Math 2025, Quiz #2 **Name:** ______________________

1) Find the average value of the numbers $1, 3, -1, 2$.

Answere: ______________________

2) Find the Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (3, 1, 2, 6)$.

Answere: ______________________

3) Assume that the Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\overrightarrow{\mathbf{s}}^0 = (\mathbf{3}, 0, -2, 1)$. Find the initial array $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$.

Answere: ______________________

Math 2025, Quiz #3

Name: ______________________________

1) Calculate the **in-place** fast Haar wavelet transform of the sample $\overrightarrow{\mathbf{s}} = (5, 1, 2, 4)$.

Answer: $\overrightarrow{\mathbf{s}^{(0)}} =$ ______________________________

2) Assume that the **in-place** fast Haar wavelet transform of a sample $\overrightarrow{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the result $(4, 1, 2, 2)$. Apply the inverse transform to reconstruct the sample $\overrightarrow{\mathbf{s}}$

Answer: $\overrightarrow{\mathbf{s}} =$ ______________________________

2) Evaluate the following:

(1) $(1 + 2i)(2 - i) =$

(2) Write the complex number $\frac{1}{1+4i}$ in the form $x + iy$. $\frac{1}{1+4i} =$

(3) $\overline{(5 + 3i)} =$

(4) What is $\mathrm{Re}(2 - 3i) =$ and $\mathrm{Im}(2 - 3i) =$

(5) What is $|3 + 4i| =$

Math 2025, Quiz #4

Name: ____________________

The inner product on $\mathbb{R}^n$ is $< \overrightarrow{\mathbf{x}}, \overrightarrow{\mathbf{y}} >= x_1y_1 + \ldots + x_ny_n$, on $\mathbb{C}^n$ is $< \overrightarrow{\mathbf{z}}, \overrightarrow{\mathbf{w}} >= z_1\overline{w_1} + \ldots + z_n\overline{w_n}$ where stands for complex conmjugation. The inner product on the space of continuous functions $C([a,b],\mathbb{C})$ is given by $< f, g >= \int_a^b f(t)\,\overline{g(t)}\,dt$.

1) Evaluate the integral $\int_0^1 (1 - it)^2\,dt =$ ____________________

2) Evaluate the inner products:

(1) $< (2, 1, -1), (3, -0, 1) >=$ ____________________

(2) $< (1 + i, 1 - i, (i, 2, -1 + i) >=$ ____________________

3) Evaluate the norm of the vector $(1, -2, 1)$. **The norm is:** ____________________

4) Let $a = 0$ and $b = 1$. Evaluate the the following inner products:

(1) $< t, 1 + it >=$ ____________________

(2) $< t, e^t >=$ ____________________

Math 2025, Quiz #5 **Name:** ______________________

1) Which of the following sets of vectors is linearly independent?

(1) $(1,0,1),(0,1,0),(1,0,-1)$. They are paar wise orthogonal and hence **linearly independent.**

(2) $(1,0,0),(1,-1,1),(1,2,2),(1,2,5)$. Four vectors in $\mathbb{R}^3$ are always l**inearly dependent**. Another solution is to notice that

$$(1,2,5) = -\frac{9}{4}(1,0,0) + \frac{3}{2}(1,-1,1) + \frac{7}{4}(1,2,4)$$

2) Which of the following sets is a basis for $\mathbb{R}^2$?

(1) $(1,0),(1,-1)$. **Basis**. The two vectors are perpenticular and hence linearly independent. In $\mathbb{R}^2$ any set of two linearly independent vectors is a basis.

(2) $(1,1),(1,0),(1,-1)$. **Not a basis**. The set is generating but not linearly independent.

3) Which of the following sets is a basis for $\mathbb{R}^3$?

(1) $(1,0,0),(1,1,0),(1,1,1)$. The set is linearly independent. The equation

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,1,1) = (c_1+c_2+c_3, c_2+c_3, c_3) = (0,0,0)$$

give:

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0\,. \end{aligned}$$

The last equation gives $c_3 = 0$. Then the second implies that $c_2 = 0$, and then finally the first gives $c_1 = 0$. Hence all the constants have to be zero.

(2) $(1,2,0),(1,0,1)$. This is **not a basis**. The vectors are linearly independent, but we need three vectors for a basis for $\mathbb{R}^3$.

4) Write the following vectors as a linear combination of the vectors $(1,1)$ and $(1,-1)$.

(1) $(2,4) = 3(1,1) - (1,-1)$

(2) $(1,-3) = -(1,1) + 2(1,-1)$

Solution: Notice that $||(1,1)||^2 = ||(1,-1)||^2 = 2$ and that $<(1,1),(1,-1)>= 1-1 = 0$. Hence they are orthogonal. Thus, if $\vec{\mathbf{u}} = (x,y)$ is a vector in $\mathbb{R}^2$ we have $(x,y) = c_1(1,1) + c_2(1,-1)$ with

$$c_1 =< (x,y),(1,1) > /2 = \frac{x+y}{2}$$

and

$$c_2 =< (x,y),(1,-1) > /2 = \frac{x-y}{2}$$

If $\vec{\mathbf{u}} = (2,4)$ this gives $c_1 = \frac{2+4}{2} = 3$ and $c_2 = \frac{2-4}{2} = -1$. If $\vec{\mathbf{u}} = (1,-3)$ we get $c_1 = \frac{1-3}{2} = -1$ and $c_2 = \frac{1+3}{2} = 2$.

Math 2025, Problems/Inner products

Recall that an inner product on $\mathbb{R}^n$ is given by $<(x_1,\ldots,x_n),(y_1,\ldots,y_n)>= x_1y_1+\ldots+x_ny_n$. The length or norm of a vector $\overrightarrow{\mathbf{x}}=(x_1,\ldots,x_n)$ is the real number

$$||(x_1,\ldots,x_n)|| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2} = \sqrt{<\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{x}}>}$$

1) Evaluate the inner product of the following vectors in $\mathbb{R}^n$:

a) $<(1,0,-3),(2,1,3)>= 1\times 2+0\times 1-3\times 3 = 2-9=-7$

b) $<(2,2,-1,-1),(1,1,2,2)>= 2\times 1+2\times 1-1\times 2-1\times 2=0$

c) $<(1,2,3,4,5),(0,2,5,3,6)>= 1\times 0+2\times 2+3\times 5+4\times 3+5\times 6=61$

2) Find the norm of the following vectors:

a) $||(1,3,1)|| = \sqrt{1+9+1}=\sqrt{11}\simeq 3.317$

b) $||(-1,2,3,5)|| = \sqrt{(-1)^2+2^2+3^2+5^2}=\sqrt{39}\simeq 6.245$

Two non-zero vectors are called perpendicular or orthogonal if the inner product $<\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{y}}>=0$.

3) Which of the following vectors are perpendicular to each other?

a) $(1,0)$, $(0-2)$. Orthogonal

b) $(1,0,1)$, $(-1,2,1)$ Orthogonal

c) $(2,-1,0)$, $(1,1,3)$ Not orthognal (inner product $=2-1=1$)

An inner product on $\mathbb{C}^n$ is given by $<(z_1,\ldots,z_n),(w_1,\ldots,w_n)>= z_1\overline{w_1}+\ldots+z_n\overline{w_n}$ where denotes complex conjugation $\overline{x+iy}=x-iy$. The norm of $\overrightarrow{\mathbf{z}}=(z_1,\ldots,z_n)$ is given by $||\overrightarrow{\mathbf{z}}|| = \sqrt{|z_1|^2+\ldots+|z_n|^2}=\sqrt{<\overrightarrow{\mathbf{z}},\overrightarrow{\mathbf{z}}>}$. Two non-zero vectors $\overrightarrow{\mathbf{z}}$ and $\overrightarrow{\mathbf{y}}$ are perpenticular or orthogonal if $<\overrightarrow{\mathbf{z}},\overrightarrow{\mathbf{y}}>=0$.

4) Evaluate the inner product of the following vectors in $\mathbb{C}^n$:

a) $<(1,1+i,2i),(-1,1-i,2+i)>= 1\times(-1)+(1+i)\times\overline{(1-i)}+2i\times\overline{(2+i)} = -1+1+2i+2i+2 = 2+4i$

b) $<(i,2i,1+4i),(1-i,3-i,3)>= i\times\overline{(1-i)}+2i\times\overline{(3-i)}+(1+4i)\times 3 = i-1+6i-2+3+12i = = 19i$

5) Which of the folowing two vectors are orthogonal to each other:

a) $(1,1,-i)$, $(1,1+i,1)$: $1\times 1+1\times\overline{(1+i)}-i\times 1 = 1+1-i-i=2-2i$ not orthogonal:

b) $(1+i,i,1-i)$, $(1-i,1,-1-i)$: $(1+i)\times\overline{(1-i)}+i\times 1+(1-i)\times\overline{(-1-i)} = 1+2i-1+i-2=-2+3i$ not orthognal

6) Find the norm of the following vectors in $\mathbb{C}^n$:

a) $||(1+i,2-3i,i)|| = \sqrt{1+1+4+9+1}=4.0$

b) $||(i,1,-1)|| = \sqrt{1+1+1}=\sqrt{3}\approx 1.732$

Let $f:[a,b]\to\mathbb{C}$ be a continuous function. Then f can be written as $f(x)=u(x)+iv(x)$ where u and v are continuous real valued functions on $[a,b]$. Then integral of f over $[a,b]$ is then defined as

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx$$

7) Evaluate the following integrals:

a) $\int_0^1 (1+it)^2\,dt = \int_0^1 1-t^2+2it\,dt = \int_0^1 1-t^2\,dt + 2i\int_0^1 t\,dt = t-\frac{t^3}{3}\Big]_0^1 + it^2]_0^1 = \frac{2}{3}+i$

b) $\int_{-1}^1 \frac{1}{1+it}dt = \int_{-1}^1 \frac{1-it}{1+t^2}\,dt = \int_{-1}^1 \frac{1}{1+t^2}\,dt - i\int_{-1}^1 \frac{t}{1+t^2}\,dt$. For the first integral we use

$$\int \frac{1}{1+t^2}\,dt = \tan^{-1}(t) + C$$

Furthermore $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$. For the second integral we use the substitution

$$u = 1+t^2, \qquad du = 2tdt$$

to get

$$\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(|u|) + C = \frac{1}{2}\ln(1+t^2) + C$$

This together gives:

$$\int_{-1}^1 \frac{1}{1+it}dt = \tan^{-1}(t)\Big]_{-1}^1 - \frac{i}{2}\ln(1+t^2)\Big]_{-1}^1 = \frac{\pi}{2}$$

c) $\int_0^2 (1+it)e^t dt = \int_0^2 dt + i\int_0^2 te^t dt$. The first integral gives

$$\int_0^2 e^t dt = e^t]_0^2 = e^2 - 1\,.$$

The second integral gives (using integration by parts) $\int udv = uv - \int vdu$, with $u = t$, $du = dt$, $dv = e^t dt$ and hence $v = e^t$.

$$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$$

Inserting the limits of integration we get

$$\int_0^2 (1+it)e^t\,dt = t]_0^2 + i\left[te^t - e^t\right]_0^2 \approx 6.389 + 8.389i$$

d) $\int_1^2 t\cos(t)\,dt$. Use integration by parts with $u = t$, $du = dt$, and $dv = \cos(t)dt$, $v = \sin(t)$. Hence

$$\int t\cos(t)\,dt = t\sin(t) - \int \sin(t)dt = t\sin(t) + \cos(t) + C\,.$$

Finally

$$\int_1^2 t\cos(t)\,dt = [t\sin(t) + \cos(t)]_1^2 \approx 0.02068$$

An inner product on the space $C([a,b],\mathbb{C})$ of continuous complex valued functions on $[a,b]$ is given by

$$< f, g >= \int_a^b f(t)\overline{g(t)}\,dt$$

The corresponding norm is

$$||f|| = \sqrt{< f, f >} = \sqrt{\int_a^b |f(t)|^2 dt}\,.$$

Two non-zero functions f and g are said to be orthogonal if $< f, g >= 0$.

8) Find the inner product of the following function in $C([0,1],\mathbb{C})$ (thus $a=0$ and $b=1$).

a) $< 1+it, -it >= -1/3 - i/2$

$$\begin{aligned} < 1+it, -it > &= \int_0^1 (1+it) \times \overline{(-it)}\, dt \\ &= \int_0^1 it - t^2\, dt \\ &= -\int_0^1 t^2\, dt + i\int_0^1 tdt \\ &= \frac{-1}{3}t^3\Big]_0^1 + \frac{i}{2}t^2\Big]_0^1 \\ &= -1/3 + i/2 \end{aligned}$$

b) $< \cos(t), t >\approx .3818$

$$\begin{aligned} < \cos(t), t > &= \int_0^1 \cos(t)t\, dt \\ &= t\sin(t) + \cos(t)]_0^1 \\ &= \sin(1) + \cos(1) - 1 \\ &\approx .3818 \end{aligned}$$

c) $< t, 1-it >= \frac{1}{2} + \frac{i}{3}$

$$\begin{aligned} \int_0^1 t\overline{(1-it)}\, dt &= \int_0^1 t(1+it)\, dt \\ &= \int_0^1 t + it^2\, dt \\ &= \frac{t^2}{2}\Big]_0^1 + i\frac{t^3}{3}\Big]_0^1 \\ &= \frac{1}{2} + \frac{i}{3}. \end{aligned}$$

9) Find the norm of the following functions (if you can not evaluate the integrals, then just set up the formulas)

a) $||\cos(t)|| = \sqrt{\int_0^1 \cos^2(t)\, dt} \approx .8528$. We can evaluate the integral in the following way

$$\cos^2(t) = \frac{1}{2}(1+\cos(2t))$$

Hence

$$\begin{aligned}\int \cos^2(t)\,dt &= \frac{1}{2}\int 1+\cos(2t)\,dt \\ &= \frac{1}{2}\left(t+\frac{1}{2}\sin(2t)+C\right) \\ &= \frac{t}{2}+\frac{\sin(2t)}{4}+C\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 \cos^2(t)\,dt &= \frac{t}{2}+\frac{\sin(2t)}{4}\Big]_0^1 \\ &= \frac{1}{2}+\frac{\sin(2)}{4} \\ &\approx .727\,3\end{aligned}$$

Finally we have

$$\sqrt{.727\,3}\approx .852\,8$$

b) $||1+it|| = \sqrt{\int_0^1 1+t^2dt} = \sqrt{t+\frac{t^3}{3}\Big]_0^1} = \sqrt{1+\frac{1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \approx 1.155$

c) $||\frac{1}{1+it}|| = \sqrt{\int_0^1 \frac{1}{1+t^2}\,dt} = \sqrt{\tan^{-1}(t)|_0^1} = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}$

d) $||te^t|| = \sqrt{\int_0^1 te^t dt} = 1.$

Here we use what we have already found out:

$$\int te^t dt = te^t - e^t + C$$

Hence

$$\int_0^1 te^t dt = te^t - e^t\big]_0^1 = e - e + 1 = 1$$

Recall the following rule for integration (what is it called?)

$$\int uv'\,dt = uv - \int vu'\,dt$$

or

$$\int udv = uv - \int vdu$$

As an example take $\int t\cos t\,dt$. Here we take $u=t$ (because then $du=dt$) and $dv=\cos(t)dt$. Thus $v=\sin(t)$. We get

$$\begin{aligned}\int t\cos(t)\,dt &= t\sin(t)-\int \sin(t)dt \\ &= t\sin(t)+\cos(t)+C\end{aligned}$$

To see if this is correct let us differentiate the function $t\sin(t)+\cos(t)+C$. Then

$$\begin{aligned}\frac{d}{dt}(t\sin(t)+\cos(t)+C) &= \sin(t)+t\cos(t)-\sin(t) \\ &= t\cos(t)\end{aligned}$$

which shows that the answer is correct.

10) Evaluate the following integrals (notice you might have to use the above rule more than once, you might also have to use substitution)

a) $\int t\sin(t)\,dt = -t\cos(t)+\sin(t)+C$

$$\begin{aligned}\int t\sin(t)\,dt &= -t\cos(t)+\int\cos(t)dt \qquad (u=t, du=dt, dv=\sin(t)dt,\ v=-\cos(t))\\ &= -t\cos(t)+\sin(t)+C\end{aligned}$$

b) $\int te^{t^2}\,dt = \frac{1}{2}e^{t^2}+C$

$$\int t^2e^{t^2}\,dt = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{t^2}+C \qquad (u=t^2,\ du=2tdt)$$

c) $\int \frac{t}{1+t^2}dt = \frac{1}{2}\ln(1+t^2)+C$. Use substitution $u=1+t^2,\ \ du=2tdt$.

d) $\int te^{2t}\,dt = \frac{1}{2}te^{2t}-\frac{1}{4}e^{2t}+C$. Use substitution $u=2t$, $du=2dt$ and what then integration by parts.

e) $\int_0^\pi t\cos(t)\,dt = -2.0$. Use integration by parts: $u=t$, $du=dt$, $dv=\cos(t)dt$, $v=\sin(t)$ to get

$$\begin{aligned}\int_0^\pi t\cos(t)\,dt &= t\sin(t)]_0^\pi - \int_0^\pi \sin(t)\,dt\\ &= \cos(t)]_0^\pi = -2\end{aligned}$$

f) $\int_{-1}^1 t^3+t\,dt = 0$. Good rules to remember are

$$\int_{-T}^T f(t)dt = 2\int_0^T f(t)\,dt$$

if f is even, i.e. $f(t)=f(-t)$, and

$$\int_{-T}^T f(t)\,dt = 0$$

if f is odd, i.e., $f(t)=-f(-t)$. Notice that the function t^3+t is odd.

g) $\int \cos(t)[\sin(t)]^2\,dt = \cos(t)[\sin(t)]^2\,dt$

Use substitution $u=\sin(t)$, $du=\cos(t)\,dt$. Then

$$\int\cos(t)[\sin(t)]^2\,dt = \int u^2\,du = \frac{u^3}{3}+C = \frac{\sin^3(t)}{3}+C$$

h) $\int\frac{\cos(t)}{\sin^2(t)}\,dt = \frac{-1}{\sin(t)}+C$

Use substitution as above $u=\sin(t)$. Then

$$\int\frac{\cos(t)}{\sin^2(t)}\,dt = \int\frac{du}{u^2} = \frac{-1}{u}+C = \frac{-1}{\sin(t)}+C$$

i) $\int t^2e^t\,dt = t^2e^t-2te^t+2e^t+C$. Use partial integration twise. First with $u=t^2$, $dv=e^tdt$, $v=e^t$. Then

$$\int t^2e^t\,dt = t^2e^t - 2\int te^t\,dt$$

Now use partial integration again with $u = t$, $du = dt$, $dv = e^t dt$, $v = e^t$. Then

$$\begin{aligned}\int t^2 e^t \, dt &= t^2 e^t - 2 \int t e^t \, dt \\ &= t^2 e^t - 2(te^t - e^t) + C \\ &= t^2 e^t - 2te^t + 2e^t + C \,.\end{aligned}$$

EXERCISES DFT

Evaluate the following values of the complex exponential function:

Recall that $e^{x+iy} = e^x e^{iy}$ and that $e^{iy} = \cos(y) + i\sin(y)$. Furthermore $e^{-z} = \frac{1}{e^z}$.

1) $e^{\pi i} = -1$ ($e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$)

2) $e^{2+\pi i/2} = ie^2$ ($e^{2+\pi i/2} = e^2 e^{\pi i/2} = e^2(\cos(\pi/2) + i\sin(\pi/2) = ie^2$ because $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$)

3) $\frac{1}{e^{2-3\pi i}} = e^{-2+3\pi i} = -e^{-2}$ (Use that $\frac{1}{e^z} = e^{-z}$. Hence $\frac{1}{e^{2-3\pi i}} = e^{-(2-3\pi i)} = e^{-2}(\cos(3\pi)+i\sin(3\pi))$

4) $e^{6\pi i} = 1$.

Find the DFT for the following samples:
Recall the following:
If $N = 2$ then we have two new basic vectors

$$\begin{aligned} \mathbf{w}_0 &= (1,1) \\ \mathbf{w}_1 &= (1,-1) \end{aligned}$$

Hence

$$\hat{f}_0 = \frac{f_0 + f_1}{2} \qquad \text{and} \qquad \hat{f}_1 = \frac{f_0 - f_1}{2}$$

In case $N = 4$ we have four new basic vectors

$$\begin{aligned} \mathbf{w}_0 &= (1,1,1,1) \\ \mathbf{w}_1 &= (1,i,-1,-i) \\ \mathbf{w}_2 &= (1,-1,1,-1) \\ \mathbf{w}_3 &= (1,-i,-1,i) \end{aligned}$$

The Fourier coefficients $\hat{f}_k$ are given by

$$\hat{f}_k = \frac{< \overrightarrow{\mathbf{f}}, \mathbf{w}_k >}{||\mathbf{w}_k||^2}.$$

Hence (remember the complex conjugation in the $\mathbf{w}'s$)

$$\begin{aligned}
\hat{f}_0 &= \frac{<\overrightarrow{\mathbf{f}}, \mathbf{w}_0>}{4} = \frac{1}{4}(f_0 + f_1 + f_2 + f_3) \\
\hat{f}_1 &= \frac{<\overrightarrow{\mathbf{f}}, \mathbf{w}_1>}{4} = \frac{1}{4}(f_0 - f_1 i - f_2 + f_3 i) \\
\hat{f}_2 &= \frac{<\overrightarrow{\mathbf{f}}, \mathbf{w}_2>}{4} = \frac{1}{4}(f_0 - f_1 + f_2 - f_3) \\
\hat{f}_3 &= \frac{<\overrightarrow{\mathbf{f}}, \mathbf{w}_3>}{4} = \frac{1}{4}(f_0 + f_1 i - f_2 - f_3 i)\,.
\end{aligned}$$

5) $\overrightarrow{\mathbf{f}} = (2, -1)$. $\hat{f}_0 = 1/2$, $\hat{f}_1 = 3/2$.
$\hat{f}_0 = \frac{2-1}{2} = 1/2$ and $\hat{f}_1 = \frac{2+1}{2} = 3/2$
6) $\overrightarrow{\mathbf{f}} = (1+i, 4)$, $\hat{f}_0 = 1/2$, $\hat{f}_1 = 3/2$.
$\hat{f}_0 = \frac{1+i+4}{2} = \frac{5}{2} + \frac{1}{2}i$ and $\hat{f}_1 = \frac{1+i-4}{2} = -\frac{3}{2} + \frac{1}{2}i$
7) $\overrightarrow{\mathbf{f}} = (1, 2, 3, 4)$. $\hat{f}_0 = \frac{5}{2}$, $\hat{f}_1 = -\frac{1}{2} + \frac{1}{2}i$, $\hat{f}_2 = -1/2$, $\hat{f}_3 = -\frac{1}{2} - \frac{1}{2}i$
Work:
$\hat{f}_0 = \frac{1}{4}(1+2+3+4) = \frac{10}{4} = \frac{5}{2} = 2.5$,
$\hat{f}_1 = \frac{1}{4}(1-2i-3+4i) = -\frac{1}{2} + \frac{1}{2}i$,
$\hat{f}_2 = \frac{1}{4}(1-2+3-4) = -1/2$,
$\hat{f}_3 = \frac{1}{4}(1+2i-3-4i) = -\frac{1}{2} - \frac{1}{2}i$
8) $\overrightarrow{\mathbf{f}} = (i, 2+2i, 3, -4)$. $\hat{f}_0 = \frac{1}{4} + \frac{3}{4}i$, $\hat{f} = -\frac{1}{4} - \frac{5}{4}i$, $\hat{f}_2 = \frac{5}{4} - \frac{1}{4}i$, and $\hat{f}_3 = -\frac{5}{4} + \frac{7}{4}i$
Work:
$\hat{f}_0 = \frac{1}{4}(i+2+2i+3-4) = \frac{1}{4} + \frac{3}{4}i$,
$\hat{f}_1 = \frac{1}{4}(i-(2+2i)i-3-4i) = \frac{1}{4}(i-2i+2-3-4i) = -\frac{1}{4} - \frac{5}{4}i$
$\hat{f}_2 = \frac{1}{4}(i-(2+2i)+3+4) = \frac{5}{4} - \frac{1}{4}i$
$\hat{f}_3 = \frac{1}{4}(i+(2+2i)i-3+4i) = \frac{1}{4}(i+2i-2-3+4i) = -\frac{5}{4} + \frac{7}{4}i$
9) $\overrightarrow{\mathbf{f}} = (-1, 2, 4, 3)$. $\hat{f}_0 = 1$, $\hat{f}_1 = -\frac{5}{4} + \frac{1}{4}i$, $\hat{f}_2 = -\frac{1}{2}$, and $\hat{f}_3 = -\frac{5}{4} - \frac{i}{4}$
Work:
$\hat{f}_0 = \frac{1}{4}(-1+2+4+3) = 1$.
$\hat{f}_1 = \frac{1}{4}\left(-1-2i-4+3i\right) = -\frac{5}{4} + \frac{1}{4}i$
$\hat{f}_2 = \frac{1}{4}\left(-1-2+4-3\right) = -\frac{1}{2}$
$\hat{f}_3 = \frac{1}{4}\left(-1+2i-4-3i\right) = -\frac{5}{4} - \frac{i}{4}$

Math 2025, Problems/linear independent, generating

Recall that a set of vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ is called linearly **independent** if

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n = 0$$

implies that all the numbers $c_1, \ldots, c_n$ are zero. If the vectors are **not linearly independent** then they are called **linearly dependent.** Remember that a set of vectors is linearly independent if and only if one of the vectors is a linear combinations of the others.

1) Which of the following set of vectors is linearly independent? If the set is linearly dependent express **one** of the vectors as a linear combination of the other.

(1) $(1,2)$, $(-1,2)$. Linearly independent. If $c_1(1,2) + c_2(-1,2) = (c_1 - 2c_2, 2c_1 + 2c_2) = (0,0)$ then

$$\begin{aligned} c_1 - 2c_2 &= 0 \\ 2c_1 + 2c_2 &= 0 \end{aligned}$$

Adding we get $3c_1 = 0$ or $c_1 = 0$. The second equation gives $c_2 = -c_1$. Whence $c_2 = 0$.

(2) $(1,0)$, $(1,1), (2,1)$. Linearly dependent. Every set of three or more vectors in $\mathbb{R}^2$ is linearly dependent.

(3) $(1,1,0)$, $(2,2,0)$, $(0,0,1)$. Linearly dependent: $(2,2,0) = 2(1,1,0)$.

(4) $(1,-1,1)$, $(1,0,-1)$, $(0,1,0)$. Linearly independent: If

$$c_1(1,-1,1) + c_2(1,0,-1) + c_3(0,1,0) = (c_1 + c_2, -c_1 + c_3, c_1 - c_2) = (0,0,0)$$

then

$$\begin{aligned} c_1 + c_2 &= 0 \\ -c_1 + c_3 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

Adding the first and the last equation gives

$$2c_1 = 0 \qquad \text{or} \qquad c_1 = 0$$

The second and the equations says that

$$c_2 = c_1 \qquad c_3 = c_1$$

Hence

$$c_2 = 0 \qquad \text{and} \qquad c_3 = 0$$

(5) $(1,0,1)$, $(1,0,-1)$, $(0,1,0)$. Linearly independent: If

$$c_1(1,0,1) + c_2(1,0,-1) + c_3(0,1,0) = (c_1 + c_2, c_3, c_1 - c_2) = (0,0,0)$$

Then

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_3 &= 0 \\ c_1 - c_2 &= 0 \end{aligned}$$

The second equation gives

$$c_3 = 0$$

Adding the first and the third gives

$$2c_1 = 0 \qquad \text{or} \qquad c_1 = 0$$

The third equation then implies that

$$c_2 = 0$$

(6) $(1,0,0)$, $(1,1,0)$, $(1,1,1)$, $(1,2,0)$. Linearly dependent. Every set in $\mathbb{R}^3$ containing more than three vectors is linearly dependent.

2) Express the following vectors as a linear combination of $(1,1)$ and $(1,0)$:

(1) $(1,2) = 2 \times (1,1) - (1,0)$. How would we solve it? If

$$(1,2) = a(1,1) + b(1,0) = (a+b, a)$$

Then

$$\begin{aligned} a + b &= 1 \\ a &= 2\,. \end{aligned}$$

The second equation gives the number a. Substraction gives the second number b:

$$b = -1\,.$$

(2) $(-1,2) = 2 \times (1,1) - 3 \times (1,0)$

(3) $(1,-1) = -(1,1) + 2 \times (1,0)$

3) Express the following vectors as a linear combination of $(2,1)$ and $(1,-1)$:

(1) $(1,0) = \frac{1}{3}(2,1) + \frac{1}{3}(1,-1)$. If

$$(1,0) = a(2,1) + b(1,-1) = (2a+b, a-b)$$

Then

$$\begin{aligned} 2a + b &= 1 \\ a - b &= 0 \end{aligned}$$

The second equation gives that

$$a = b$$

Adding the second equation to the first gives

$$3a = 1 \qquad or \qquad a = 1/3$$

(2) $(0,1) = \frac{1}{3}(2,1) - \frac{2}{3}(1,-1)$. Just as above we get

$$\begin{aligned} 2a + b &= 0 \\ a - b &= 1 \end{aligned}$$

Adding gives $3a = 1$ or $a = 1/3$. Then the second equation gives

$$b = a - 1 = \frac{1}{3} - 1 = -\frac{2}{3}$$

(3) $(2,2) = \frac{4}{3}(2,1) - \frac{2}{3}(1,-1)$ We have

$$\begin{aligned} 2a + b &= 2 \\ a - b &= 2 \end{aligned}$$

Or

$$3a = 4 \qquad a = 4/3$$

Hence

$$b = a - 2 = \frac{4}{3} - 2 = \frac{-2}{3}$$

(4) $(2,-2) = 2 \times (1,-1)$

4) Express the following vectors as a linear combination of $(2,1,0)$, $(1,0,1)$, $(0,1,0)$. The equations that we get if we set

$$\begin{aligned} (x,y,z) &= a(2,1,0) + b(1,0,1) + c(0,1,0) \\ &= (2a + b, a + c, b) \end{aligned}$$

are

$$\begin{aligned} 2a + b &= x \\ a + c &= y \\ b &= z \end{aligned}$$

From the third equation we get

$$b = z$$

Putting this information into the first equation gives

$$2a = x - b = x - z$$

or

$$a = \frac{1}{2}(x - z)$$

Now put this into the second equation to get

$$c = y - a = y - \frac{1}{2}(x - z)$$

(1) $(1,0,0) = \frac{1}{2}(2,1,0) - \frac{1}{2}(0,1,0)$. Using the above we get

$$\begin{aligned} a &= \frac{1}{2}(x-z) = \frac{1}{2} \\ b &= z = 0 \\ c &= y - \frac{1}{2}(x-z) = -\frac{1}{2} \end{aligned}$$

(2) $(0,1,0) = (0,1,0)$ Here

$$a = b = 0 \qquad c = 1$$

(3) $(0,0,1) = -\frac{1}{2}(2,1,0) + (1,0,1) + \frac{1}{2}(0,1,0)$. Here

$$\begin{aligned} a &= \frac{1}{2}(x-z) = -\frac{1}{2} \\ b &= z = 1 \\ c &= y - \frac{1}{2}(x-z) = \frac{1}{2} \end{aligned}$$

(4) $(2,2,-1) = \frac{3}{2}(2,1,0) - (1,0,1) + \frac{1}{2}(0,1,0)$.

$$\begin{aligned} a &= \frac{1}{2}(2+1) = \frac{3}{2} \\ b &= -1 \\ c &= 2 - \frac{1}{2}(2+1) = 2 - \frac{3}{2} = \frac{1}{2} \end{aligned}$$

We can also solve this by noticing that the solution is 2×(the solution in 1)+2×(the solution in 2)−(the solution from 3)

(5) $(0,1,1) = -\frac{1}{2}(2,1,0) + (1,0,1) + \frac{3}{2}(0,1,0)$

Recall that a set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in a vector space is called **generating** if every vector $\mathbf{u}$ in the space can be written as a combination of those vectors, i.e., we can find scalars $c_1, \ldots, c_n$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_n\mathbf{u}_n$$

5) Which of the following set of vectors is generating for $\mathbb{R}^2$?

(1) $(1,0)$, $(0,1)$. Generating

(2) $(1,1), (2,2)$. Not generating

(3) $(2,1), (-2,1)$. Generating

(4) $(2,0), (1,1), (3,1)$. Generating, but we only need two of them $(3,1) = (2,0) + (1,1)$

6) Which of the following set of vectors is generating for $\mathbb{R}^3$?

(1) $(1,0,2), (0,1,0), (0,0,2)$. Generating

(2) $(1,-1,2), (1,0,0)$. Not generating. We need at least three vectors to generate $\mathbb{R}^3$.

(3) $(1,2,3),(1,-1,0),(0,1,1)$. Not generating because

$$(1,2,3)-3(0,1,1)=(1,-1,0)$$

So those three vectors are linearly dependent, but we need three linearly **independent** vectors to generate $\mathbb{R}^3$.

Recall that a set is called a **basis** if it is generating **and** linearly independent.

7) Which of the following set is a basis for $\mathbb{R}^2$?

(1) $(1,0),(0,1)$. Basis
(2) $(1,2),(-1,1)$. Basis
(3) $(1,1),(2,2)$. Not a basis, because linearly dependent.
(4) $(-1,1)$. Not a basis, because not generating. A basis contains exactly two elements!
(5) $(1,2),(-1,1),(1,0)$. Not a basis. It is generating but not linearly independent.

8) Which of the following set is a basis for $\mathbb{R}^3$?

(1) $(1,0,2),(1,1,0),(0,0,2)$
(2) $(2,0,1),(0,1,0)$
(3) $(2,0,1),(-1,1,0),(1,0,1),(0,1,1)$
(4) $(1,2,-1),(0,-1,2),(1,0,3)$

9) Let V be a vector space with an inner product and let $\mathbf{u}_1,\ldots,\mathbf{u}_n$ be a set of non-zero vectors of length one which are orthogonal to each other. Thus $<\mathbf{u}_i,\mathbf{u}_j>=0$ if $i\neq j$ and $<\mathbf{u}_i,\mathbf{u}_i>=1$. Are then the following statement correct or not?

(1) Then the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_n$ are linearly independent.
(2) If the vector $\mathbf{u}$ is in the span of $\mathbf{u}_1,\ldots,\mathbf{u}_n$, i.e., $\mathbf{u}$ can be written as a combination of $\mathbf{u}_1,\ldots,\mathbf{u}_n$ then we much have

$$\mathbf{u}=<\mathbf{u},\mathbf{u}_1>\mathbf{u}_1+\ldots+<\mathbf{u},\mathbf{u}_n>\mathbf{u}_n$$

SOLUTION: Both statements are correct.

Assume that

$$c_1\mathbf{u}_1+c_2+\ldots+c_n\mathbf{u}_n=\mathbf{u}$$

for some vector $\mathbf{u}$. Take the inner product to get:

$$\begin{aligned}<\mathbf{u},\mathbf{u}_i> &= <\sum_{j=1}^{n}c_j\mathbf{u}_j,\mathbf{u}_i> \\ &= \sum_{j=1}^{n}c_j<\mathbf{u}_j,\mathbf{u}_i> \\ &= c_i||\mathbf{u}_i||^2 \\ &= c_i\end{aligned}$$

where we have used in the third line that $< \mathbf{u}_j, \mathbf{u}_i >= 0$ if $j \neq i$. In the fourth line we used that $< \mathbf{u}_i, \mathbf{u}_i >= 1$. This show that the second statement is correct.

In the first statement we assume that $\mathbf{u} = 0$. Then $< \mathbf{u}, \mathbf{u}_i >= 0$ for all $i = 1, \ldots, n$. Hence all the numbers c_i are zero. But that means that the vectors are linearly independent.

Math 2025, Problems/Inner products

Recall that an inner product on $\mathbb{R}^n$ is given by $<(x_1,\ldots,x_n),(y_1,\ldots,y_n)>= x_1y_1+\ldots+x_ny_n$. The length or norm of a vector $\overrightarrow{\mathbf{x}}=(x_1,\ldots,x_n)$ is the real number

$$||(x_1,\ldots,x_n)|| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2} = \sqrt{<\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{x}}>}$$

1) Evaluate the inner product of the following vectors in $\mathbb{R}^n$:

a) $<(1,0,-3),(2,1,3)>= 1\times 2+0\times 1-3\times 3 = 2-9=-7$

b) $<(2,2,-1,-1),(1,1,2,2)>= 2\times 1+2\times 1-1\times 2-1\times 2=0$

c) $<(1,2,3,4,5),(0,2,5,3,6)>= 1\times 0+2\times 2+3\times 5+4\times 3+5\times 6=61$

2) Find the norm of the following vectors:

a) $||(1,3,1)|| = \sqrt{1+9+1}=\sqrt{11}\simeq 3.317$

b) $||(-1,2,3,5)|| = \sqrt{(-1)^2+2^2+3^2+5^2}=\sqrt{39}\simeq 6.245$

Two non-zero vectors are called perpendicular or orthogonal if the inner product $<\overrightarrow{\mathbf{x}},\overrightarrow{\mathbf{y}}>=0$.

3) Which of the following vectors are perpendicular to each other?

a) $(1,0)$, $(0-2)$. Orthogonal

b) $(1,0,1)$, $(-1,2,1)$ Orthogonal

c) $(2,-1,0)$, $(1,1,3)$ Not orthognal (inner product $=2-1=1$)

An inner product on $\mathbb{C}^n$ is given by $<(z_1,\ldots,z_n),(w_1,\ldots,w_n)>= z_1\overline{w_1}+\ldots+z_n\overline{w_n}$ where denotes complex conjugation $\overline{x+iy}=x-iy$. The norm of $\overrightarrow{\mathbf{z}}=(z_1,\ldots,z_n)$ is given by $||\overrightarrow{\mathbf{z}}|| = \sqrt{|z_1|^2+\ldots+|z_n|^2}=\sqrt{<\overrightarrow{\mathbf{z}},\overrightarrow{\mathbf{z}}>}$. Two non-zero vectors $\overrightarrow{\mathbf{z}}$ and $\overrightarrow{\mathbf{y}}$ are perpenticular or orthogonal if $<\overrightarrow{\mathbf{z}},\overrightarrow{\mathbf{y}}>=0$.

4) Evaluate the inner product of the following vectors in $\mathbb{C}^n$:

a) $<(1,1+i,2i),(-1,1-i,2+i)>= 1\times(-1)+(1+i)\times\overline{(1-i)}+2i\times\overline{(2+i)} = -1+1+2i+2i+2 = 2+4i$

b) $<(i,2i,1+4i),(1-i,3-i,3)>= i\times\overline{(1-i)}+2i\times\overline{(3-i)}+(1+4i)\times 3 = i-1+6i-2+3+12i = = 19i$

5) Which of the folowing two vectors are orthogonal to each other:

a) $(1,1,-i)$, $(1,1+i,1)$: $1\times 1+1\times\overline{(1+i)}-i\times 1 = 1+1-i-i=2-2i$ not orthogonal:

b) $(1+i,i,1-i)$, $(1-i,1,-1-i)$: $(1+i)\times\overline{(1-i)}+i\times 1+(1-i)\times\overline{(-1-i)} = 1+2i-1+i-2=-2+3i$ not orthognal

6) Find the norm of the following vectors in $\mathbb{C}^n$:

a) $||(1+i,2-3i,i)|| = \sqrt{1+1+4+9+1}=4.0$

b) $||(i,1,-1)|| = \sqrt{1+1+1}=\sqrt{3}\approx 1.732$

Let $f:[a,b]\to\mathbb{C}$ be a continuous function. Then f can be written as $f(x)=u(x)+iv(x)$ where u and v are continuous real valued functions on $[a,b]$. Then integral of f over $[a,b]$ is then defined as

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx$$

7) Evaluate the following integrals:

a) $\int_0^1 (1+it)^2\,dt = \int_0^1 1-t^2+2it\,dt = \int_0^1 1-t^2\,dt + 2i\int_0^1 t\,dt = t - \frac{t^3}{3}\Big]_0^1 + it^2]_0^1 = \frac{2}{3}+i$

b) $\int_{-1}^1 \frac{1}{1+it}dt = \int_{-1}^1 \frac{1-it}{1+t^2}\,dt = \int_{-1}^1 \frac{1}{1+t^2}\,dt - i\int_{-1}^1 \frac{t}{1+t^2}\,dt$. For the first integral we use

$$\int \frac{1}{1+t^2}\,dt = \tan^{-1}(t) + C$$

Furthermore $\tan^{-1}(1) = \frac{\pi}{4}$ and $\tan^{-1}(-1) = -\frac{\pi}{4}$. For the second integral we use the substitution

$$u = 1+t^2, \qquad du = 2tdt$$

to get

$$\int \frac{t}{1+t^2}\,dt = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(|u|) + C = \frac{1}{2}\ln(1+t^2) + C$$

This together gives:

$$\int_{-1}^1 \frac{1}{1+it}dt = \tan^{-1}(t)\big]_{-1}^1 - \frac{i}{2}\ln(1+t^2)\Big]_{-1}^1 = \frac{\pi}{2}$$

c) $\int_0^2 (1+it)e^t dt = \int_0^2 dt + i\int_0^2 te^t dt$. The first integral gives

$$\int_0^2 e^t dt = e^t]_0^2 = e^2 - 1\,.$$

The second integral gives (using integration by parts) $\int udv = uv - \int vdu$, with $u=t$, $du=dt$, $dv = e^t dt$ and hence $v = e^t$.

$$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t + C$$

Inserting the limits of integration we get

$$\int_0^2 (1+it)e^t\,dt = t]_0^2 + i\left[te^t - e^t\right]_0^2 \approx 6.389 + 8.389i$$

d) $\int_1^2 t\cos(t)\,dt$. Use integration by parts with $u=t$, $du=dt$, and $dv = \cos(t)dt$, $v = \sin(t)$. Hence

$$\int t\cos(t)\,dt = t\sin(t) - \int \sin(t)dt = t\sin(t) + \cos(t) + C\,.$$

Finally

$$\int_1^2 t\cos(t)\,dt = [t\sin(t) + \cos(t)]_1^2 \approx 0.02068$$

An inner product on the space $C([a,b],\mathbb{C})$ of continuous complex valued functions on $[a,b]$ is given by

$$< f,g >= \int_a^b f(t)\overline{g(t)}\,dt$$

The corresponding norm is

$$||f|| = \sqrt{< f,f >} = \sqrt{\int_a^b |f(t)|^2 dt}\,.$$

Two non-zero functions f and g are said to be orthogonal if $< f,g >= 0$.

8) Find the inner product of the following function in $C([0,1],\mathbb{C})$ (thus $a = 0$ and $b = 1$).

a) $< 1 + it, -it >= -1/3 - i/2$

$$
\begin{aligned}
< 1 + it, -it > &= \int_0^1 (1 + it) \times \overline{(-it)}\, dt \\
&= \int_0^1 it - t^2\, dt \\
&= -\int_0^1 t^2\, dt + i \int_0^1 t dt \\
&= \left.\frac{-1}{3} t^3 \right]_0^1 + \left.\frac{i}{2} t^2 \right]_0^1 \\
&= -1/3 + i/2
\end{aligned}
$$

b) $< \cos(t), t > \approx .3818$

$$
\begin{aligned}
< \cos(t), t > &= \int_0^1 \cos(t) t\, dt \\
&= t \sin(t) + \cos(t)]_0^1 \\
&= \sin(1) + \cos(1) - 1 \\
&\approx .3818
\end{aligned}
$$

c) $< t, 1 - it >= \frac{1}{2} + \frac{i}{3}$

$$
\begin{aligned}
\int_0^1 t\overline{(1 - it)}\, dt &= \int_0^1 t(1 + it)\, dt \\
&= \int_0^1 t + it^2\, dt \\
&= \left.\frac{t^2}{2}\right]_0^1 + i \left.\frac{t^3}{3}\right]_0^1 \\
&= \frac{1}{2} + \frac{i}{3}\,.
\end{aligned}
$$

9) Find the norm of the following functions (if you can not evaluate the integrals, then just set up the formulas)

a) $||\cos(t)|| = \sqrt{\int_0^1 \cos^2(t)\, dt} \approx .852\,8$. We can evaluate the integral in the following way

$$
\cos^2(t) = \frac{1}{2}(1 + \cos(2t))
$$

Hence

$$\begin{aligned}\int \cos^2(t)\,dt &= \frac{1}{2}\int 1+\cos(2t)\,dt \\ &= \frac{1}{2}\left(t+\frac{1}{2}\sin(2t)+C\right) \\ &= \frac{t}{2}+\frac{\sin(2t)}{4}+C\end{aligned}$$

Hence

$$\begin{aligned}\int_0^1 \cos^2(t)\,dt &= \left.\frac{t}{2}+\frac{\sin(2t)}{4}\right]_0^1 \\ &= \frac{1}{2}+\frac{\sin(2)}{4} \\ &\approx .727\,3\end{aligned}$$

Finally we have

$$\sqrt{.727\,3}\approx .852\,8$$

b) $||1+it|| = \sqrt{\int_0^1 1+t^2 dt} = \sqrt{t+\frac{t^3}{3}\Big]_0^1} = \sqrt{1+\frac{1}{3}} = \sqrt{\frac{4}{3}} = \frac{2}{\sqrt{3}} \approx 1.155$

c) $||\frac{1}{1+it}|| = \sqrt{\int_0^1 \frac{1}{1+t^2}\,dt} = \sqrt{\tan^{-1}(t)|_0^1} = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}$

d) $||te^t|| = \sqrt{\int_0^1 te^t dt} = 1.$

Here we use what we have already found out:

$$\int te^t dt = te^t - e^t + C$$

Hence

$$\int_0^1 te^t dt = te^t - e^t\Big]_0^1 = e - e + 1 = 1$$

Recall the following rule for integration (what is it called?)

$$\int uv'\,dt = uv - \int vu'\,dt$$

or

$$\int udv = uv - \int vdu$$

As an example take $\int t\cos t\,dt$. Here we take $u=t$ (because then $du=dt$) and $dv=\cos(t)dt$. Thus $v=\sin(t)$. We get

$$\begin{aligned}\int t\cos(t)\,dt &= t\sin(t)-\int\sin(t)dt \\ &= t\sin(t)+\cos(t)+C\end{aligned}$$

To see if this is correct let us differentiate the function $t\sin(t)+\cos(t)+C$. Then

$$\begin{aligned}\frac{d}{dt}(t\sin(t)+\cos(t)+C) &= \sin(t)+t\cos(t)-\sin(t) \\ &= t\cos(t)\end{aligned}$$

which shows that the answer is correct.

10) Evaluate the following integrals (notice you might have to use the above rule more than once, you might also have to use substitution)

a) $\int t\sin(t)\,dt = -t\cos(t) + \sin(t) + C$

$$\begin{aligned}\int t\sin(t)\,dt &= -t\cos(t) + \int \cos(t)dt \qquad (u = t, du = dt, dv = \sin(t)dt,\ v = -\cos(t)) \\ &= -t\cos(t) + \sin(t) + C\end{aligned}$$

b) $\int te^{t^2}\,dt = \frac{1}{2}e^{t^2} + C$

$$\int t^2e^{t^2}\,dt = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{t^2} + C \qquad (u = t^2,\ du = 2tdt)$$

c) $\int \frac{t}{1+t^2}dt = \frac{1}{2}\ln(1+t^2) + C$. Use substitution $u = 1 + t^2,\ \ du = 2tdt$.

d) $\int te^{2t}\,dt = \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + C$. Use substitution $u = 2t$, $du = 2dt$ and what then integration by parts.

e) $\int_0^{\pi} t\cos(t)\,dt = -2.0$. Use integration by parts: $u = t$, $du = dt$, $dv = \cos(t)dt$, $v = \sin(t)$ to get

$$\begin{aligned}\int_0^{\pi} t\cos(t)\,dt &= t\sin(t)]_0^{\pi} - \int_0^{\pi}\sin(t)\,dt \\ &= \cos(t)]_0^{\pi} = -2\end{aligned}$$

f) $\int_{-1}^{1} t^3 + t\,dt = 0$. Good rules to remember are

$$\int_{-T}^{T} f(t)dt = 2\int_0^T f(t)\,dt$$

if f is even, i.e. $f(t) = f(-t)$, and

$$\int_{-T}^{T} f(t)\,dt = 0$$

if f is odd, i.e., $f(t) = -f(-t)$. Notice that the function $t^3 + t$ is odd.

g) $\int \cos(t)[\sin(t)]^2\,dt = \cos(t)[\sin(t)]^2\,dt$

Use substitution $u = \sin(t)$, $du = \cos(t)\,dt$. Then

$$\int \cos(t)[\sin(t)]^2\,dt = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sin^3(t)}{3} + C$$

h) $\int \frac{\cos(t)}{\sin^2(t)}\,dt = \frac{-1}{\sin(t)} + C$

Use substitution as above $u = \sin(t)$. Then

$$\int \frac{\cos(t)}{\sin^2(t)}\,dt = \int \frac{du}{u^2} = \frac{-1}{u} + C = \frac{-1}{\sin(t)} + C$$

i) $\int t^2e^t\,dt = t^2e^t - 2te^t + 2e^t + C$. Use partial integration twise. First with $u = t^2$, $dv = e^tdt$, $v = e^t$. Then

$$\int t^2e^t\,dt = t^2e^t - 2\int te^t\,dt$$

Now use partial integration again with $u = t$, $du = dt$, $dv = e^t dt$, $v = e^t$. Then

$$\begin{aligned}\int t^2 e^t\, dt &= t^2 e^t - 2\int t e^t\, dt \\ &= t^2 e^t - 2(te^t - e^t) + C \\ &= t^2 e^t - 2te^t + 2e^t + C\,.\end{aligned}$$

Math 2025, Problems/Inner products

Recall: Let V be a vector space with inner product $< \cdot, \cdot >$. Let $W \subset V$ be a finite dimensional subspace. Let $w_1, \ldots, w_n$ be an orthogonal basis for W. Then the **orthogonal projection** $P_w : V \to W$ is given by

$$P_W(v) = \frac{< v, w_1 >}{||w_1||^2} w_1 + \ldots + \frac{< v, w_n >}{||w_n||^2} w_n \,.$$

If $v \in V$ then the point in W closest to v is exactly $P_W(v)$. Let us recall how we get this result:

1) If the piont $w = s_1 w_1 + s_2 w_2 + \ldots + s_n w_n \in W$ is such that

$$< v - w, w_j >= 0 \quad \text{for all } j = 1, \ldots n$$

Then the coefficients s_j are given by

$$s_j = \frac{< v.w_j >}{||w_j||^2} \,.$$

Whe? Let us see what our assumption $< v - w, w_j >= 0$ for all $j = 1, \ldots, n$ implies: We take the inner product with each of the vectors w_j and get **using that the inner product is linear in the first factor:**

$$< v - w, w_j >=< v, w_j > - \sum_{i=1}^{n} s_i < w_i, w_j >$$

But $< w_i, w_j >= 0$ if $i \neq j$ and $< w_j, w_j >= ||w_j||^2$. Hence the above reduces to

$$0 =< v - w, w_j >=< v, w_j > -s_j \, ||w_j||^2$$

or

$$s_j = \frac{< v, w_j >}{||w_j||^2}$$

which is exactly what the above formula says.

2) If $v - w \perp w_j$, $j = 1, \ldots, n$ then $v - w$ is perpenticular to all points in the plane W.

Why? How does an arbitary point u in the plain W looks like? By our assumption the vectors $w_1, \ldots, w_n$ are a basis. Hence $u = a_1 w_1 + \ldots + a_n w_n$ for some numbers $a_j \in \mathbb{R}$ (or $a_j \in \mathbb{C}$). Hence

$$\begin{aligned} < v - w, u > &=< v - w, \sum_{j=1}^{n} a_j w_j > \\ &= \sum_{j=1}^{n} \overline{a_j} < v - w, w_j > \\ &= 0 \end{aligned}$$

because the inner product is *conjugate* linear in the second variable.

3) Why is the orthogonal projection $P_W(v)$ the point in the plane W closest to v? Let u be another point in the plain. We want to show that

$$||v - P_W(v)|| \leq ||v - u||$$

or – which is the same (why?) –

$$||v - P_W(v)||^2 \leq ||v - u|| \ .$$

Let us just calculate this:

$$\begin{aligned} ||v-u||^2 &= ||v - P_W(v) + (P_W(v) - u)||^2 \\ &=< v - P_W(v) + (P_W(v) - u), v - P_W(v) + < P_W(v) - u, P_W(v) - u > \\ &=< v - P_W(v), v - P_W(v) > + < P_W(v) - u, v - P_W(v) > \\ &+ < P_W(v) - u, v - P_W(v) > + < P_W(v) - u, P_W(v) - u > \\ &= ||v - P_W(v)||^2 + ||P_W(v) - u||^2 \\ &\geq ||v - P_W(v)||^2 \end{aligned}$$

Where we have used that $P_W(v) - u$ is orthogonal to the plain W and that the lenght of a vector is always positive.

1) Let $V = \mathbb{R}^2$ and let W be the line $\mathbb{R}(1,1)$. Find the orthogonal projection of the following vectors to W:

Notice first that $||(1,1)||^2 = 1 + 1 = 2$
a) $P_W(1,2) = \frac{<(1,2),(1,1)>}{||(1,1)||^2}(1,1) = \frac{3}{2}(1,1) = (1.5, 1.5)$
b) $P_W(3,-5) = \frac{<(3,-5),(1,1)>}{2}(1,1) = \frac{3-5}{2}(1,1) = -(1,1)$
c) $P_W(8,1) = \frac{<(8,1),(1,1)>}{2}(1,1) = \frac{9}{2}(1,1) = (4.5, 4.5)$
2) Find the point on the line $\mathbb{R}(2,-1)$ closest to the point $(5,2)$.
Solution: The point on the line closest to $(5,2)$ is the orthogonal projection of $(5,2)$ onto the line. Hence this point is given by

$$\begin{aligned} \frac{<(5,2),(2,-1)>}{||(2,-1)||^2}(2,-1) &= \frac{10-2}{2^2+(-1)^2}(2,-1) \\ &= \frac{8}{5}(2,-1) \\ &= \left(\frac{16}{5}, -\frac{8}{5}\right) \end{aligned}$$

The final answere is therefore: The point on the line, closest to $(5,2)$ is the point $(16/5, -8/5)$.
3) Find the point on the line $3y = 2x$ closest to the point $(-4,3)$.
Solution: The first thing we need to do is to find *one* vector – call it w – on the line. If we take $x = 1$ then $y = \frac{2}{3}$. Hence $(1, 2/3)$ is on the line. But then also $(3,2)$ is on the line. Let $w = (3,2)$.

Then $||w||^2 = 9 + 4 = 13$. Hence the point on the line, closest to $(-4, 3)$ is given by

$$\begin{aligned} \frac{<(-4,3),(3,2)>}{13}(3,2) &= \frac{-12+6}{13}(3,2) \\ &= \frac{-6}{13}(3,2) \end{aligned}$$

4) Let $V = \mathbb{R}^3$ and let W be the plane generated by the vectors $(1, 0, 1)$ and $(0, 1, 0)$. Find the orthogonal projection of the following vectors to W:

a) $P_W(1, 3, -1) =$

b) $P_W(1, 1, 1) =$

c) $P_W(4, -2, 1) =$

Solution: Notice first that the vectors $(1, 0, 1)$ and $(0, 1, 0)$ are orthogonal. Furthermore

$$||(1,0,1)||^2 = 2$$

and

$$||(0,1,0)||^2 = 1$$

Hence for (a)

$$\begin{aligned} P_W(1,3,-1) &= \frac{<(1,3,-1),(1,0,1)>}{2}(1,0,1) + \frac{<(1,3,-1),(0,1,0)>}{1}(0,1,0) \\ &= \frac{1-1}{2}(1,0,1) + 3(0,1,0) \\ &= (0,3,0) \end{aligned}$$

b) We have

$$\begin{aligned} P_W(1,1,1) &= \frac{1+1}{2}(1,0,1) + 1(0,1,0) \\ &= (1,0,1) + (0,1,0) \\ &= (1,1,1) \end{aligned}$$

Hence we get the point back again. That simply meens, that $(1, 1, 1) = (1, 0, 1) + (0, 1, 0)$ is in the planin spanned by the vectors $(1, 0, 1)$ and $(0, 1, 0)$.

c) We get in the same way:

$$\begin{aligned} P_W(4,-2,1) &= \frac{4+1}{2}(1,0,1) - 2(0,1,0) \\ &= \frac{5}{2}(1,0,1) - (0,2,0) \\ &= \left(\frac{5}{2}, -2, \frac{5}{2}\right). \end{aligned}$$

5) Find the point on the line $\mathbb{R}(1, -1, 2)$ closest to the points:

a) $(4, 2, -1)$
b) $(3, -1, 10)$.

Solution: Recall that the point on the line generated by $(1, -1, 2)$ is given by the orthogonal projection, which we will denote by P. Next we calculate

$$||(1, -1, 2)||^2 = 1 + 1 + 4 = 6\,.$$

a) We have

$$\begin{aligned} P(4, 2, -1) &= \frac{< (4, 2, -1), (1, -1, 2)}{6}(1, -1, 2) \\ &= \frac{4 - 2 - 2}{6}(1, -1, 2) \\ &= 0\,. \end{aligned}$$

b) In this case we get:

$$\begin{aligned} P(3, -1, 10) &= \frac{< (3, -1, 10), (1, -1, 2) >}{6}(1, -1, 2) \\ &= \frac{3 + 1 + 10}{6}(1, -1, 2) \\ &= \frac{7}{3}(1, -1, 2) \\ &= \left(\frac{7}{3}, -\frac{7}{3}, \frac{14}{3}\right)\,. \end{aligned}$$

6) Let W be the subspace in $\mathbb{R}^3$ generated by the vectors $(1, 1, 0)$ and $(1, -1, 1)$. Find the point in W closest to the points:
a) $(1, 2, 3)$
b) $(-1, 0, 1)$
b) $(1, 0, 1)$
Solution: First we notice that

$$< (1, 1, 0), (1, -1, 1) >= 1 - 1 = 0\,.$$

Hence the vectors $(1, 1, 0)$ and $(1, -1, 1)$ are orthogonal. Then we calculate the lenght of the generating vectors. Those are

$$||(1, 1, 0)||^2 = 1 + 1 = 2$$

and

$$||(1, -1, 1)||^2 = 3\,.$$

a) The point is given by

$$\begin{aligned}\frac{<(1,2,3),(1,1,0)>}{2}(1,1,0)+\frac{<(1,2,3),(1,-1,1)>}{3}(1,-1,1) &= \frac{1+2}{2}(1,1,0)+\frac{1-2+3}{3}(1,-1,1)\\ &= \frac{3}{2}(1,1,0)+\frac{2}{3}(1,-1,1)\\ &= \left(\frac{3}{2}+\frac{2}{3},\frac{3}{2}-\frac{2}{3},\frac{2}{3}\right)\\ &= \left(\frac{13}{6},\frac{5}{6},\frac{2}{3}\right)\end{aligned}$$

b) Same calculation gives $\left(\frac{-1}{2},\frac{-1}{2},0\right)$.

c) Here we get:

$$\frac{1}{2}(1,1,0)+\frac{2}{3}(1,-1,1)=\left(\frac{7}{6},\frac{1}{6},\frac{2}{3}\right).$$

7) Let $V = PC([0,1])$ be the space of (real valued) piecewise continuous function on the closed interval $[0,1]$. Let W be the wavelet space generated by $\varphi_0^{(1)}$ and $\varphi_1^{(1)}$. Thus W is the space of all functions of the form $s_0\varphi(2t)+s_1\varphi(2t-1)$. Find the orthogonal projection of the following functions $f(t)$ onto W:

a) $f(t)=1$ for all t.

b) $f(t)=2t$

c) $f(t)=4t+2$

d) $f(t)=\varphi(4t)+3\varphi(4t-1)$.

e) $f(t)=2\varphi(4t)+4\varphi(4t-1)+3\varphi(4t-2)+9\varphi(4t-3)$

Solution: Let us first recall that the inner product in $PC([0,1])$ is given by

$$<f,g>=\int_0^1 f(t)\,g(t)\,dt$$

In particular the norm of a function in $PC([0,1])$ is given by

$$||f||^2=\int_0^1 f(t)^2\,dt\,.$$

Let us start by finding the norm of $\varphi_0^{(1)}(t)=\varphi(2t)$ and $\varphi_1^{(1)}(t)=\varphi(2t-1)$. A simple calculation show that

$$\varphi_0^{(1)}(t)^2=\varphi_0^{(1)}(t)=\begin{cases}1 & 0\le t<0.5\\ 0 & \text{else}\end{cases}$$

and

$$\varphi_1^{(1)}(t)^2=\varphi_1^{(1)}(t)=\begin{cases}1 & 0.5\le t<1\\ 0 & \text{else}\end{cases}$$

Hence

$$\left|\left|\varphi_0^{(1)}\right|\right|^2=\int_0^{1/2}1\,dt=t\big|_0^{1/2}=\frac{1}{2}$$

and

$$\left|\left|\varphi_1^{(1)}\right|\right|^2 = \int_{1/2}^{1} 1dt = t|_{1/2}^{1} = \frac{1}{2}.$$

a) Here we simply notice (or evaluate) that $f = 1 = \varphi_0^{(1)} + \varphi_1^{(1)} \in V_1$.

b) First we notice that

$$2t\varphi_0^{(1)}(t) = \begin{cases} 2t & 0 \le t < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

and

$$2t\varphi_1^{(1)}(t) = \begin{cases} 2t & \frac{1}{2} \le t < 1 \\ 0 & \text{else} \end{cases}$$

Hence

$$\begin{aligned} < f, \varphi_0^{(1)} > &= \int_0^{1/2} 2t\,dt \\ &= t^2\Big]_0^{1/2} = \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} < f, \varphi_1^{(1)} > &= \int_{1/2}^{1} 2t\,dt \\ &= t^2\Big]_{1/2}^{1} \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

Hence the projection $P(f)$ is given by

$$\begin{aligned} P(f)(t) &= \frac{1/4}{1/2}\varphi(2t) + \frac{3/4}{1/2}\varphi(2t-1) \\ &= \frac{1}{2}\varphi(2t) + \frac{3}{2}\varphi(2t-1). \end{aligned}$$

c) There are at least two different ways to solve this part. One way is to do the calculation directly or notice that the projection operator is **linear**. Hence by using (a) and (b) we get

$$\begin{aligned} P(4t+2) &= 2P(2t) + 2P(1) \\ &= 2\left(\frac{1}{2}\varphi(2t) + \frac{3}{2}\varphi(2t-1)\right) + 2\left(\varphi(2t) + \varphi(2t-1)\right) \\ &= 3\varphi(2t) + 4\varphi(2t-1). \end{aligned}$$

d) Let us first figure out what the products $\varphi(4t)\varphi(2t)$, $\varphi(4t-1)\varphi(2t)$, $\varphi(4t)\varphi(2t-1)$, and $\varphi(4t-1)\varphi(2t-1)$. By drawing the graphs are evaluate directly we get

$$\varphi(4t)\varphi(2t) = \varphi(4t) = \begin{cases} 1 & 0 \le t < \frac{1}{4} \\ 0 & \text{else} \end{cases}$$

$$\varphi(4t-1)\varphi(2t) = \varphi(4t-1) = \begin{cases} 1 & \frac{1}{4} \le t < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$

and

$$\varphi(4t)\varphi(2t-1) = \varphi(4t-1)\varphi(2t-1) = 0\,.$$

The inner products are therefore

$$< f, \varphi_0^{(1)} >= \int_0^{1/4} 1\,dt + \int_{1/4}^{1/2} 1\,dt = \frac{1}{4} + \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$$

$$< f, \varphi_1^{(1)} >= 0\,.$$

The projection is therefore given by

$$P(f) = \frac{1}{2}\varphi(2t)\,.$$

e) The same analysis gives in this case

$$P(f) = 3\varphi(2t) + 6\varphi(2t-1)\,.$$

(Do you recognize those numbers from the wavelet transform theory?)

8) Let $V = V_2$ be the space of functions of the form

$$s_0\varphi(4t) + s_1\varphi(4t-1) + s_2\varphi(4t-2) + s_3\varphi(4t-3),$$

$s_0, s_1, s_2, s_3 \in \mathbb{R}$. Let V_1 be the subspace generated by $\varphi(2t)$ and $\varphi(2t-1)$. Finally let W_1 be the subspace of V_2 of the form $t_0\psi(2t) + t_1\psi(2t-1)$.

a) Find the orthogonal projection of

$$f(t) = s_0\varphi(4t) + s_1\varphi(4t-1) + s_2\varphi(4t-2) + s_3\varphi(4t-3)$$

onto V_1.

b) Find the orthogonal projection of

$$s_0\varphi(4t) + s_1\varphi(4t-1) + s_2\varphi(4t-2) + s_3\varphi(4t-3)$$

onto W_1.

Solution: Here is just the final answere:

a) The projection gives the Haar wavelet transform

$$P_{V_1}(f)(t) = \frac{s_0 + s_1}{2}\varphi(2t) + \frac{s_2 + s_3}{2}\varphi(2t-1)\,.$$

b) This is the detail-transform

$$P_{W_1}(f)(t) = \frac{s_0 - s_1}{2}\psi(2t) + \frac{s_2 - s_3}{2}\psi(2t-1)\,.$$

Math 2025, Problems, 2-dim Haar wavelet transform

1) Let $f = 4\varphi \otimes \psi$. What is the 2×2-matrix corresponding to f?

Solution: Recall that $\varphi(0) = \varphi(1/2) = 1$ and $\psi(0) = 1$ and $\psi(1/2) = -1$. Hence

$$\begin{aligned} f \quad &\leftrightarrow \quad \begin{pmatrix} f(0,0) & f(0,1/2) \\ f(1/2,0) & f(1/2,1/2) \end{pmatrix} = \begin{pmatrix} 4 \times 1 \times 1 & 4 \times 1 \times (-1) \\ 4 \times 1 \times 1 & 4 \times 1 \times (-1) \end{pmatrix} \\ &= \quad \begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix} \end{aligned}$$

2) Let $f = -3\varphi \otimes \varphi + 2\psi \otimes \varphi$. What is the 2×2-matrix corresponding to f?

Solution: We proceed as in (1). Hence:

$$\begin{pmatrix} f(0,0) & f(0,1/2) \\ f(1/2,0) & f(1/2,1/2) \end{pmatrix} = \begin{pmatrix} -3+2 & -3+2 \\ -3-2 & -3-2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -5 & -5 \end{pmatrix}$$

3) Let $f = 3\psi \otimes \varphi + 5\psi \otimes \psi$. What is the 2×2-matrix corresponding to f?

Solution: We proceed as in (1). Hence:

$$\begin{pmatrix} f(0,0) & f(0,1/2) \\ f(1/2,0) & f(1/2,1/2) \end{pmatrix} = \begin{pmatrix} 3+5 & 3-5 \\ -3-5 & -3+5 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ -8 & 2 \end{pmatrix}$$

5) Assume that the signal f corresponds to the matrix

$$\begin{pmatrix} f(0,0) & f(0,1/2) \\ f(1/2,0) & f(1/2,1/2) \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 0 \end{pmatrix}.$$

Evaluate the Haar wavelet transform of f.

Solution: We do the Haar wavelet transform in two steps. The first is horizontal:

$$\begin{pmatrix} 2 & 4 \\ 6 & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{2+4}{2} & \frac{2-4}{2} \\ \frac{6+0}{2} & \frac{6-0}{2} \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 3 & 3 \end{pmatrix}$$

The next step is vertical:

$$\begin{pmatrix} 3 & -1 \\ 3 & 3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{3+3}{2} & \frac{-1+3}{2} \\ \frac{3-3}{2} & \frac{-1-3}{2} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}.$$

Hence the Haar transform gives

$$\begin{pmatrix} 2 & 4 \\ 6 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 3 & 1 \\ 0 & -2 \end{pmatrix}.$$

6) What is the Haar wavelet transform of the matrix $\begin{pmatrix} 9 & 11 \\ -1 & 7 \end{pmatrix}$?

Solution: We proceed as in problem (5). Thus

Step one:

$$\begin{pmatrix} 9 & 11 \\ -1 & 7 \end{pmatrix} \mapsto \begin{pmatrix} \frac{9+11}{2} & \frac{9-11}{2} \\ \frac{-1+7}{2} & \frac{-1-7}{2} \end{pmatrix} = \begin{pmatrix} 10 & -1 \\ 3 & -4 \end{pmatrix}.$$

Step two:

$$\begin{pmatrix} 10 & -1 \\ 3 & -4 \end{pmatrix} \mapsto \begin{pmatrix} \frac{10+3}{2} & \frac{-1-4}{2} \\ \frac{10-3}{2} & \frac{-1+4}{2} \end{pmatrix} = \begin{pmatrix} \frac{13}{2} & \frac{-5}{2} \\ \frac{7}{2} & \frac{3}{2} \end{pmatrix}.$$

Math 2025, Quiz #2 **Name:** ____________________

1) The following table lists eight sample points corresponding to a step function $p(t)$.

j	0	1	2	3	4	5	6	7
r_j	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{7}{8}$
s_j	**5**	**1**	**3**	**1**	**8**	**6**	**9**	**7**

a) Draw the graph of the function $p(t)$.

b) Write $p(t)$ using the functions $\varphi(8t - r)$.

c) Apply the Haar wavelet transform to write $p(t)$ as a linear combination of the functions $\varphi(4t-r)$ and $\psi(4t - r)$ using **averages** and **details**.

Math 2025, Quiz #1

Name: ____________________

1) a) Draw the graph of the function defined by $f(t) = \begin{cases} t+1 & \text{if} \quad t < -1 \\ t^2 & \text{if} \quad -1 \leq t \leq 1 \\ 1 & \text{if} \quad 1 < t \end{cases}$

b) Is $f(t)$ continuous at the point $t = -1$? Is $f(t)$ continuous at $t = 1$?

2) Let $f(t) = \begin{cases} t & \text{if} \quad -1 \leq t < 1 \\ 0 & \text{else} \end{cases}$. Draw the graph of the functions $f(2t-1)$ and $f(4t+2)$.

Math 2025, Quiz #6 **Name:** ______________________________

1) Let $\overrightarrow{\mathbf{f}} = (1, 3, -4, 1, 3, 0, 10, 3)$. Write down all the arrays of length two that you get in the first step of the FFT.

2) Use the FFT to find the Fourier transform of the array $\overrightarrow{\mathbf{f}} = (2, 4, -8, 6)$.

Math 2025, Test # 1, Tuesday March 7, 2001

1)[7pts] Plot and write a formula – using the dilated and translated Haar scaling functions $\varphi(2^k x - j)$ – for the step function $\tilde{p}$ given by the table:

j	0	1	2	3
r_j	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
s_j	5	1	3	−1

Answer: (Only the formula) $\tilde{p}(t) = 5\varphi(4t) + \varphi(4t-1) + 3\varphi(4t-2) - \varphi(4t-3)$

2)[5pts] Find the average value of the numbers $1, 2, 3, 4, 5, 6, 7$.

Answer: 4

We have

$$\text{Average} = \frac{1+2+3+4+5+6+7}{7} = \frac{28}{7} = 4$$

(Remember the formula $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Then average of $1, 2, \ldots, n$ is therefore $\frac{n(n+1)}{2n} = \frac{n+1}{2}$)

3)[15pts] Find the **ordered** Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (3, 5, 2, 6)$. **Answer:** $(4, 0, -1, -2)$

1th-step:

$$(3,5,2,6) \mapsto \left(\frac{3+5}{2}, \frac{2+6}{2}, \frac{3-5}{2}, \frac{2-6}{2}\right) = (4, 4; -1, -2)$$

2th-step:

$$(4,4,-1,-2) \mapsto \left(\frac{4+4}{2}; \frac{4-4}{2}; -1, -2\right) = (4, 0, -1, -2)$$

4)[15pts] Assume that the **ordered** Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\overrightarrow{\mathbf{s}}^0 = (\mathbf{3}, 0, -2, 1)$. Find the initial array $\overrightarrow{\mathbf{s}}^2$. **Answer** $\overrightarrow{\mathbf{s}}^2 = (1, 5, 4, 2)$

1th-step:

$$(\mathbf{3}, 0, -2, 1) \mapsto (3+0, 3-0; -2, 1) = (3, 3; -2, 1)\,.$$

2th-step:

$$(3, 3; -2, 1) \mapsto (3-2, 3-(-2), 3+1, 3-1) = (1, 5, 4, 2)$$

5)[15pts] Evaluate the **in-place** fast Haar wavelet transform of the sample $\overrightarrow{\mathbf{s}} = (5, 7, 9, 1)$.

Answer: $\overrightarrow{\mathbf{s}^{(0)}} = \left(\frac{11}{2}, -1, \frac{1}{2}, 4\right)$

1th-step:

$$(5,7,9,1) \mapsto \left(\frac{5+7}{2}, \frac{5-7}{2}, \frac{9+1}{2}, \frac{9-1}{2}\right) = (6, -1, 5, 4)$$

2th-step:

$$(6,-1,5,4) \mapsto \left(\frac{6+5}{2}, -1, \frac{6-5}{2}, 4\right) = \left(\frac{11}{2}, -1, \frac{1}{2}, 4\right)$$

6)[15pts] Assume that the **in-place** fast Haar wavelet transform of a sample $\overrightarrow{\mathbf{s}} = (s_0, s_1, s_2, s_3)$ produces the result $(10, -1, 2, 4)$. Apply the inverse transform to reconstruct the sample $\overrightarrow{\mathbf{s}} = (11, 13, 12, 4)$

1^{th}-step:

$$(10, -1, 2, 4) \mapsto (10 + 2, -1, 10 - 2, 4) = (12, -1, 8, 4)$$

2^{th}-step:

$$(12, -1, 8, 4) \mapsto (12 - 1, 12 + 1, 8 + 4, 8 - 4) = (11, 13, 12, 4)$$

7)[6pts] Evaluate the following complex numbers:

a) $(2 - i)(4 + 3i) = 2 \times 4 + (-i) \times (3i) + (-i) \times 4 + 2 \times 3i = 8 + 3 - 4i + 6i = 11 + 2i$

b) $\dfrac{1}{2+3i} = \dfrac{2-3i}{(2+3i)(2-3i)} = \dfrac{2-3i}{4+9} = \dfrac{2}{13} - \dfrac{3}{13}i$

8)[9pts] Determine which of the following maps is linear is which one is not linear. **Explain your answer**.

(1) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x, y, z) = (x + 2y, x - 3y^2, x)$. **Not** linear because of the factor y^2

(2) $T : C^1(\mathbb{R}) \to C(\mathbb{R})$, $T(f) = f' + f^2$. **Not** linear because of the factor f^2.

(3) $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x+y, x-y)$. Linear because of the standard form $(ax+by, cx+dy)$

9)[6pts] Evaluate the following linear maps at the given point:

(1) $T(f) = f'$. $T(\cos(x) + \sin(x)) = -\sin(x) + \cos(x)$

(2) $T(x, y) = (2x - y, 3x + 2y)$, $T(2, 1) = (4 - 1, 6 + 2) = (3, 8)$

10) [7pts] Multiply the matrices

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -2 & 7 & 5 \end{bmatrix} = \begin{bmatrix} 6 & -7 & -7 \\ -6 & 14 & 11 \\ 6 & 0 & -3 \end{bmatrix}$$

Math 2025, Test # 3, Tuesday May 1, 2001 Name ______________________________

1)[**8pts**] Find the **ordered** Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^3 = (3, 7, 2, 6, 1, -2, 3, 1)$.

Solution: 1th-step:

$$\begin{aligned}\mathbf{s}^2 &= \left(\frac{3+7}{2}, \frac{2+6}{2}, \frac{1-2}{2}, \frac{3+1}{2}; \frac{3-7}{2}, \frac{2-6}{2}, \frac{1+2}{2}, \frac{3-1}{2}\right) \\ &= (5, 4, -1/2, 2; -2, -2, 3/2, 1)\end{aligned}$$

2th-step:

$$\begin{aligned}\mathbf{s}^1 &= \left(\frac{5+4}{2}, \frac{-1/2+2}{2}; \frac{5-4}{2}, \frac{-1/2-2}{2}; -2, -2, 3/2, 1\right) \\ &(9/2, 3/4; 1/2, -5/4; -2, -2, 3/2, 1)\end{aligned}$$

Last step:

$$\begin{aligned}\mathbf{s}^0 &= \left(\frac{1}{2}\left(\frac{9}{2}+\frac{3}{4}\right); \frac{1}{2}\left(\frac{9}{2}-\frac{3}{4}\right); 1/2, -5/4; -2, -2, 3/2, 1\right) \\ &= \left(\frac{21}{8}, \frac{15}{8}; 1/2, -5/4; -2, -2, 3/2, 1\right) .\end{aligned}$$

Final answer:$\mathbf{s}^0 = \left(\frac{21}{8}, \frac{15}{8}; \frac{1}{2}, -\frac{5}{4}; -2, -2, \frac{3}{2}, 1\right)$

2)[**8pts**] Assume that the **ordered** Haar wavelet transform of the initial data $\overrightarrow{\mathbf{s}}^2 = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\overrightarrow{\mathbf{s}}^0 = (\mathbf{6}, 1, 5, 7)$. Find the initial array $\overrightarrow{\mathbf{s}}^2$:

Solution: 1th-step:

$$\mathbf{s}^1 = (6+1, 6-1, 5, 7) = (7, 5, 5, 7) .$$

2th-step:

$$\mathbf{s}^2 = (7+5, 7-5, 5+7, 5-7) = (12, 2, 12, -2)$$

3)[**6pts**] On the vector space $C[-1, 1]$ of continuous functions on the interval $[-1, 1]$ consider the inner product $< f, g >= \frac{1}{2}\int_{-1}^{1} f(t)\overline{g(t)}\, dt$. Evaluate the inner product of the following functions:

(1) $f(t) = 1$ for all t and $g(t) = t$. $< f, g >= \int_{-1}^{1} t\, dt = \left[\frac{t^2}{2}\right]_{-1}^{1} = 0$
(2) $f(t) = \cos(\pi t)$ and $g(t) = \sin(\pi t)$. $< f, g >= 0$
(3) $f(t) = t$ and $g(t) = t^3$. $< f, g >= 1/5$

4)[**10pts**] Find the DFT of the array $\overrightarrow{\mathbf{f}} = (18, 9, 2, -5)$.

Solution:

$$\hat{f}_0 = \frac{1}{4}(18+9+2-5) = \frac{24}{4} = 6$$
$$\hat{f}_1 = \frac{1}{4}(18-9i-2-5i) = \frac{16}{4} - \frac{14}{4}i = 4 - \frac{7}{2}i$$
$$\hat{f}_2 = \frac{1}{4}(18-9+2+5) = 4$$
$$\hat{f}_3 = 4 + \frac{7}{2}i$$

Thus

$$\widehat{\overrightarrow{\mathbf{f}}} = \left(6, 4 - \frac{7}{2}i, 4, 4 + \frac{7}{2}i\right)$$

5)[10pts] Given the array $\widehat{\overrightarrow{\mathbf{f}}} = (1, 1-i, 3, 1+i)$ find the initial sample array $\overrightarrow{\mathbf{f}}$ using the inverse DFT.

Solution:

$$f_0 = 1 + (1-i) + 3 + (1+i) = 6$$
$$f_1 = 1 + (1-i)i - 3 + (1+i)(-i) = 0$$
$$f_2 = 1 - (1-i) + 3 - (1+i) = 2$$
$$f_3 = 1 + (1-i)(-i) - 3 + (1+i)i = -4$$

Thus

$$\overrightarrow{\mathbf{f}} = (6, 0, 2, -4)$$

6)[10pts] Use the FFT to evaluate the Fourier transform of the array $\overrightarrow{\mathbf{f}} = (1, 0, 3, 2, 0, 5, 4, 3)$..Problem like this will not be on the final. The maximal length of a array will be **four**.

7)[8pts] Which of the following functions have period 2π?

(1) $\cos(2x)\sin(x) + 3\frac{1}{4+\cos(x)}$

[trim=0.000000in 0.000000in 0.175530in 0.070841in, height=2.5097in, wid

Periodic with period 2π.

(2) $\sin(x/2) + \cos(x)$. Periodic with period 4π (and

[height=2.6351in, width=3.6616in]t3f2.eps

not 2π).

(3) $t\cos(x)$.

[natheight=2.000300in, natwidth=3.000000in, height=2.0003in, width=3in]GCXLDZ0E
because t is a constant independent of x.

(4) $e^x \sin(x)$. . Not periodic

[height=2.6792in, width=3.6893in]t3f3.eps

8)[6pts] Which of the following sets is a basis for $\mathbb{R}^3$?

(1) $(1,0,1),(1,1,-1),(1/2,-1,-1/2)$. Orthogonal, Hence linear independent. Hence BASIS.
(2) $(1,1,0),(1,0,1),(2,1,1)$. Not a basis because not linear independent: $(1,1,0)+(1,0,1) = (2,1,1)$.
(3) $(1,1,0),(1,2,3),(3,-1,4),(1,4,3)$. Not a basis, four vectors in $\mathbb{R}^3$ are always linearly dependent.

Project on Fourier series[24pts]: Let f be a periodic function with period 2π. Let $c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-itk}\,dt$ be the Fourier coefficient. Let

$$S_N(f)(t) = \sum_{k=-N}^{N} c_k e^{ikt}$$

be the N^{th} partial sum of the Fourier series for f. Consider the function $f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 \le t \le \pi \\ 0 & else \end{cases}$

[natheight=3.466200in, natwidth=4.944100in, height=2.9499in, width=4.1969in]probe.EPS

Evaluate the Fourier coefficients c_0, c_1, and c_2. Notice that the function f is defined in two ways, as -1 on the interval $]-\pi,0[$ and as $+1$ on the interval $[0,\pi]$. It is therefore natural to split all integrals involving f into an integral over $]-\pi,0[$ where f is replaced by -1 and an integral over $[0,\pi]$ where f is replaced by $+1$.

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{0}(-1)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} 1\,dt = 0$$

(1)

$$
\begin{aligned}
c_1 &= \frac{1}{2\pi}\int_{-\pi}^{0}(-1)\,e^{-it}\,dt + \frac{1}{2\pi}\int_{0}^{\pi} e^{it}dt \\
&= \frac{-1}{2\pi}\int_{-\pi}^{0}\cos(t) - i\sin(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi}\cos(t) - i\sin(t)\,dt \\
&= \frac{-1}{2\pi}\left[\sin(t) + i\cos(t)\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\sin(t) + i\cos(t)\right]_{0}^{\pi} \\
&= \frac{-1}{2\pi}\left[\sin(0) - \sin(-\pi) + i(\cos(0) - \cos(-\pi)\right] + \frac{1}{2\pi}\left[\sin(\pi) - \sin(0) + i(\cos(\pi) - \cos(0)\right] \\
&= \frac{-4i}{2\pi} \\
&= \frac{-2i}{\pi}
\end{aligned}
$$

$$c_2 = 0$$

(2) Find a general formula for all the coefficients c_k, $k \neq 0$.

$$
\begin{aligned}
c_k &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-itk}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -1 \times (\cos(-kt) + i\sin(-kt))\,dt + \frac{1}{2\pi}\int_{0}^{\pi} 1 \times (\cos(-kt) + i\sin(-kt))\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} -\cos(kt) + i\sin(kt)\,dt + \frac{1}{2\pi}\int_{0}^{\pi}\cos(kt) - i\sin(kt)\,dt \\
&= \frac{1}{2\pi}\left[-\frac{\sin(kt)}{k} - i\frac{\cos(kt)}{k}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{\sin(kt)}{k} + i\frac{\cos(kt)}{k}\right]_{0}^{\pi} \\
&= \frac{1}{2\pi}\left[-i\frac{\cos(0)}{k} + i\frac{\cos(k\pi)}{k} + i\frac{\cos(kt)}{k} - i\frac{\cos(0)}{k}\right] \\
&= \frac{-2i}{2\pi k}\left(1 - (-1)^k\right) \\
&= \begin{cases} 0 & \text{if } k \text{ even} \\ \frac{-2i}{\pi k} & \text{if } k \text{ odd} \end{cases}
\end{aligned}
$$

where we have used that $\cos(k\pi) = (-1)^k$.

(3) Use a graphic calculator (or any other computer graphic software) to draw the graph of $S_3(f)$, $S_5(f)$ and $S_9(f)$. (Enclose that as a part of your solution)

Using that $c_{-k} = -c_k$ we write for $N = 2M + 1$ odd

$$\begin{aligned} S_N(f)(t) &= \sum_{k=-N}^{N} c_k e^{ikt} \\ &= c_0 + \sum_{k=1}^{N} c_k (e^{ikt} - e^{-ikt}) \\ &= \sum_{n=0}^{M} \frac{-2i}{\pi(2n+1)} 2i \sin((2n+1)t) \\ &= \frac{4}{\pi} \sum_{n=0}^{M} \frac{\sin((2n+1)t)}{2n+1} \end{aligned}$$

where we have used that $e^{iy} - e^{-iy} = (\cos(y) + i\sin(y)) - (\cos(y) + i\sin(-y)) = 2i\sin(y)$ for $y \in \mathbb{R}$. In particular we get

$$S_3(f)(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3}\right)$$

[height=2.4414in, width=3.6694in]t3f4.eps

$$S_5(f)(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5}\right)$$

[height=2.4699in, width=3.6893in]t3f5.eps

$$S_9(f)(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \frac{\sin(7t)}{7} + \frac{\sin(9t)}{9}\right)$$

[height=2.4699in, width=3.6893in]t3f6.eps

(1) Compare and discuss the graph of f, $S_3(f)$, $S_5(f)$ and $S_9(f)$. In particular discuss what happens around the points $-\pi$, 0, and π. What is the value of f and the partial sums $S_3(f)$, $S_5(f)$ and $S_9(f)$ at those points. The graph of $S(f)(t)$ becomes closer and closer to the graph of $f(t)$ as N increases. The frequency increases as N increases, but at the same time the difference from the line $y = 1$ (and $y = -1$ for $-\pi < t < 0$) becomes less and less. The

only exception are the points $x = -\pi, 0, \pi$. Here we still have a deviation around ± 0.2 all the time. This shows up even if N becomes much bigger. Here is the graph for $N = 101$: $\frac{4}{\pi} \sum_{n=0}^{50}$

$$[height = 4.8845in, width = 7.3111in]t3f7.eps$$

This is the so-called Gibbs-phenomenon.

13)[10pts] Write few words about **one** of the following topics:

(1) The two dimensional Haar wavelet transform (Y. Nievergelt, Wavelets Made Easy)
(2) The two dimensional DFT. (Y. Nievergelt)
(3) The application of the Haar wavelet in data compression (Discovering Wavelets, by E. Aboufadel and S. Schlicker)
(4) Wavelets and the FBI-fingerprints.
(Discovering Wavelets, http://www.c3.lanl.gov/~brislawn/FBI/FBI.html,
http://www.ams.org/notices/199511/brislawn.pdf)
(5) Why study wavelets and Fourier transform?
(6) Wavelets and computer graphics or choose any other application you think is interesting and important.
(7) Orthogonal projections (Y. Nievergelt)
(8) Who was Fourier?

Some good web-pages

http://www.cosy.sbg.ac.at/~uhl/wav.html

http://www.amara.com/current/wavelet.html

http://www.crc.ricoh.com/CREW/index.html

http://www.amara.com/current/wavelet.html#soundfun

http://www.gvsu.edu/mathstat/wavelets/wavelnk.htm

http://www.siam.org/siamnews/mtc/mtc1193.htm

http://grail.cs.washington.edu/projects/wavelets/article/wavelet1.pdf

Math 2025, Final, Thursday May 10, 2001, 3–5, PM
Name ______________________________

1)[17pts] Find the **ordered** Haar wavelet transform of the initial data $\vec{\mathbf{s}}^{2} = (4, 10, 15, 3)$.

$\vec{\mathbf{s}}^{0} =$ ______________________

2)[17pts] Find the **in place** Haar wavelet transform of the initial data $\vec{\mathbf{s}}^{2} = (1, -1, 3, 5)$. $\vec{\mathbf{s}}^{0} =$ ______________________

3)[17pts] Assume that the **in place** Haar wavelet transform of the initial data $\vec{\mathbf{s}}^{2} = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\vec{\mathbf{s}}^{0} = (8, 1, 2, 4)$. Find the initial array $\vec{\mathbf{s}}^{2} =$ ______________________

4)[17pts] Assume that the **ordered** Haar wavelet transform of the initial data $\vec{\mathbf{s}}^{2} = (a_0^{(2)}, a_1^{(2)}, a_2^{(2)}, a_3^{(2)})$ produces the result $\vec{\mathbf{s}}^{0} = (5, 1, -2, 3)$. Find the initial array $\vec{\mathbf{s}}^{2} =$ ______________________

5)[16pts] Wich one of the following maps is linear:

a) $T(f) = f'$
b) $T((x, y, z)) = (2x - 3y + z, -3x + z)$
c) $T(x, y) = (x + yx, x^2)$
d) $T(a_0\varphi(4t) + a_1\varphi(4t-1) + a_2\varphi(4t-2) + a_3\varphi(4t-3)) = \frac{a_0+a_1}{2}\varphi(2t) + \frac{a_2+a_3}{2}\varphi(2t-1)$.

6)[15pts]Which one of the following sets of vectors is a basis for the corresponding vectorspace V.

a) $V = \mathbb{R}^2$, $(1, 0, -1)$, $(1, 2, 1)$, $(1, -1, 1)$.
b) $V = \mathbb{R}^2$, $(1, 2)$, $(2, -1)$.
c) $V = \mathbb{R}^3$, $(1, 0, 0)$, $(0, 1, 1)$, $(0, 1, -1)$, $(1, 1, 0)$.

7)[15pts] Evaluate the following complex numbers:

a) $\frac{1}{3-2i} =$

b) $(4+7i)\times(-3+2i) =$

c) $e^{-\pi/2} =$

8)[10pts] Evaluate the following inner products:

a) $V = \mathbb{C}^3$, $<(2, 2-i, 3i), (1+i, 2-i, 1+i)>=$

b) $V = C([0,1])$, $<1+2it, it-1>=$

9[7pts] What is the norm of the vector $(3, 4, 5i)$? $\|(3,4,5i)\| =$

10[8pts] Multiply the matricies

$$\begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & -1 \\ 1 & 4 & 5 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 2 & 4 \\ 9 & 1 \end{pmatrix} =$$

11)[17pts] Find the DFT of the array $\overrightarrow{\mathbf{f}} = (4, 3, -1, 5)$. $\widehat{\overrightarrow{\mathbf{f}}} =$ ______________________________

12)[17pts] Given the array $\widehat{\overrightarrow{\mathbf{f}}} = (1, 1-i, 3, 1+i)$ find the initial sample array $\overrightarrow{\mathbf{f}}$ using the inverse DFT. $\overrightarrow{\mathbf{f}} =$ ______________________________

13)[17pts] Use the Fast Fourier Transform to evaluate the Fourier transform of the sample $\overrightarrow{\mathbf{f}} = (1, 3, -1, 3)$.

$\widehat{\overrightarrow{\mathbf{f}}} =$

14)[10pts] Let $f(t)$ be the 2π-periodic function given by $f(t) = \begin{cases} 1 & \text{if} \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \text{if} \quad \pi/2 \leq |t| \leq \pi \end{cases}$

Find the Fourier coefficients $c_k =$