Math 7311, Analysis 1, Homework #5. Due Monday, Sept 24, at 11:30 in Class

1) Let $(X, \mathcal{A})$ be a measurable space and Y a non-empty set. If $f : X \to Y$ is a function define $\mathcal{B} = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{A}\}$. Show that $\mathcal{B}$ is a σ-algebra.

2) (Exercise 3.27, p. 52). For $\epsilon > 0$ construct a open dense subset $U_\epsilon \subset \mathbb{R}^n$ such that $\lambda(U_\epsilon) < \epsilon$ where λ is the Lebesgue measure on $\mathbb{R}^n$. (Think about $\mathbb{Q}^n \subset \mathbb{R}^n$.)

3) Let $(X, \mathcal{A})$ be a measurable space such that $\mathcal{A} \neq \mathcal{P}(X)$. Construct a function $f : X \to \mathbb{R}$ which is not measurable but $|f|$ is measurable.

4) (Compare to Bartle: The Elements of Integration and Lebesgue Measure, p. 26.)
a) Let $E \subset \mathbb{R}$ be open. Then $\lambda(E) = 0$ if and only if $E = \emptyset$.
b) If $K \subset \mathbb{R}$ is compact, then $\lambda(K) < \infty$.

Solutions # 5

1) We know that $f^{-1}(\bigcup_{j=1}^{\infty} E_j) = \bigcup_{j=1}^{\infty} f^{-1}(E_j)$ and $f^{-1}(A^c) = f^{-1}(A)^c$. Therefore $\bigcup_{j=1}^{\infty} E_j \in \mathcal{B}$ if each $E_j \in \mathcal{B}$ and $A^c \in \mathcal{B}$ if $A \in \mathcal{B}$ (because $\mathcal{A}$ is a σ-algebra).

2) Number the rationals as $\mathbb{Q}^n = \{q_1, q_2, \dots\} = \{q_j\}_{j \in \mathbb{N}}$ and recall that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. For each $q_j \in \mathbb{Q}^n$, $\varepsilon' > 0$, let $I_j^{\varepsilon} = (q_{j,1} - \frac{1}{2}\sqrt[n]{\frac{\varepsilon'}{2^j}}, q_{j,1} + \sqrt[n]{\frac{\varepsilon'}{2^j}}) \times \dots \times (q_{j,n} - \sqrt[n]{\frac{\varepsilon'}{2^j}}, q_{j,n} + \sqrt[n]{\frac{\varepsilon'}{2^j}})$

where we have written

$$q_j = (q_{j,1}, \dots, q_{j,n}).$$

Then $I_j^{\varepsilon'}$ is open and $\lambda(I_j^{\varepsilon'}) = \frac{\varepsilon'}{2^j}$. It follows that $I^{\varepsilon'} = \bigcup_{j=1}^{\infty} I_j^{\varepsilon'}$ is open and

$$\lambda(I^{\varepsilon'}) \leq \sum_{j=1}^{\infty} \lambda(I_j^{\varepsilon'}) = \varepsilon' \sum_{j=1} 2^{-j} = \varepsilon'$$

For a given $\varepsilon > 0$ let $0 < \varepsilon' < \varepsilon$ and $U^{\varepsilon} = I^{\varepsilon'}$. Then U^{ε} is dense (because it contains the dense set $\mathbb{Q}^n$) and open.

3) Let $A \in \mathcal{P}(X) \setminus \mathcal{A}$ and define

$$f(x) = 1_A(x) - 1_{X \setminus A}(x).$$

Then f is not measurable $\left(f^{-1}((\tfrac{1}{2}, \tfrac{3}{2})) = A \notin \mathcal{A}\right)$ but $|f| = 1_X$ is measurable.

a) If $E \subseteq \mathbb{R}$ is open and $E \neq \emptyset$, there there exists $a_i < b_i$ (i in same indexset I) such that

$$E = \bigcup_{i \in I} (a_i, b_i)$$

In particular, there exists an open interval $(a,b) \subset E$. But then $\lambda(E) \geq \lambda(a,b) = b-a > 0$.

[Another solution: As $E \neq \emptyset$, there exists $x \in E$. As E is open there exists $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset E$. Hence

$$\lambda(E) \geq \lambda(x-\varepsilon, \varepsilon+x) = 2\varepsilon > 0.$$

b) Let $\eta > 0$. For each x in K let $U_x = (x-\eta, x+\eta)$. Then U_x is open and $K \subseteq \bigcup_{x \in K} U_x$. As K is compact there is an $N \in \mathbb{N}$ and $x_1, \dots, x_N \in K$ s.t.

$$K \subset \bigcup_{j=1}^{N} U_{x_j}.$$

Thus $$\lambda(K) \leq \sum_{j=1}^{N} \lambda(U_{x_j}) = \sum_{j=1}^{N} 2\eta = 2N\eta < \infty.$$

Math 7311, Analysis 1, fall 2012
Homework due Monday, Oct 1, 11:30

In the following problems $(X, \mathcal{A}, \mu)$ will always denote a measure space. Thus X is a non-empty set, $\mathcal{A}$ is a σ-algebra of subsets of X and $\mu : \mathcal{A} \to [0, \infty]$ is a measure.

1) (Exercise 4.5) Assume that $\mu(X) > 0$. Suppose that $f : X \to \bar{\mathbb{R}}$ is finite μ-almost every where (same as: there exists a null set $N \in \mathcal{A}$ such that $f(x) \in \mathbb{R}$ for all $x \in N^c$). Show that there exists a set $Y \in \mathcal{A}$ such that $\mu(Y) > 0$ and f is bounded on Y.

2) Let $f : X \to \bar{\mathbb{R}}$. Show that the following are equivalent:

1. f is measurable.
2. $\{x \in X \mid f(x) > r\}$ is measurable for all $r \in \mathbb{Q}$.
3. $\{x \in X \mid f(x) \leq r\}$ is measurable for all $r \in \mathbb{Q}$.

3) Denote by $(X, \mathcal{A}', \mu')$ the completion of $(X, \mathcal{A}, \mu)$. Assume that $f : X \to \bar{\mathbb{R}}$ is $\mathcal{A}'$ measurable. Show that there exists a $\mathcal{A}$-measureable function $g; X \to \bar{\mathbb{R}}$ such that $g = f$ μ-almost every where. (Hint: For each $r \in \mathbb{Q}$ consider $A_r = \{x \in X \mid f(x) > r\}$. Then $A_r \in \mathcal{A}'$. Then write $A_r = B_r \cup Z_r$ where $B_r \in \mathcal{A}$ and $Z_r \subseteq N_r$ for some null set $N_r \in \mathcal{A}$. Use that $\mathbb{Q}$ is countable to construct g.)

4) (From the Comprehensive/Qualifying Exam August 2011.) Let f_n be a sequence of continuous functions on $[0, 1]$ that converges pointwise for all $x \in [0, 1]$. Given $\epsilon > 0$ show there is a subset $A \subset [0, 1]$, $\lambda(A) < \epsilon$ (λ the Lebesgue measure), and a positive number M such that

$$|f_n(x)| \leq M,$$

for all $x \in [0, 1] \setminus A$. (Hint: Let f be the pointwise limit of f_n. Since f_n is continuous on $[0, 1]$ there is a number M_n such that $|f(x)| \leq M_n$ for all $x \in [0, 1]$. Use Egoroff's Theorem.)

Solutions # 6

1) Let N be so that $\mu(N)=0$ and $f(x)\in\mathbb{R}$ for $x\notin N$.
For $n\in\mathbb{N}$ let

$$A_n = f^{-1}([n,n+1)\cup(-n-1,-1])$$
$$A_0 = f^{-1}(-1,1).$$

Then the sets A_j, $j=0,1,\ldots$, are disjoint and

$$X = \dot{\bigcup} A_n \cup N$$

It follows that

$$0<\mu(X)=\sum_{j=0}^{\infty}\mu(A_j)$$

Therefore, there must be a n s.t. $\mu(A_n)>0$. It follows that $\mu(\bigcup_{j=0}^{m} A_j)>0$ and $|f(x)|\le n+1<\infty$ for all $x\in\bigcup_{j=1}^{m} A_j$.

2) "(1)$\Leftarrow$(2)" Let $\alpha\in\mathbb{R}$. Let $r_j\searrow\alpha$ with $r_j\in\mathbb{Q}$. Then

$$f^{-1}((\alpha,\infty])=\bigcup f^{-1}((\alpha_j,\infty])\in\mathcal{A}$$

"(2)$\Leftarrow$(3) $\{x\mid f(x)\le r\}^c=\{x\mid f(x)>r\}$

"(3)$\Leftarrow$(1) obvious.

3) For $r\in\mathbb{Q}$ let $A_r=f^{-1}(r,\infty)$. Write $A_r=B_r\cup Z_r$ with $B_r, Z_r\in\mathcal{A}$ and $\mu(Z_r)=0$. (We can, by replacing B_r by $B_r\setminus(B_r\cap Z_r)$, assume that $B_r\cap Z_r=\emptyset$) Let $A=\bigcup B_r\in\mathcal{A}$, $Z=\bigcup Z_r$. By replacing A by $A\setminus Z\cap A$ we can assume that A and Z are disjoint. Note that Z is a null-set as it is a (finite) countable union of null-sets.

Define $g: X \to \overline{\mathbb{R}}$ by

$$g(x) = \begin{cases} f(x) & x \in A \\ 0 & x \in Z \end{cases}$$

Then $f = g$ a.e. Let $r \in \mathbb{Q}$. Then

$$g^{-1}([r,\infty]) = B_r \setminus B_r \cap Z \in \mathcal{A} \quad \text{for } r > 0$$

$$g^{-1}(r,\infty] = B_r \cup Z \in \mathcal{A} \quad \text{for } r \leq 0$$

Thus g is measurable by #2.

4) Let $f(x) := \lim_{n\to\infty} f_n(x)$. Then there exist $B \subset [0,1]$ such that $f_n(x) \to f(x)$ uniformly on B and $\lambda(B) < \varepsilon$. But then $\{f_n\}$ is uniformly Cauchy on B ($\forall\, \eta > 0\ \exists\, n_0 \in \mathbb{N}$: $\forall n \geq n_0$ $|f_n(x) - f_m(x)| < \eta$ for all $n, m \geq n_0$ and all $x \in B$)

Let n_0 be so that $|f_n(x) - f_m(x)| < 1$ for all $x \in B$ and all $n, m \geq n_0$. Let $\tilde{M} = \max\{M_1, \dots, M_{n_0}\}$.

Then for $n \geq n_0$:

$$|f_n(x)| \leq |f_n(x) - f_{n_0}(x)| + |f_{n_0}(x)|$$
$$< 1 + \tilde{M}$$

Hence the claim holds with $M = \tilde{M} + 1$.

Math 7311, Analysis 1, Homework #7.
Due Monday, Oct, 8, at 11:30 in Class

If $X \subset \mathbb{R}$ then the σ-algebra will always be the restriction of the Lebesgue σ-algebra to X and the measure λ is the Lebesgue measure. $(X, \mathcal{A}, \mu)$ will stand for a general measur space.
If $f \in \mathcal{S}_0(X)$ and $A \in \mathcal{A}$ then

$$\int_A f \, d\mu = \int_X f 1_A \, d\mu \,.$$

1) (From previous exam) Let $A \subset (0,1)$ be a null-set. Show that $A^2 = \{x^2 \mid x \in A\}$ is a null-set. (Hint: For $\epsilon > 0$ take an open set $O = \dot{\bigcup}(a_i, b_i)$ such that $A \subset O$ and $\lambda(O \setminus A) < \epsilon/2$.)

2) (Almost from the book, problem 4.10) Assume that $\mu(X) < \infty$. Assume that the sequence $\{f_n\}_{=1}^{\infty}$ of measurable functions $f_n : X \to \mathbb{R}$ converges almost everywhere to the measureable function $f : X \to \mathbb{R}$. Show that $f_n \to f$ in measure.

3) (Similar to the book, Exercise 5.3, p. 71) If $f \in \mathcal{S}_o(X)$, $f \geq 0$, then $\mu_f : \mathcal{A} \to [0, \infty]$ defined by

$$A \mapsto \int_A f \, d\mu$$

is a finite measure on $\mathcal{A}$. (Hint: You have to show that if $\{E_j\}_{j=1}^{\infty}$ is a disjoint sequence of measurable subsets of X, then $\mu_f(\cup E_j) = \sum_j \mu_f(E_j)$.)

4) Let $f, g : X \to \mathbb{R}$ be simple functions. Show that $f \wedge g$ and $f \vee g$ are simple functions and write them as a sum of indicator functions.

1) Take $\mathcal{O} \subseteq (0,1)$ open, $A \subseteq \mathcal{O}$, $\lambda(A) + \frac{\varepsilon}{2} > \lambda(\mathcal{O})$.
As $\mathcal{O}$ is open in $\mathbb{R}$ we can write

$$\mathcal{O} = \dot{\bigcup_{i \in I}} (a_i, b_i) \quad , \text{ disjoint},$$

where I is a finite or countably infinite index set.
Then $\lambda(\mathcal{O}) = \sum_{i \in I} (b_i - a_i)$. Assume now that A is a null-set. Then $\sum (b_i - a_i) < \frac{\varepsilon}{2}$. We have
$(a_i, b_i)^2 = \{x^2 \mid x \in (a_i, b_i)\} = (a_i^2, b_i^2)$. ~~Note that the sets $\{(a_i^2, b_i^2)\}_{i \in I}$ are not always disjoint~~
We have then

$$\lambda(A^2) \leq \lambda(\mathcal{O}^2) \leq \sum_{i \in I} b_i^2 - a_i^2$$

$$= \sum_{i \in I} (b_i - a_i)(b_i + a_i)$$

$$\leq 2 \sum (b_i - a_i) < \varepsilon$$

As this holds for all $\varepsilon > 0$ it follows that $\lambda(A) = 0$.

2) Let $\varepsilon > 0$. Define

$$A_n = \{x \in X \mid \exists m \geq n_0 : |f_n(x) - f(x)| \geq \varepsilon\}$$

As $f_n \to f$ pointwise, it follows that $\bigcap A_n = \emptyset$.
Hence $\lim \mu(A_n) = 0$. In particular there exists $n_0 \in \mathbb{N}$ s.t. for all $n \geq n_0$, $\mu(A_n) < \varepsilon$, which is exactly the definition of convergence in measure.

3) Write $f = \sum_{\nu=0}^{N} \alpha_\nu 1_{A_\nu}$ with $A_0 = f^{-1}(0)$.

Let $E = \bigcup_{j=1}^{\infty} E_j$. Then, using that $1_C \cdot 1_D = 1_{C \cap D}$

$$\int_E f = \sum_{\nu=1}^{N} \alpha_\nu \int 1_{A_\nu \cap E}$$

$$= \sum_{\nu=1}^{N} \alpha_\nu \mu(A_\nu \cap E)$$

$$= \sum_{\nu=1}^{N} \alpha_\nu \sum_{j=1}^{\infty} \mu(A_\nu \cap E_j)$$ (because μ is a measure and $\{A_\nu \cap E_j\}$ disjoint

$$= \sum_{j=1}^{\infty} \sum_{\nu=1}^{N} \alpha_\nu \mu(A_\nu \cap E_j)$$ (all numbers are ≥ 0)

$$= \sum_{j=1}^{\infty} \int_{\cancel{A}E_j} f = \sum_{j=1}^{\infty} \mu(E_j) \quad \square$$

$X = \bigcup A_\nu = \bigcup B_\nu$

4) Write $f = \sum_{\nu=1}^{N} \alpha_\nu 1_{A_\nu}$ and $g = \sum_{\mu=1}^{M} \beta_\mu 1_{B_\mu}$.

Then

$$f \vee g = \sum_{\nu=1}^{N} \sum_{\mu=1}^{M} \max\{\alpha_\nu, \beta_\mu\} 1_{A_\nu \cap B_\mu}$$

$$f \wedge g = \sum_{\nu=1}^{N} \sum_{\mu=1}^{M} \min\{\alpha_\nu, \beta_\mu\} 1_{A_\nu \cap B_\mu}$$