Wavelets, Problems due Fr. March 28

1) Suppose that $\mathbf{H}$ is a finite dimensional Hilbert space. Show that a finite set $\{f_n\}$ in $\mathbf{H}$ is a frame if and only if $\{f_n\}$ is generating.

2) Let $\mathbf{H}$ be a separable Hilbert space. Let $\{f_n\}$ be a sequence in $\mathbf{H}$. Then $\{f_n\}$ is called a *Bessel* sequence if there exists a $B > 0$ such that

$$\sum_n |(x, f_n)|^2 \leq B\|x\|^2$$

for all $x \in \mathbf{H}$. Define the *Gram* matrix associated to $\{f_n\}$ by $G = ((f_k, f_j))_{j,k}$. Show that $\{f_n\}$ is a Bessel sequence with bound B if and only if G defines a bounded linear operator $(x_n) \mapsto (\sum_n x_n(f_n, f_j))_j$ on $\ell^2 = \{(c_n) \mid c_n \in \mathbb{C} \sum_n |c_n|^2 < \infty\}$.

3) Assume that $\{f_n\}$ is a Bessel sequence with bound B. Prove that the following holds:

a) $\|f_n\|^2 \leq B$ for all $n \in \mathbb{N}$.

b) If $\|f_n\| = B$ for some $n \in \mathbb{N}$, then $(f_n, f_k) = 0$ for all $k \neq n$.