PROBLEMS FROM CHAPTER 6

#1 Suppose f is decreasing on $[a,b]$. Then $V_a^b(f) = |f(b) - f(a)|$, so $f \in \mathcal{BV}[a,b]$.

Proof Let $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then

$$P(f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

$$= \sum_{i=1}^{n} f(x_{i-1}) - f(x_i)$$

$$= f(a) - f(b)$$

$$= |f(b) - f(a)|$$

Thus $\sup_P P(f) = |f(b) - f(a)|$. ∎

#2) ~~Let P =~~ If $f \in \mathcal{BV}[a,b]$ then f is bounded on $[a,b]$.

Proof. $f \in \mathcal{BV}[a,b]$. Thus, by Theorem 6.1.3 there are monotone increasing function $h, k : [a,b] \to \mathbb{R}$ such that

$$f = h - k.$$

But then

$$|f(x)| = |h(x) - k(x)| \le |h(x)| + |k(x)|$$
$$\le \|h\|_\infty + \|k\|_\infty$$
$$= \max\{|h(a)|, |h(b)|\}$$
$$+ \max\{|k(a)|, |k(b)|\}.$$

4) Let P be any partition of $[a,b]$, $x' \in [a,b]$ and $f: [a,b] \to \mathbb{R}$. Let $P^* = P \cup \{x'\}$. Prove $P(f) \le P^*(f) \le V_a^b(f)$.

<u>Proof</u>. a) If $x' \in P$, then $P^* = P$ and the statement is clear ($P(f) = P^*(f)$).

(b) Assume that $x' \notin P = \{x_0 = a < x_1 < \cdots < x_n = b\}$. Let $j \in \{1, \dots, n\}$ be such that $x' \in (x_{j-1}, x_j)$. Then

$$P(f) = \sum_{k=1}^{j-1} |f(x_k) - f(x_{k-1})| + |f(x_j) - f(x_{j-1})|$$
$$+ \sum_{k=j+1}^{n} |f(x_k) - f(x_{k-1})|$$

where the first (in case $j=1$) or the last sum (in case $j=n$) might not be there.

But

$$|f(x_j) - f(x_{j-1})| \le |f(x_j) - f(x')| + |f(x') - f(x_{j-1})|$$

Hence

$$P(f) \le \sum_{k=1}^{j-1} |f(x_k) - f(x_{k-1})| + |f(x_j) - f(x')| + |f(x') - f(x_{j-1})|$$
$$+ \sum_{k=j+1}^{n} |f(x_k) - f(x_{k-1})|$$
$$= P^*(f) \le V_a^b(f)$$

∎

6) Show that $V_a^b(cf+g) \leq |c| V_a^b(f) + V_a^b(g)$.

<u>Proof.</u> Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition.

Note that

$$|cf(x_j) + g(x_j) - cf(x_{j-1}) g(x_{j-1})| \leq |c| |f(x_j) - f(x_{j-1})| + |g(x_j) - g(x_{j-1})|$$

Hence

$$P(cf+g) = \sum |cf(x_j) + g(x_j) - cf(x_{j-1}) - g(x_{j-1})|$$

$$\leq \sum |c| |f(x_j) - f(x_{j-1})| + \sum |g(x_j) - g(x_{j-1})|$$
$$= |c| P(f) + P(g)$$
$$\leq |c| V_a^b(f) + V_a^b(g)$$

As this holds for all partitions it holds also for the sup, so

$$V_a^b(cf+g) \leq |c| V_a^b(f) + V_a^b(g) \qquad \blacksquare$$

9) $f, g \in BV[a,b] \Rightarrow fg \in BV[a,b]$.

<u>Proof.</u> Let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition

Note that

$$|f(x_j)g(x_j) - f(x_{j-1})g(x_{j-1})| = |f(x_j)g(x_j) - f(x_j)g(x_{j-1}) + f(x_j)g(x_{j-1}) - f(x_{j-1})g(x_{j-1})|$$

$$\leq |f(x_j)| |g(x_j) - g(x_{j-1})| + |g(x_{j-1})| |f(x_j) - f(x_{j-1})|$$
$$\leq \|f\|_\infty |g(x_j) - g(x_{j-1})| + \|g\|_\infty |f(x_j) - f(x_{j-1})|$$

and by #2 we know that $\|f\|_\infty, \|g\|_\infty < \infty$. Hence

$$P(fg) = \sum_j |f(x_j)g(x_j) - f(x_{j-1})g(x_{j-1})|$$

$$\leq \|f\|_\infty \sum_j |g(x_j) - g(x_{j-1})| + \|g\| \sum_j |f(x_j) - f(x_{j-1})|$$

$$\leq \|f\|_\infty V_a^b(g) + \|g\|_\infty V_a^b(g) < \infty.$$

Thanking the sup over all P, we get

$$V_a^b(fg) \leq \|f\|_\infty V_a^b(g) + \|g\|_\infty V_a^b(f) < \infty. \qquad \blacksquare$$

\# 14) Suppose $f'(x)$ exists on $[a,b]$ and $f' \in \mathcal{R}[a,b]$. Use the Fundamental Theorem of calculus to prove that $f \in BV[a,b]$ and $V_a^b(f) \leq \int_a^b |f'(t)|dt$.

Proof By the FTC we have

$$|f(x_i) - f(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} f'(t)\,dt\right|$$
$$\leq \int_{x_{i-1}}^{x_i} |f'(t)|\,dt.$$

Hence, if $P = \{x_0 = a < x_1 < \cdots < x_n = b\}$:

$$P(f) = \sum_j |f(x_j) - f(x_{j-1})| \leq \sum_j \int_{x_{j-1}}^{x_j} |f'(t)|\,dt$$
$$= \int_a^b |f'(t)|\,dt < \infty$$

(Here we need to use problem 6 page 78 to show $f' \in \mathcal{R}[a,b] \Rightarrow |f'| \in \mathcal{R}[a,b]$.)

As this holds for all partitions P we get

$$V_a^b(f) \leq \int_a^b |f'(t)|\,dt < \infty \quad \blacksquare$$

6.2

3) $\int_1^2 x\,d(\log x) = \int_1^2 x \frac{1}{x}\,dx$ (Thm 6.2.1)

$= \int_1^2 dx = 2-1 = \underline{\underline{1}}$

4) $\int_1^2 (x+x^3)\,d(\tan^{-1}x) = \int_1^2 x(1+x^2)\frac{1}{1+x^2}\,dx$ (Thm 6.2.1)

$= \int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = 2-\frac{1}{2} = \underline{\underline{\frac{3}{2}}}$

5) Recall that

$[x] = \sup\{n \mid n \in \mathbb{N}\cup\{0\},\ n \le x\}$

Hence $[x]$ has a jump at each $x=n$.

do the detail $\rightarrow \int_0^3 x\,d[x] = 1+2+3 = \underline{\underline{6}}$

1) $f \in C[a,b]$, $t\in(a,b)$, $g(x) = \begin{cases} c_1, & a\le x<t \\ c, & x=t \\ c_2, & t<x\le b \end{cases}$

<u>Solution</u>: Let $P=\{x_0<\dots<x_n\}$ be a partition. We can assume that $t\in P$ (otherwise replace P by $P^* = P\cup\{t\}$ and use that $P(f) \le P^*(f)$.) Assume that $t=x_j$. Then

$P(f,g,\mu) = f(\mu_j)(c-c_1) + f(\mu_{j+1})(c_2-c)$

where $\mu_j \in [x_{j-1},t]$ and $\mu_{j+1}\in[x_{j+1},t]$. As f is continuous, it follows that

$\lim_{\|P\|\to 0} P(f,g,\mu) = f(t)(c-c_1)+f(t)(c_2-c)$

$= f(t)(c_2-c_1)$ ∎

#7) Let $f \in C[a,b]$, $f \in RS([a,b],g)$. Let $p \in (a,b)$ and $h(x) = g(x)$ for all $x \in [a,b] \setminus \{p\}$. Then $f \in RS([a,b],h)$ and $\int_a^b f\,dh = \int_a^b f\,dg$.

Proof: Define

$$k(x) = h(x) - g(x) = \begin{cases} 0 & a \le x < p \\ h(p) - g(p) & x = p \\ 0 & p < x \le b \end{cases}$$

Then $f \in R([a,b],k)$ and $\int_a^b f\,dk = 0$ by problem # 1. It follows by Thm 6.2 (b) that – using that $h = g + k$ –

$f \in RS([a,b],h)$ and

$$\int_a^b f\,dh = \int_a^b f\,dg + \int_a^b f\,dk = \int_a^b f\,dg. \quad \blacksquare$$

6.3

1) $\int_{-1}^{1} x\,d(|x| + [x]) = \int_{-1}^{1} x\,d|x| + \int_{-1}^{1} x\,d[x]$ (6.2.2)

$= -\int_{-1}^{1} |x|\,dx + 1 + 1 - \int_{-1}^{1} [x]\,dx + 1 - (-1)\cdot(-1)$

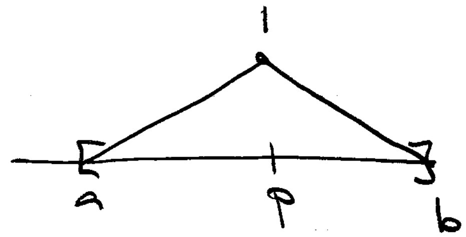
$= 1 + 1 = \underline{\underline{2}}$ (what did I use?)

2) $\int_0^{\pi/2} x\,d(\cos x) = -\int_0^{\pi/2} \cos x\,dx + 0 \cdot \frac{\pi}{2} - 0 \cdot 1$

$= -\sin x \Big]_0^{\pi/2} = \underline{\underline{-1}}$

3) $T : C[a,b] \to \mathbb{R}$, $f \mapsto f(p)$. Then T is bounded and $\|T\| = 1$.

Proof. $|Tf| = |f(p)| \leq \|f\|_\infty$. Hence T is bounded and $\|T\| \leq 1$. Define f by the following graph:



Then $1 = f(p) = \|f\|_\infty$. Hence $|Tf| = \|f\|_\infty$
So $\|T\| = 1$.

4) Let $g(x) = \begin{cases} 0 & \text{if } a \leq x < p \\ 1 & \text{if } p \leq x \leq b \end{cases}$

Then $V_a^b(g) = 1 = \|T\|$ and

$$T_g f = \int_a^b f\,dg = f(p) = Tf \qquad \blacksquare$$