1. Functions, Wavelets, and Signals

Analysis is about functions! So our first question will be: What is a function? In text books functions are usually given by an explicit formula like

$$f(x) = 2x + 3$$

or

$$g(t) = e^t + \cos(3t)\,.$$

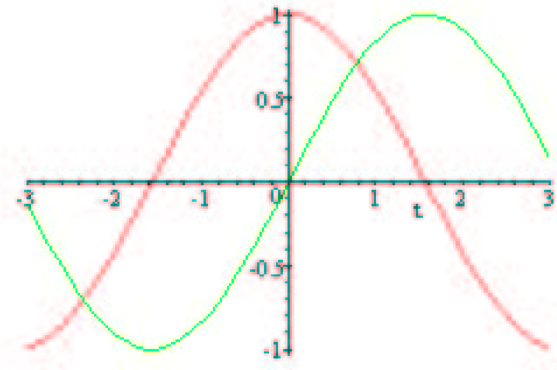
But in real life this is usually not possible! Most functions are given as a solution to a differential equation like

$$y' + py = q$$

or as finitely many numbers collected as a result of an measurement or another form of an experiment. In our digital age this is even more important because *any form of digital information is discrete and finite.* Therefore we do not have all the information on the function that we are working with only finitely many sample values. What does this mean? Every computer or any other form of digital storage has only limited amount of space. Even if that amount is counted in Gigabits, only finitely many bites can be stored that way. Even worse, even if we have a formula for a function, and are then able to write a computer algorithms to evaluate it, that calculation will only result in finitely many numerical values, because a computer can only deal with number of finite length and there is always some limitation of the size. In this class we mostly think of functions as signals. There are two main classes of signals:

(1) Analog signals
(2) Digital signals.

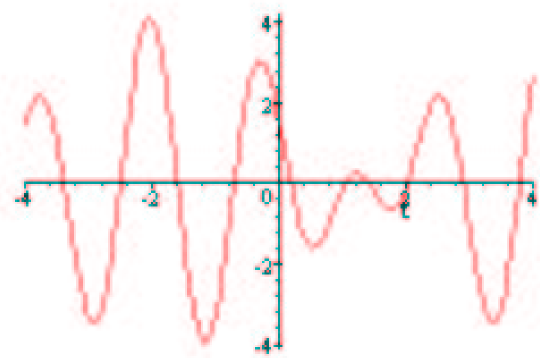
If you think about it for a moment you will probably know the difference. You probably know analog signals from playing a tape, listening to a tape, using an analog phone, or from a TV. Those signals usually changes continuously by time - think of a tape, music on a LP, TV signals. Those signals are build up using simple sin and cos waves:



By changing the amplitude and the frequency, and by adding several waves more complicated signals results. As an example the superposition of the simple waves $2\cos(3t)$, $-3\sin(4t)$, and $\cos(4t-2)$ results in the function

$$2\cos(3t) - 3\sin(4t) + \cos(4t-2)$$

which corresponds to the graph



Digital signals are different, they are discrete, and finite. We divide a computer monitor into finitely many pixels, and each pixel has a given color which is encoded using a number say between 0 and 255. Think of this as 256 different *intensity levels*, or *scales*, of **gray**, ranging from 0 (black) and 255 (white). A digital image will be created by dividing part of the x, y-*plane* into grid of squares called **pixels** and coloring each pixel with some shade of gray or color. This information can be put together to form a **function** of two variable. The value of the function at each point in the plane corresponds then to the value of the gray scale at that point. Notice that this function will in general **not be continuous** but build up from elementary jump functions. **Analog to digital** is in **some cases** to **approximate** a continuous function like the one in picture 1 by step functions. Here the important word is **approximate**. It is an important part of our analysis to understand in what **sense** the step functions approximate the true signal and what the **error** or true **difference** is. One of the practical problems is then to :

(1) Convert continuous analog signa/music into a discrete finite digital signal;
(2) Convert the signal back to analog signal to play it;
(3) Compress the information to store it or to speed up the transformation of the signal on the internet, over a satellite in case we are talking about TV or digital phone;

One way to do this is to use **wavelets** a relatively new branch of mathematics. A good example is the FBI fingerprint files and **image compression standard** the so-called *wavelet transform/scalar quantization (WSQ)* image coding, which was developed by project leader Tom Hopper of the FBI's criminal Justice Information Service Division and Jonathan Bradley and Chris Brislawn from the Computer Research and Application Group at Los Alamos National Laboratory. See Chris Brislwn's homepage:

http://www.c3.lanl.gov/˜brislawn/FBI/FBI.html

Here are some of the main facts:

- The FBI is digitizing fingerprints at 500 dots per inch
- With 8 bits of grayscale resolution
- A single fingerprint card turns into about **10 MB of data**! For each person we need 10 cards

Question: How long does it take to transform one such card with a 56KB-modem?

- And this with about 25 millions persons or about 250 millions cards: This is **2500000000 MB** of images!

Question: How many Hard drive (say 16MB) do we need to store this information

Without some sort of compression of data, the size of this database would make **sorting**, **storing**, and **searching** nearly impossible. But the compression has to be such that we still have a "true" image of the fingerprints. Let us look at some information from

http://www.c3.lanl.gov/~brislawn/FBI/FBI.html

Other tasks or problems where wavelets are used are:

- **noise**, a extra information in a signal that is introduced during transmission of data. Wavelets and other integral transforms can be used to filter out the noise.
- How do we make a CD out of a LP, i.e., how do we digitalize analog information?
- How do we transform live interviews or TV-reportage as fast as possibly?

We will not discuss all of those questions. What we will do is to discuss the **basic mathematics** behind the theory of wavelets and other similar **integral transforms**, some of which we will discuss in this class.

2. Functions

Functions are one of the most important objects in mathematics and all of its applications. We are confronted with functions every time where *one quantity - the dependent variable - depends on one or more other quantities - the independent variables.*

(1) The volume V of a box depends of the length, the with, and the height of the box. Denoting the length by the letter l, the with by w and the height by h, we have

$$V = l \cdot w \cdot h\,.$$

Thus the volume V is a function of three variables, $V = V(l, w, h)$.

(2) The area A of a circle depends on the radius r:

$$A = 2\pi \cdot r\,.$$

Thus the area is a function of *one* variable, $A = A(r)$.

(3) According to Newtons law there is a simple relation between force and acceleration given by

$$F = ma$$

or

$$a = \frac{F}{m}\,.$$

Thus we can view the force as a function of the acceleration or the acceleration as a function of the force. For example if we hold a 1 gram ball 10 meters above the surface of the earth, then a force

$$F = -9.8/1000$$

The velocity of the ball at time t after we let it loose is given by

$$v = at = -9.8t\mathrm{m/sec}$$

and then the height above the earth is given by

$$\begin{aligned} h &= 10 + \int_0^t v(t)\,dt \\ &= 10 - 4.95t^2 \,. \end{aligned}$$

In particular both the velocity and the height depends on the time. Notice that both the height function and the velocity is not defined for all time. First of all our formula is only valid for *positive time* and furthermore the height can not be smaller than zero. To find the upper time limit for our height and velocity function we have to solve the equation

$$10 - 4.9t^2 = 0$$

or

$$t = \sqrt{\frac{10}{4.9}} \approx 1.429\,.$$

We can therefore write

$$v(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ -9.8t & \text{if} \quad 0 < t < \sqrt{\frac{10}{4.9}} \\ 0 & \text{if} \quad \sqrt{\frac{10}{4.9}} \leq t \end{cases}$$

and

$$h(t) = \begin{cases} 10 & \text{if} \quad t \leq 0 \\ 10 - 4.9t^2 & \text{if} \quad 0 < t < \sqrt{\frac{10}{4.9}} \\ 0 & \text{if} \quad \sqrt{\frac{10}{4.9}} \leq t \end{cases}$$

(4) Consider a family driving in a car from Baton Rouge to Houston. The distance traveled at time t depends on the velocity of the car. It is very unlikely that they are driving at a constant velocity, so the velocity depends on the time. Thus the velocity is a function of the time $v = v(t)$, where we can measure t in minutes, hours, or even seconds. Let us measure the time in

minutes Then the distance from Baton Rouge after T minutes is

$$d(T) = \int_0^T v(t)\,dt\,.$$

is then and the velocity might depend on the time.

Definition 1. *A* ***function*** *f defined on a non-empty set A with a value in a set B is a* ***rule*** *that assigns to each element in A* ***exactly one*** *element in the set B.*

Thus

$$f(t) = \sqrt{1-t^2}, -1 \leq t \leq 1,$$

is a function, but

$$g(t) = \pm\sqrt{1-t^2}, \ -1 \leq t \leq 1$$

is not a function, because it assigns to each t in the interval $[-1,1]$ **two** values, the **positive** and the **negative** square root of the number $1-t^2$. We will often use the world **signal** for functions. This is motivated by many of the applications. Here the independent variable t stands for **time** and $f(t)$ is a time dependent signal.
In this course our set A will be a subset of the real line $\mathbb{R}$ or the plane $\mathbb{R}^2$ and B will be the real line. Later we will also consider subsets of the complex plane $\mathbb{C}$. The set A is called **the domain** of the function f. If $x \in A$ then $f(x)$ is called the **value** of the function at the point x. The **range** of f is the set of all possible values, $\{f(x) \mid x \in A\}$. We say that the elements in A are the **independent variables** and the values $f(x)$ **the dependent variables**. The **graph** of the function $f : A \to \mathbb{R}$ is the set of points in the plain

$$G(f) = \left\{(x,y) \in \mathbb{R}^2 \mid x \in A, y = f(x)\right\}$$

Notice that usually we will not be able to picture all of the graph because A might be an infinite interval.

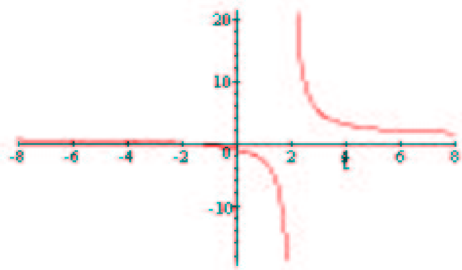
Example 1. *Let A be the set of all real numbers except 2. Thus*

$$A = \{t \in \mathbb{R} \mid t \neq 2\} = \mathbb{R} \setminus \{2\}\ .$$

Let

$$f(t) = \frac{t+2}{t-2}$$

Then $f(t)$ is defined for all real numbers except 2 and the range of f is the set $\mathbb{R}\setminus\{1\}$ (why?). The graph of the function f looks like:



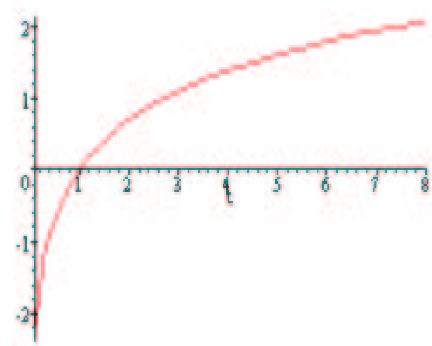
$$f(t) = \frac{t+2}{t-2}, \qquad t \neq 2$$

with vertical a asymptote at $t = 2$ and a horizontal asymptote $y = 1$.

Example 2. *Let $f(t) = \ln(t)$. Then we have to take*

$$A = \{t \in \mathbb{R} \mid t > 0\} = (0, \infty)$$

*because f is **not** defined for 0 or negative numbers. The range of f is the set of all real numbers $\mathbb{R}$. The graph is given by*



$$f(t) = \ln(t), \quad t > 0$$

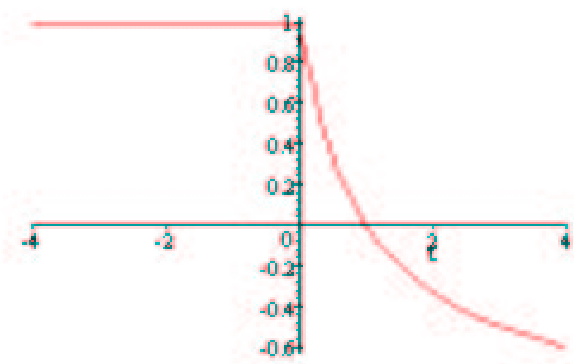
There are several ways to define a functions. What we need to know is a **rule** how to evaluate the dependent variable $y = f(x)$ for a given numerical value of the independent variable x. This can happen in may different ways. This can be done in many different ways. The most useful form is if the function is given by an **explicit equation** like:

(1) $f(t) = \cos(t)$, $t \in \mathbb{R}$.
(2) $f(x) = e^x$, $x \in \mathbb{R}$
(3) $p(x) = 2x^2 + 3x - 2$, $x \in \mathbb{R}$,
(4) $g(u) = \frac{2u+3}{u-1}$, $u \neq 1$
(5) $f(t) = e^t \cos(t) + \ln(t-2)$, $t > 2$,

In this case we can **evaluate** or **sample** the function at any point in the domain of definition. We can also graph the function, put it into a graphical calculator or any other graphical or computational device, we can integrate the function numerically or if possible using an explicit formula, and finally we can differentiate the function. Many times the equation consists of more than one part. As an example let us take the function

$$f(t) = \begin{cases} 1 & t \leq 0 \\ \frac{1-t}{1+t} & t > 0 \end{cases}$$

The graph of f then looks like



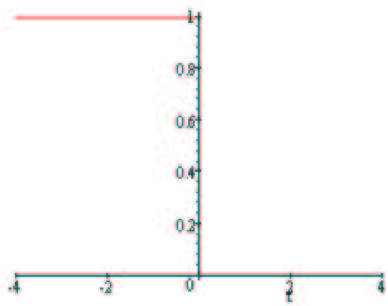
To draw the graph we notice that the function is build up from two functions. The first function is

$$t \mapsto 1$$

valid for all negative t. The other function is

$$t \mapsto \frac{1-t}{1+t}$$

valid for all $t \geq 0$. The result is:



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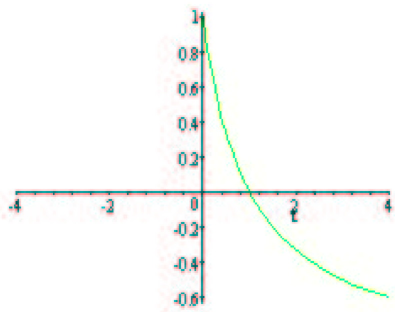

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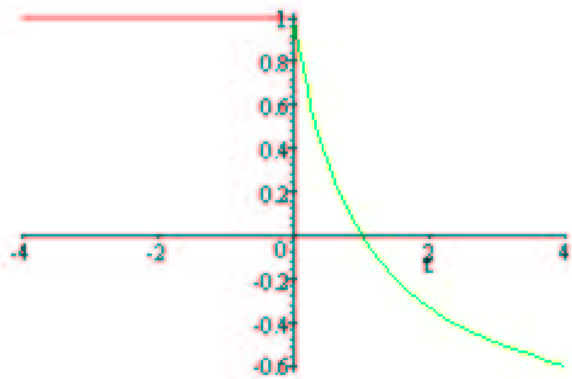


We notice that in this case the *limit from the left*

$$\lim_{t \to o^-} f(t) = 1$$

is the same as the *limit from the right*

$$\lim_{t \to o^+} f(t) = \lim_{t \to o^+} \frac{1-t}{1+t} = \frac{1-0}{1+0} = 1\,.$$

Hence the function f is *continuous.*

Example 3. *Evaluate the above function* $f(t) = \begin{cases} 1 & t \leq 0 \\ \frac{1-t}{1+t} & t > 0 \end{cases}$ *at the points* $t = -1$, $t = 0$, *and* $t = 1$.
Solution: *Starting with* $t = -1$ *we observe first that* $-1 < 1$. *Thus we have to use the part of the definition valid for* ***negative*** *numbers. Thus*

$$f(-1) = 1\,.$$

*The value 1 is assigned to all numbers less or **equal** to 0. Hence*

$$f(0) = 1\,.$$

For $t = 1$ we notice that $1 > 0$, hence the second part of the definition is in force. Thus

$$f(1) = \frac{1-1}{1+1} = 0\,.$$

The final answer is:

t	$f(t)$
-1	1
0	1
1	0

Definitions of this kind requires the following steps:

(1) Make clear which of the intervals in the definition of f that we are using;
(2) Make sure we know what the definition of f is on each interval;
(3) Draw the graph of each part according to the definition.

Example 4. *Define the function $f : \mathbb{R} \to \mathbb{R}$ by*

$$f(t) = \begin{cases} 1 - t^2 & \text{if } \ t < 1 \\ t\ln(t) & \text{if } \ t \geq 1 \end{cases}$$

Evaluate the function at the points $t = -2, -1, 0, 1, 2$. Draw the graph of f and decide where f is continuous.
Solution: *We first notice that the definition consists of two parts. The first of them*

$$t \mapsto 1 - t^2$$

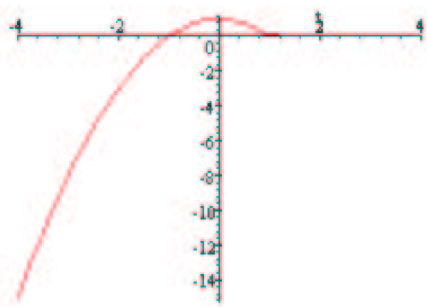
is valid for all $t < 1$. The other part

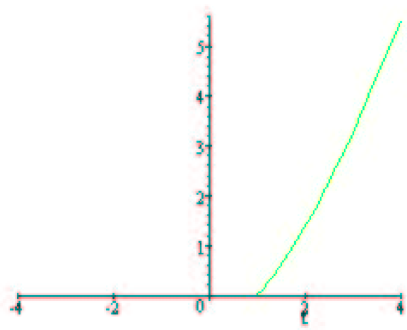
$$t \mapsto t\ln(t)$$

is valid for all $t \geq 1$. In particular this definition has to be used for the endpoint $t = 1$. The result is that we have to use the first part of the definition for the points $t = -2, -1, 0$ and the second part for the points $t = 1, 2$. Thus

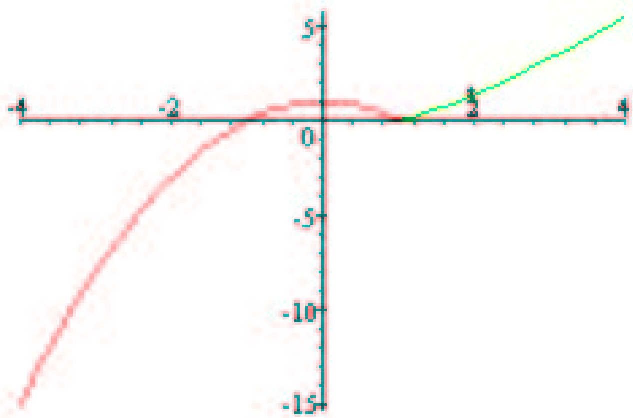
t	*we use*	*calculation*	$f(t)$
-2	$1 - t^2$	$1 - (-2)^2$	-3
-1	$1 - t^2$	$1 - (-1)^2$	0
0	$1 - t^2$	$1 - 0^2$	1
1	$t\ln(t)$	$1 \cdot \ln(1)$	0
2	$t\ln(t)$	$2 \cdot \ln(2)$	≈ 1.386

The graph of the function is build up from two parts:





for $t < 1$ *and* $t \geq 1$*, Put together into one graph this gives:*



Next we notice that $t \mapsto 1-t^2$ *is a polynomial and hence continuous. As* t *and* $t \mapsto \ln(t)$ *are continuous it follows that* $f(t)$ *is continuous except possibly at the point* $t = 1$ *where the two definitions come together. But*

$$\lim_{t \to 1^-} f(t) = \lim_{t \to 1^-} (1 - t^2) = 1 - 1^2 = 0$$

and

$$\lim_{t \to 1^+} f(t) = \lim_{t \to 1^+} t \ln(t) = 1 \cdot 0 = 0\,.$$

Hence f *is continuous at* $t = 1$*. It follows that* t *is continuous on the real line.*

Example 5. *Define the function* $f : \mathbb{R} \to \mathbb{R}$ *by*

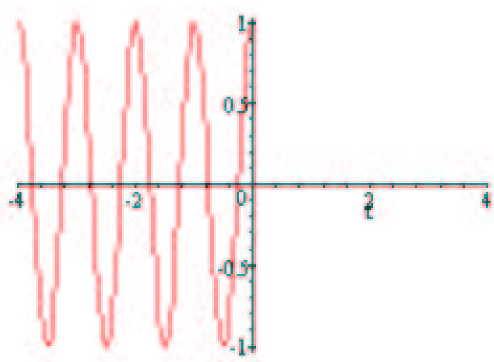
$$f(t) = \begin{cases} \cos(2\pi t) & \text{if} \quad t < 0 \\ 2t & \text{if} \quad 0 \leq t < 1 \\ \frac{t^2-2}{1+2t+t^2} & \text{if} \quad 1 \leq t \end{cases}\,.$$

Evaluate the function at the points $t = -2, -1, 0, 1, 2$. Draw the graph of f and decide if f is continuous or not.
Solution: *In this case we three different parts in the definition of f.*

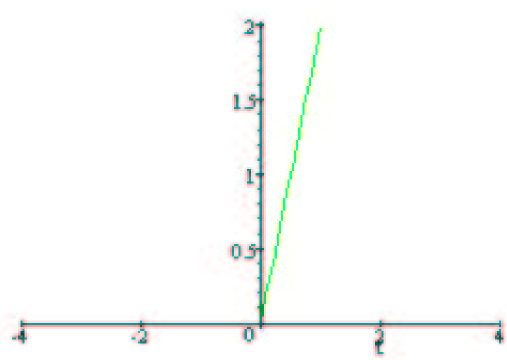
t	*we use*	*calculation*	$f(t)$
-2	$\cos(2\pi t)$	$\cos(-2\pi(-2))$	1
-1	$\cos(2\pi t)$	$\cos(-2\pi(-1))$	1
0	$2t$	$2 \cdot 0$	0
1	$\frac{t^2-2}{1+2t+t^2}$	$\frac{1^2-2}{1+2\cdot 1+1^2}$	$-\frac{1}{4}$
2	$\frac{t^2-2}{1+2t+t^2}$	$\frac{2^2-2}{1+2\cdot 2+2^2}$	$\frac{2}{9}$

The graph consists of three parts each valid for the corresponding piece of the definition:

$t \mapsto \cos(2\pi t),\ t < 0,$

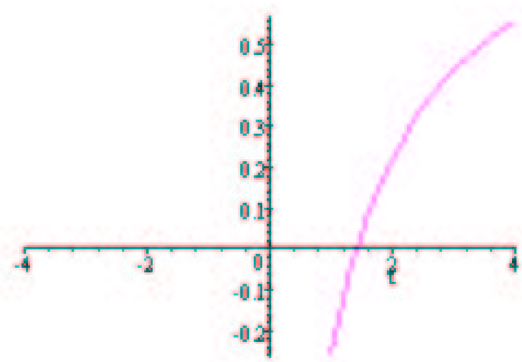


$t \mapsto 2t,\ 0 \le t < 1,$

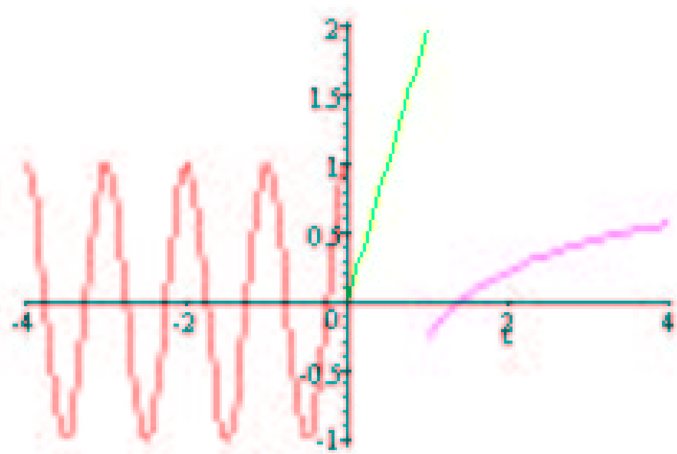


and finally

$$t \mapsto \frac{t^2-2}{1+2t+t^2}\,,\ 1 \le t,$$



The result is the graph



To find out where f is continuous we first remark that:

(1) *The function* $\cos(2\pi t)$ *is continuous;*
(2) *The function* $2t$ *is a polynomial and hence continuous;*
(3) *The function* $\frac{t^2-2}{1+2t+t^2}$ *is a quotient of two polynomials and hence continuous where ever* $1+2t+t^2 = (1+t)^2 \neq 0$. *As* $(1+t)^2 = 0$ *only for* $t = -1$ *it follows that* $\frac{t^2-2}{1+2t+t^2}$ *is continuous for all* $t \geq 1$.

It therefore follows that f is continuous at every point except possibly the points $t = 0$ and $t = 1$ where the different definitions meets each other. At those points we will always have to work out the limit from the left and the limit from the right.

t=0: *We have*

$$\lim_{t\to o^-} f(t) = \lim_{t\to o^-} \cos(2\pi t) = \cos(0) = 1$$

and

$$\lim_{t\to o^+} f(t) = \lim_{t\to o^+} 2t = 2\cdot 0 = 0\,.$$

As

$$\lim_{t\to o^-} f(t) = 1 \neq 0 = \lim_{t\to o^+} f(t)$$

it follows that f is ***not*** *continuous at $t = 0$.*
t=1: *In this case we get*

$$\lim_{t\to 1^-} f(t) = \lim_{t\to 1^-} 2t = 2\cdot 1 = 2$$

and

$$\lim_{t\to 1^+} f(t) = \lim_{t\to 1^+} \frac{t^2-2}{1+2t+t^2} = \frac{1^2-2}{1+2\cdot 1+1^2} = \frac{-1}{4} = -.25$$

Thus f is ***not*** *continuous at $t = 1$ because the limit from the left is different from the limit from the right.*

Exercise

(1) Let $f(t) = \frac{1+t}{1-t^2}$. Determine the set of points where f is defined.
(2) Where is the function $f(t) = \frac{\cos(t)}{1-t-2t^2}$ defined?
(3) Define the function $f(t)$ by

$$f(t) = \begin{cases} 2t-1 & \text{if } \; t \leq 1 \\ 1+\ln(t) & \text{if } \; 1 < t \end{cases}$$

Evaluate f at the points $t = -1, 0, 1, 2$. Draw the graph of f and determine where f is continuous.
(4) Draw the graph of the function

$$f(t) = \begin{cases} 0 & \text{if } \; t < 0 \\ e^{-1/t^2} & \text{if } \; t \geq 0 \end{cases}$$

Evaluate f at the points $t = -1, 0, 1, 2$. Draw the graph of f and determine where f is continuous.
(5) Draw the graph of the function

$$f(t) = \begin{cases} -1 & \text{if } \; t \leq -1 \\ t & \text{if } \; -1 < t < 1 \\ \frac{1-t}{1+t^2} & \text{if } \; 1 \leq t \end{cases}$$

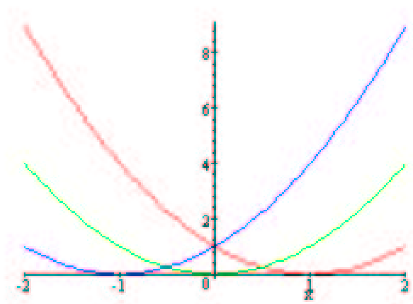
Evaluate f at the points $t = -2, -1, 0, 1, 2$. Draw the graph of f and determine where f is continuous.

3. Translation of Functions

In this section we discuss two operations of function that will be one of our main tool later on. Those are **translation** and **dilation** of function.
Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For a real number r define a new function by $x \mapsto f(x-r)$. We call the function $f(x-r)$ the translated function. To evaluate $f(x-r)$ we first have to replace x by $x-r$ and then plug the result into f. Thus if $f(x) = x^2$ and $r = 1$ then

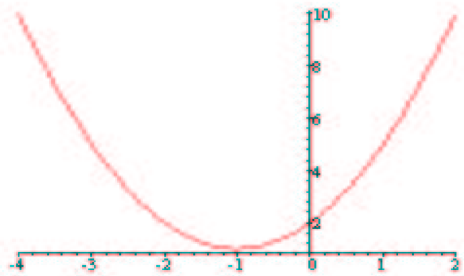
x	$x-1$	$f(x-1)$
-1	-2	$(-2)^2 = 2$
0	-1	$(-1)^2 = 1$
1	0	$0^2 = 0$
2	1	$1^2 = 1$

The effect on the graph of f is that the graph is shifted to left by r if r is positive and to the right by $|r|$ if r is negative. The following picture shows the graph of $t \mapsto (t+1)^2$ (blue), $t \mapsto t^2$ (green), and $t \mapsto (t-1)^2$ (red):

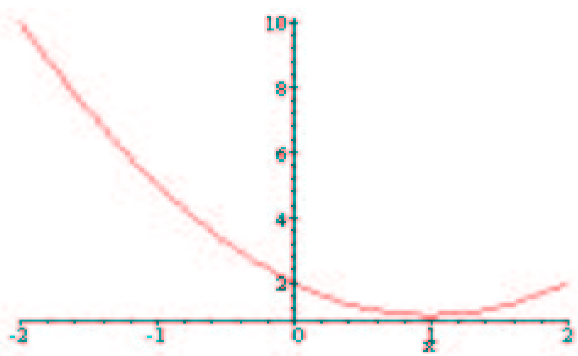


Example 6. *Let $f(t) = t^2 + 2t + 2$. Draw the graph of the function $t \mapsto f(t-2)$ for t in the interval* $[-2, 2]$

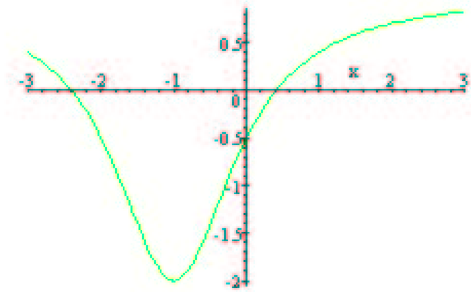
Solution: *The graph of the function f is given by*



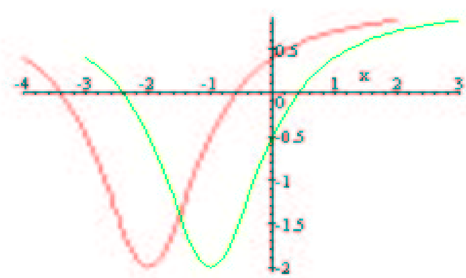
The graph of $t \mapsto f(t-2)$ is gotten by moving this graph ***two*** *units to the right. Hence the graph of $f(t-2)$ is given by*



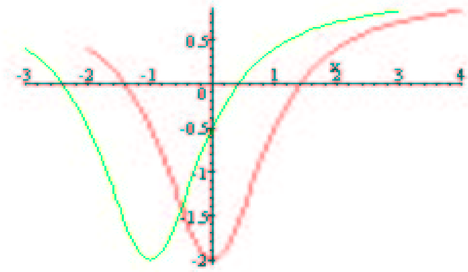
Example 7. *If the graph of $f(t)$ is given by*



draw the graph of $f(t+1)$ and $f(t-1)$.
Solution: *To draw the graph of $f(t+1)$ we have to translate the graph of $f(t)$ to the **left** by one. Thus*



To draw the graph of $f(t-1)$ we have to take the original graph of $f(t)$ and translate it by one unite to the ***right***



Example 8. *Let $f(x) = \frac{x+2}{x^2+1}$. Evaluate $f(x-1)$ at the points $x = -1, 0, 1$ and draw the graph of $f(x-1)$ for x in the interval $[-1, 1]$.*
Solution:

(1) *If $t = -1$, then $t - 1 = -2$ and hence*

$$f(-1-1) = \frac{-2+2}{(-2)^2+1} = 0\,.$$

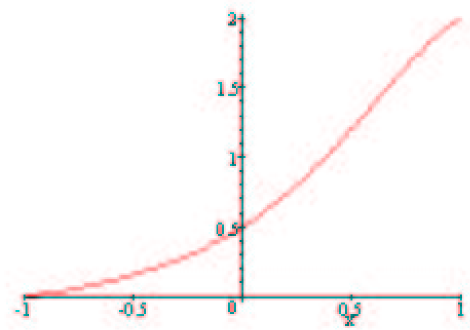
(2) *If $t = 0$, then $t - 1 = -1$ and*

$$f(0-1) = \frac{-1+2}{(-1)^2+1} = \frac{1}{2}\,.$$

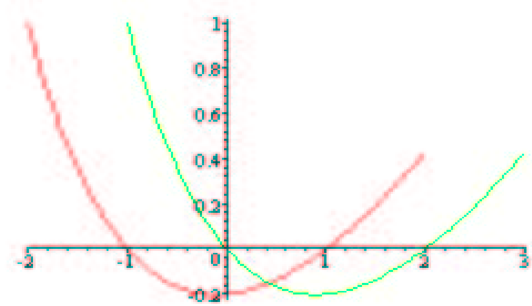
(3) *If $t = 1$, then $t - 1 = 0$ and*

$$f(1-1) = \frac{0+2}{0^2+1} = 2\,.$$

For the graph of $f(t-1)$ *we get*



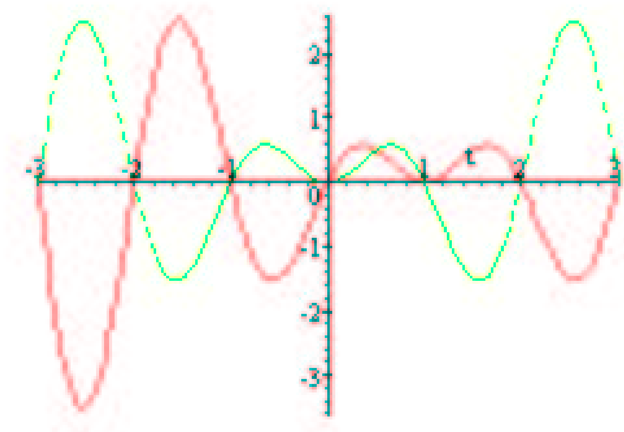
Example 9. *Consider the following two graphs where the red line denotes the graph of* $f(t)$ *and the green denotes the graph of a function* $g(t)$*:*



Find a number r *such that* $g(t) = f(t-r)$.
Solution: *We notice that the for* f *the* t*-intercept is at* $t = -1$ *and for* g *the* t*-intercept is at* $t = 0$. *To move the* t*-intercept of* f *into that of* g *we have to translate the graph of* f *by* 1 *to the right. Hence* $r = 1$ *and* $g(t) = f(t-1)$.

Exercise

(1) Let $f(t) = \frac{t^2}{t+4}$. Evaluate $f(t-2)$ at the points $t = -1, 0, 1$, and 2. Then draw the graph of $f(t-2)$ over the interval $-1 \leq t \leq 2$.
(2) Consider the two following graphs where the green is the graph of f and the read line denotes the graph of g. Find $r > 0$ such that $g(t) = f(t-1)$.



(3) Draw the graph of the functions $\cos(t)$, $\sin(t)$, $\cos(t-\pi)$, and $\cos(t-2\pi)$.

4. Dilation of Functions

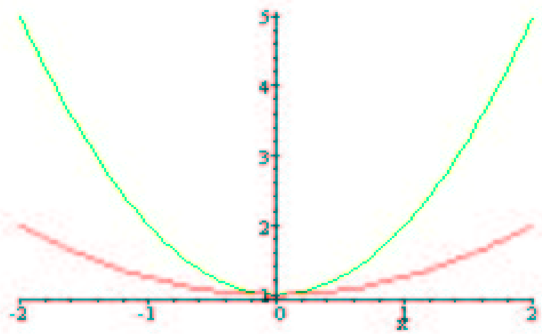
The translation moved the graph of a function to the left or to the right. We will now discuss how to stretch the graph or compress it. Let $\lambda > 0$ and let f be a given function. We consider then a new function given by

$$t \mapsto f(\lambda t) .$$

Example 10. *Let $f(t) = t^2 + 1$ and $\lambda = .5$. Let us tabulate $f(t)$ and $f(\lambda t)$ for few values of t :*

t	λt	$f(t)$	$f(\lambda t)$
-2	-1	5	2
-1	-0.5	2	1.25
0	0	1	1
1	0.5	2	1.25
2	1	5	2
4	2	17	5

Let us then compare the graphs of the two functions:

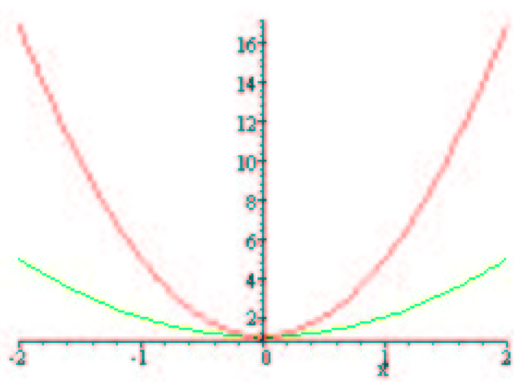


Here the green line denotes the graph of $f(t)$ and the red the graph of $f(\lambda t)$.

Example 11. *Let us now use the same function but replace* 0.5 *by* 2. *Then the corresponding table is:*

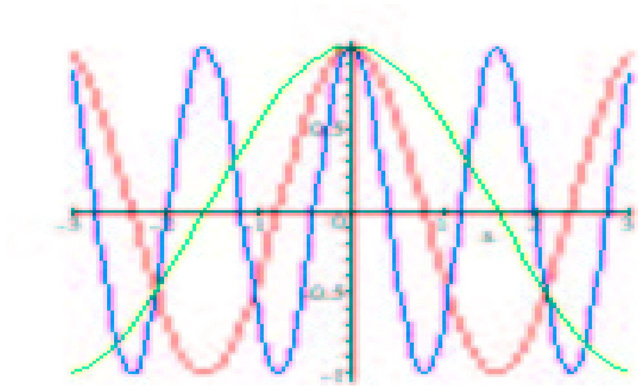
x	λx	$f(x)$	$f(\lambda x)$
-2	-4	5	17
-1	-2	2	5
0	0	1	1
1	2	2	5
2	4	5	17
4	8	17	65

and the graphs are



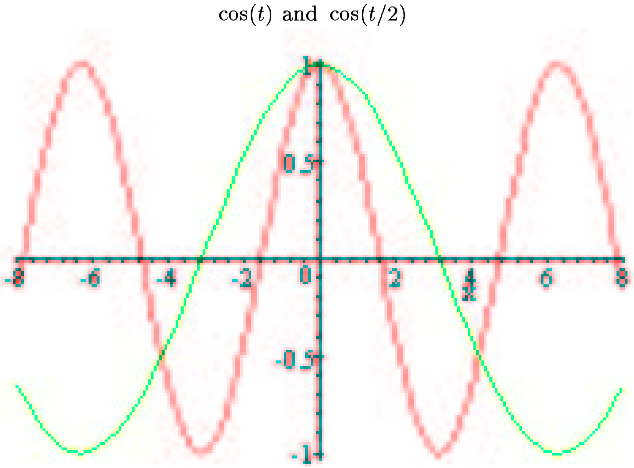
In general the graph of $f(\lambda t)$ is stretch out if $0 < \lambda < 1$ and compressed if $1 < \lambda$.

Example 12. *One important example is if $f(t)$ is a simple wave like $f(x) = cox(Ax)$. Then $A/2\pi$ is called the* ***frequency*** *of the wave. Dilating the functions means changing the frequency by a factor of λ. Thus if λ is small we* ***decrease*** *the frequency, if λ is big, then we* ***increase*** *the frequency*



green $\cos(t)$, red $\cos(2t)$, blue $\cos(4t)$.

$\cos(t)$ and $\cos(t/2)$



In many application we need to use both the translation **and** the dilation. Thus we both compress/stretch the graph and translate it to the left or to the right.

Example 13. *Let* $f(t) = t^2 + t - 1$. *Draw the graph of the function* $f(2t-1)$ *over the interval* $[-2, 2]$.
Solution: *There are more than one way to solve the problem. Let us start by drawing the graph of the original function* f.
Next we notice that the factor of 2 *increases the "speed" by two, hence the graph is compressed by factor of* 2. *Then factor* -1 *implies that we need to translate the graph to the right. The question is:* ***How far?*** *To see that, let us write* $f(2t-1) = f(2(t-1/2))$. *Hence we need to translate it* 0.5*-units to the left. The result is:*