## Math 2025, What to learn for the final

- Time: Monday, Dec. 9 at 7:30–9:30 AM.
- Where: The usual class room, Lockett 239

- Look at all the old tests, quizzes, and homework!

- Functions, draw the graph of simple functions.
- Translation and dilation of functions. Draw the graph of $f(t)$ and $f(\lambda t - r)$.
- Find $\lambda$ and $r$ using the graph.
- The ordered Haar wavelet transform and its inverse.
- The in place Haar wavelet transform and its inverse.
- The Haar wavelet transfrom of a function of two variables given by a $2 \times 2$ matrix and by a $4 \times 4$ matrix.
- Tensorproduct of functions, $f \otimes g(x, y) = f(x)g(y)$.
- How to represent a function on $[0, 1) \times [0, 1)$ by a matrix ($2 \times 2$ and $4 \times 4$ matrix).
- Vector spaces. Be able to decide if a given set is a vector space or not. Examples:
  (1) The kernel of a linear map $\{\mathbf{v} \in \mathbf{V} \mid T(\mathbf{v}) = \mathbf{0}\}$.
  (2) The space of solutions to a system of linear equations like

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + \dots + a_n x_n = 0\}$$

  where $a_1, \dots, a_n$ are real numbers. If there is **anything** other than zero in the right hand side, then the set is **not** a vector space.
  (3) The subspace spanned by elements $\mathbf{v}_1, \dots, \mathbf{v}_k$, i.e.,

$$\mathbf{W} = \left\{ \sum_{j=1}^{k} s_j \mathbf{v}_j \mid s_1, \dots, s_k \in \mathbb{R} \right\}.$$

  The space of polynomials of degree $\leq k - 1$ is an examples of this, and so are the wavelet spaces $V_N$.
- Linear map. Be able to decide if a given map is linear or not. Examples are:
  (1) All maps of the form $A\mathbf{x}$ where $A$ is a $m \times n$ matrix. Notice that those maps have the form

$$[a_{ij}] \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j} x_j \\ \cdot \\ \cdot \\ \cdot \\ \sum_{j=1}^n a_{mj} x_j \end{bmatrix}.$$

  (2) Differentiation and integration,
  (3) If you see any products like $x_i x_j$ or $f' f$, or powers like $x_j^k$, $k > 1$, then the map is not linear,
  (4) If you see a nonzero constant added like $x + 2y + 3x + 1$ then the map is not linear. (The zero vector is not mapped to the zero vector).
- Inner products $< \cdot, \cdot >$ and norms in

(1) $\mathbb{R}^n$, $< \mathbf{x}, \mathbf{y} >= \sum_{j=1}^n x_j y_j$;
(2) $\mathbb{C}^n$, $< \mathbf{z}, \mathbf{w} >= \sum_{j=1}^n z_j \overline{w_j}$;
(3) and on spaces of functions on the interval $[0, 1)$. If the functions are real valued then $< f, g >= \int_0^1 f(t)g(t)\,dt$. If they are complex valued then $< f, g >= \int_0^n f(t)\overline{g(t)}\,dt$.
(4) Evaluate the inner product using the wavelet functions $\varphi_j^N$;
(5) Evaluate the norm of functions like $\varphi_j^N$, polynomials, and simple functions like cos and sin.

Recall that $\|\mathbf{x}\| = \sqrt{< \mathbf{x}, \mathbf{x} >}$ and that only the zero vector has norm zero. Furthermore the norm is always nonnegative.

- Orthogonal vectors, $< \mathbf{v}, \mathbf{u} >= 0$.
- Orthogonal spanning sets; the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is orthogonal and every vector in $\mathbf{W}$ can be written as a linear combination $\mathbf{v} = \sum_{j=1}^k s_k \mathbf{v}_j$. Recall that the constants $s_j$ are then given by

$$s_j = \frac{< \mathbf{v}, \mathbf{v}_j >}{\|\mathbf{v}_j\|^2} \, .$$

- Orthogonal projections: If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is an orthogonal spanning set for the subspace $\mathbf{W} \subset \mathbf{V}$, then the orthogonal projection $P_{\mathbf{W}} : \mathbf{V} \to \mathbf{W}$ is given by

$$P_{\mathbf{W}}(\mathbf{v}) = \sum_{j=1}^n \frac{< \mathbf{v}, \mathbf{v}_j >}{\|\mathbf{v}_j\|^2} \mathbf{v}_j \, .$$

(1) Be able to work out the orthogonal projection of vectors in $\mathbb{R}^n$;
(2) Be able to work out the orthogonal projection of functions on $[0, 1)$ onto the wavelet spaces $V_N$. Recall that the wavelet space $V_N$ is the space spanned by the **orthogonal** Haar wavelet functions $\varphi_j^N$. Thus

$$V_N = \{ \sum_{j=0}^{2^N-1} s_j \varphi_j^N \mid s_0, \ldots, s_{2^N-1} \in \mathbb{R} \} \, .$$

- The Gram-Schmidt orthogonalization in $\mathbb{R}^n$ and spaces of polynomials. Recall that the Gram-Schmidt orthogonalization works as follows:
Given a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for a subspace $\mathbf{W}$, then we can construct an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k$ for $\mathbf{W}$ in the following way:

(1) Let $\mathbf{v}_1 = \mathbf{u}_1$;
(2) Let

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{< \mathbf{u_2}, \mathbf{v}_1 >}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 \, .$$

(3) If we have already constructed $\mathbf{v}_1, \ldots, \mathbf{v}_j$ and $j < k$, then we construct $\mathbf{v}_{j+1}$ by

$$\mathbf{v}_{j+1} = \mathbf{u}_{j+1} - \sum_{s=1}^j \frac{< \mathbf{u}_{j+1}, \mathbf{v}_s >}{\|\mathbf{v}_s\|^2} \mathbf{v}_s$$

(4) Proceed until $j = k$.

- The discrete Fourier transform and its inverse for $N = 2$ and $N = 4$.