Math 2025 Quiz #3 (Fall 2004) **Name:** ____________________

Inner product on functions on $[a,b]$ (piecwise continuous, continous, etc.) is given by $(f,g) = \int_a^b f(t)g(t)\,dt$. The norm is

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_a^b f(t)^2\,dt}\,.$$

If A is a subset of $\mathbb{R}^n$ then the indicator function of A is the function $\chi_A : \mathbb{R}^n \to \mathbb{R}$ given by

$$\chi_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A \end{cases}$$

In the following we will always use $a = 0$ and $b = 1$.

A) Evaluate the following inner products:

(1) $((1,-1,2),(1,2,1)) =$ 1 ; $[1 - 2 + 2 = 1]$

(2) Let $f(t) = t^2$ and $g(t) = t + 1$. Then $(f,g) =$ 7/12 ;

(3) Let $f(t) = \chi_{[0,1/2)}(t)$ and $g(t) = \chi_{[0,1/4)}(t) - \chi_{[1/4,1/2)}(t)$. What is $(f,g) =$ 0

B) Determine which of the following pair of vectors are orthogonal.

(1) $(1,-1,2)$, $(1,1,-1)$; No

(2) $f(t) = \sin(2\pi t)$ and $g(t) = \cos(2\pi t)$; Yes

(3) $f(t) = t^2 - 1$ and $g(t) = t$; No

C) Evaluate the following norms:

(1) $\|(1,-1)\| =$ $\sqrt{2}$;

(2) $\|(2,3,4)\| =$ $\sqrt{29}$;

(3) $\|te^t\| =$ $\frac{1}{2}\sqrt{e^2 - 1}$

A)

1. $((1,-1,2),(1,2,1)) = 1 - 2 + 2 = 1$

2. $(f,g) = \int_0^1 t^2(t+1)\,dt$
$= \int_0^1 t^3 + t^2\,dt = \frac{1}{4}t^4 + \frac{1}{3}t^3\Big]_0^1 = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$

3. $(f,g) = \int_0^1 \chi_{[0,\frac{1}{2})}(t)\left(\chi_{[0,\frac{1}{4})}(t) - \chi_{[\frac{1}{4},\frac{1}{2})}(t)\right)dt$
$= \int_0^{\frac{1}{4}} dt - \int_{\frac{1}{4}}^{\frac{1}{2}} dt = \frac{1}{4} - \frac{1}{4} = 0$

B) 1) $((1,-1,2),(1,1,-1)) = 1 - 1 - 2 = -2$
Not orthogonal.

2) $\int_0^1 \sin 2\pi t \cos 2\pi t\,dt$

$u = \sin(2\pi t)$
$du = 2\pi\cos(2\pi t)\,dt$

$= \frac{1}{2\pi}\int_0^0 u\,du = 0$
orthogonal.

3) $\int_0^1 (t^2-1)t\,dt = \int_0^1 t^3 - t\,dt = \frac{1}{4} - \frac{1}{2} = \frac{-1}{4} \neq 0$
not orthogonal

C) A) $\|(1,-1)\|^2 = (1)^2 + (-1)^2 = 2$

B) $\|(2,3,4)\|^2 = 4 + 9 + 16 = 29$

C) $\|te^t\|^2 = \int_0^1 t^2e^{2t}\,dt = \frac{1}{2}t^2e^{2t}\Big]_0^1 - \int_0^1 te^{2t}\,dt$
$= \frac{1}{2}e^2 - \frac{1}{2}te^{2t}\Big]_0^1 + \frac{1}{2}\int_0^1 e^{2t}\,dt$
$= \frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{4}e^{2t}\Big]_0^1 = \frac{1}{4}\left[e^2 - 1\right]$