math 2065 - Information

8.1, 1, 5, 7, 15, 19,

8.2: 1, 3, 5, 9, 11-17 odd, 19, 25

8.3: 1, 3, 13

9.2: 3, 5, 7-19 odd, 23, 25, 27, 29, 31, 33, 35

9.3: 9, 11.

9.5: 1, 3, 5, 7, 9

What kind of questions can be on the final

1) Solve the initial value problem

$$ty' + y = t, \quad y(1) = 1$$

2) Solve the differential equation

$$y' = \frac{y^2 - yt}{t^2 + yt}$$

3) Solve the differential equation

$$(t - y)y' = t + y.$$

4) Find the general solution to

$$ty' + y = e^t, \quad t > 0.$$

5) Mixing problems, #27, 29 p. 60.

6) Solve the following differential equations

(a) $y' - 4y = 0$, $y(0) = 2$

(b) $y' + 2y = 3e^{t}$, $y(0) = 0$

(c) $y'' + 3y' + 2y = 0$, $y(0) = 3$, $y'(0) = -6$

(d) $y'' + 8y' + 16y = 0$, $y(0) = 0$, $y'(0) = 1$

(e) $y'' + 2y' + y = e^{-t}$, $y(0) = 0$, $y'(0) = 0$

(f) $y'' + 2y' + 5y = t + e^{t}$, $y(0) = 0$, $y'(0) = 1$

(g) $y'' + 2y' + 5y = \cos 2t$, $y(0) = y'(0) = 0$

(h) $y''' + 2y'' + y' = t + e^{-t}$, $y(0) = y'(0) = 0$

7) Solve the system of differential equations

$$y_1' = y_2$$
$$y_2' = -y_1$$

and $\vec{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

8) Solve the system of differential equations:

$$y_1' = y_1 + 2y_2 \quad , \; y_1(0) = 1$$

$$y_2' = y_2 \qquad y_2(0) = 1$$

9) Find the Laplace transform of the function:

(a) $\mathcal{L}(\cos 2t + \sin 2t) =$

(b) $\mathcal{L}(t^2 e^t) =$

(c) $\mathcal{L}(e^{8t}(\cos 2t + 3\sin 4t))$

10) Find the inverse Laplace transform:

a) $\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s-1)}\right)$

b) $\mathcal{L}^{-1}\left(\frac{s+2}{s(s^2+1)}\right)$

c) $\mathcal{L}^{-1}\left(\frac{s-1}{s^2+4s+4}\right)$

11) Find the convolution

a) $t * t^4$
b) $t * e^t$

12) Let $L = D^4 + 5D^2 + 4$. Find

a) $L(e^{2t}) =$

b) $L(\cos t) =$

13) Solve

$$y_1' = 3y_1 - 4y_2 \qquad y_1(0) = 0$$
$$y_2' = y_1 - y_2 \qquad y_2(0) = 1.$$

14) Solve the differential equation

$$t^2 y'' - 3ty' + 4y = 0$$

15) Solve the system

$$3x + 2y + z = 4$$
$$2x + 2y + z = 3$$
$$x + y + z = 0$$

16) Row reduce the matrix

$$\begin{bmatrix} 2 & 3 & 8 & 0 & 4 \\ 3 & 4 & 11 & 1 & 8 \\ 1 & 2 & 5 & 1 & 6 \\ -1 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

to row reduced echelon form.

17) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Find e^{tA}.

18) Verify that

$$\vec{y}(t) = \begin{bmatrix} e^{t} - e^{3t} \\ 2e^{t} - e^{3t} \end{bmatrix}$$

is a solution to the initial value problem

$$\vec{y}\,' = \begin{bmatrix} 5 & -2 \\ 4 & -1 \end{bmatrix} \vec{y}(t)$$

$$\vec{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

MATH 2065, Test 2, Tuesday, July 21, 2015, Name: ______________________

For partial credit, show all your work!

Correct change of variable changes the Euler equation $at^2y''(t)+bty'(t)+cy(t) =$ into the constant coefficient differential equation $aY''(x)+(b-a)Y'(x)+cY(x)=0$.

1[20P]) Find $t * e^t =$ $e^t - t - 1$

$$\int_0^t u e^{t-u}\,du = e^t \int_0^t u e^{-u}\,du = e^t\left(\left[-ue^{-u}\right]_0^t + \int_0^t e^{-u}\,du\right)$$

$$= e^t\left(-te^{-t} - e^{-u}\Big]_0^t\right) = -t - 1 + e^t$$

2[20P]) Solve the initial value problem $y'' - y = 4e^{2t}$, $y(0) = 0$ and $y'(0) = 0$: $y(t) =$ $-2e^t + \frac{2}{3}e^{-t} + \frac{4}{3}e^{2t}$

homogeneous eq:

$y_h(t) = c_1 e^t + c_2 e^{-t} + \frac{4}{3}e^{2t}$

$$\left.\begin{aligned} y_p(t) &= Ae^{2t} \\ y_p'(t) &= 2Ae^{2t} \\ y_p''(t) &= 4Ae^{2t} \end{aligned}\right\} \quad \begin{aligned} 3Ae^{2t} &= 4e^{2t} \\ A &= \frac{4}{3} \end{aligned}$$

$c_1 + c_2 = -\frac{4}{3}$

$c_1 - c_2 = -8/3$

$2c_1 = -\frac{12}{3} = -4,\ c_1 = -2$

$c_2 = -\frac{4}{3} - c_1 = -\frac{4}{3} + 2 = 2/3$

3[20P]) Find the general solution to the differential equation $y'' + 2y' + 5y = 0$. $y(t) =$ $e^{-t}(c_1\cos 2t + c_2 \sin 2t)$

$$s^2 + 2s + 5 = (s+1)^2 + 4$$

$$= (s+1)^2 + 2^2$$

4[20P]) Find the general solution to the differential equation $y''' + 4y' = 6$. $y(t) =$ $C_1 + C_2 \cos 2t + C_3 \sin 2t +$

$\frac{3}{2}$

$s^3 + 4s = s(s^2+4)$

$c_1 + c_2 \cos 2t + c_3 \sin 2t = y_h(t)$

$y_p(t) = At, \; 4A = 6 \; = A = \frac{6}{4} = \frac{3}{2}$

5[20P]) Find the general solution to the Euler equation $t^2y'' + ty' - y = 0$ $y(t) =$ $C_1 t + C_2 t^{-1}$

$z'' - z = 0 \qquad s^2 - 1 = (s-1)(s+1)$

$z(x) = c_1 e^{x} + c_2 e^{-x}, \quad t = \ln(x)$

$y(t) = c_1 t + c_2 t^{-1}$

MATH 2065, Test 1, Thursday, June 25, 2015, Name: Solutions

For partial credit, show all your work!

1[15P]) Find the general solution (implicit or explicit) to the differential equation $y' = \dfrac{e^t}{1+2y}$.

Solution: $y + y^2 = e^t + C$

This is a separable equation and $(1+2y)dy = e^t dt$. Thus

$(y+y^2) = e^t + C$

2[15P]) Solve the initial value problem $t^2y' + ty = 1$, $y(1) = 0$.

$y(t) = \dfrac{\ln t}{t}$

First order + linear. Standard form $y' + \frac{1}{t}y = \frac{1}{t^2}$. $\int \frac{1}{t}dt = \ln t$ so $\mu(t) = t$. General solution $\frac{1}{\mu(t)}\left(\int \mu f dt + C\right)$ a $y(t) = \frac{1}{t}\left[\ln t + C\right] = \frac{\ln t}{t} + \frac{C}{t}$. Let $t=1$ gives

$0 = 0 + \frac{C}{1}$ so $C = 0$

3[20P]) A tank holds 10 L of pure water. A brine solution is poured into the tank at a rate of 2 L/min. The concentration of the incoming brine is 10 g of salt per liter. The mixture in the tank is kept well stirred and mixture leaves the tank also at the rate of 2 L/min. Determine the amount of salt, denoted by $y(t)$, in the tank at time t.

Solution: $y(t) = 100 - 100e^{-t/5}$

$y' = \text{Input} - \text{output} = 20 - \frac{2}{10}y(t) = 20 - \frac{1}{5}y$.

$y' + \frac{1}{5}y = 20$. $y(t) = e^{-t/5}\left[\int 20 e^{\frac{t}{5}} + C\right]$

$= e^{-\frac{t}{5}}\left[100 e^{t/5} + C\right] = 100 + Ce^{-t/5}$

$t=0$ gives $y(0) = 0 = 100 + C$. Hence $C = -100$

4[14P]) Compute the Laplace transform $\mathcal{L}\left(t\sin(t) + t^5 e^{2t}\right)(s) = \frac{2s}{(s^2+1)^2} + \frac{5!}{(s-2)^6}$

Using the table:

$\mathcal{L}(t\sin t) = \frac{2s}{(s^2+1)^2}$ 7

$\mathcal{L}(t^5 e^{2t}) = \frac{5!}{(s-2)^6}$ 7

5[16P]) Compute the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{4}{s^2+2s-3}\right) = e^{t} - e^{-3t}$ 8

$s^2+2s-3 = (s+3)(s-1)$

$\frac{4}{(s+3)(s-1)} = \frac{1}{s-1} - \frac{1}{s+3}$ 8

or $= \mathcal{L}^{-1}\left(\frac{4}{(s+1)^2-4}\right) = \frac{2}{i}\mathcal{L}^{-1}\left(\frac{2i}{(s+1)^2+(2i)^2}\right) = \frac{2}{i}e^{-t}\sin(2it)$

6[20P]) Use the Laplace transform to solve the initial value problem $y'' + 9y = e^{-t}$, $y(0) = 0$ and $y'(0) = 0$.

This gives

$(s^2+9)Y(s) = \frac{1}{s+1}$ 5 or $Y(s) = \frac{1}{(s+1)(s^2+9)}$ 5 $= \left(\frac{1}{s+1} - \frac{s-1}{s^2+9}\right)\frac{1}{10}$

$= \frac{1}{10}\left[\frac{1}{s+1} - \frac{s}{s^2+9} + \frac{1}{3}\frac{3}{s^2+3^2}\right]$ 5

Applying $\mathcal{L}^{-1}$ gives

$y(t) = \frac{1}{10}\left[e^{-t} - \cos 3t + \frac{1}{3}\sin 3t\right]$

5

A short table of Laplace transforms and inverse Laplace transform

1	$\mathcal{L}(af(t)+bg(t))(s)=aF(s)+bG(s)$
2	$\mathcal{L}(e^{at}f(t))(s)=F(s-a)$
3	$\mathcal{L}(-tf(t))(s)=\frac{d}{ds}F(s)$
4	$\mathcal{L}(1)(s)=\frac{1}{s}$
5	$\mathcal{L}(t^n)(s)=\frac{n!}{s^{n+1}}$
6	$\mathcal{L}(e^{at})(s)=\frac{1}{s-a}$
7	$\mathcal{L}(\cos(bt))(s)=\frac{s}{s^2+b^2}$
8	$\mathcal{L}(\sin(bt))(s)=\frac{b}{s^2+b^2}$
9	$\mathcal{L}(f'(t))(s)=sF(s)-f(0)$
10	$\mathcal{L}(f*g(t))(s)=F(s)G(s)$
11	$\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)^2}\right)(t)=\frac{1}{2}(\sin(t)-t\cos(t))$
12	$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right)(t)=\frac{1}{2}t\sin(t)$