Wavelets, Problems #3

Due, Friday Feb. 28

1) Let $g_{m,n}(x) = e^{2\pi imx}\chi_{[0,1)}(x-n)$, $m,n \in \mathbb{Z}$. Show that $\{g_{m,n}\}_{m,n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

2) Show in all details (and without using complex analysis) that if $f \in L^2(\mathbb{R})$ is Ω-bandlimited for some $\Omega > 0$, then f is analytic, i.e., there exists complex numbers a_n, $n \in \mathbb{N}_0$, such that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \ .$$

Find a formula for the numbers a_n.

3) Denote the elements in $\mathbb{R}^n \times \mathbb{R}$ by (x,t) where $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Let

$$\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_n}\right)^2$$

be the Laplace operator on $\mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is smooth and solves the initial value problem

$$\begin{aligned} \Delta u(x,t) &= (\partial/\partial t)^2 u(x,t) \\ u(x,0) &= 0 \\ \frac{\partial u}{\partial t}(x,0) &= f(x) \end{aligned}$$

where $f \in S(\mathbb{R}^n)$. (Thus u is a solution to the wave equation with initial condition $u(x,0) = 0$ and $\partial u/\partial t(x,0) = f(x)$.) Show that

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^n} \hat{f}(\omega)\frac{\sin(2\pi|\omega|t)}{|\omega|}\ e^{2\pi i x\cdot\omega}\ d\omega$$