbf{H}(u) = [\rho(G)u]$ as before. Then $\mathbf{H}(u)$ is cyclic and perpendicular to all $\mathbf{U} \in \mathcal{U}$. Hence $\mathcal{U} \cup \{\mathbf{H}(u)\} \in \mathcal{S}$ and $\mathcal{U} \subsetneqq \mathcal{U} \cup \{\mathbf{H}(u)\}$ contradicting the maximality of $\mathcal{U}$. Hence $\mathbf{W} = \mathbf{H}$. □

9. Finite dimensional representations and G-bundles

Constructions with vector bundles: $Hom(E, F)$, E^*, $E \otimes F$, $\otimes^r E$, $\wedge^r E$, differential forms, integration on manifolds, invariant measure on Lie groups, the modular function. Invariant measure on G/K, G compact, $K \subset G$, close, G-bundles, basic representation theory, hermitian structure, induced representations, representations of compact groups and compact symmetric spaces. Spherical harmonics.