Math 2025, Quiz #4 (Fall 2003) Name: ____________________

1) Determine, if the following set of vectors are linearly independent or not. Give a reason for your answer. Functions are elements of the space of piecewise continuous functions on $[0, 1)$.

a) $(1, 0, 1), (1, 1, 0), (1, 1, 1)$ LI

b) $(1, 1), (1, -1)$ Orthogonal, hence LI

c) $(1, 1), (2, 1), (2, 2)$. 3 vectors in $\mathbb{R}^2$ are always LD

d) $\chi_{[0,1)}, \chi_{[0,1/2)}$ LI

e) x, x^2 and x^3 LI

2) Which of the following sets of vectors is generating for $\mathbb{R}^3$?

a) $(1, 0, 1), (1, 0, -1), (0, 1, 0)$. Generating

b) $(1, 1, 0), (1, 2, 1), (0, 1, 1)$. LD, hence not generating (need 3 LI vector in $\mathbb{R}^3$ to have generating set)

c) $(1, 1, 1), (1, 0, -1), (0, 1, 0), (-1, 0, 2)$. Generating

3) Show that the vectors $(1, -1, 1), (1, 1, 0), (1, -1, -2)$ form an orthogonal basis for $\mathbb{R}^3$ and then determine the constants $a, b, c \in \mathbb{R}$ such that

$$(2, -4, 3) = a(1, -1, 1) + b(1, 1, 0) + c(1, -1, -2)\,.$$

$$a = \frac{(2,-4,3)\cdot(1,-1,1)}{3} = \frac{2+4+3}{3} = 3$$

$$b = \frac{(2,-4,3)\cdot(1,1,0)}{2} = \frac{2-4}{2} = -1$$

$$c = \frac{(2,-4,3)\cdot(1,-1,-2)}{6} = \frac{2+4-6}{6} = 0$$

Math 2025, Quiz #5 (Fall 2003) Name: ______________________

In the following we use the inner product $((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = x_1y_1+\ldots+x_ny_n$ on $\mathbb{R}^n$. All vector spaces of functions are viewed as subspaces of piecewise continuous functions on $[0,1)$ with the inner product

$$(f,g) = \int_0^1 f(x)g(x)\,dx\,.$$

In all of the problems use the Gram-Schmidt orthogonalization to construct of orthogonal set of vectors with the same linear span as the given vectors.

1) $\mathbf{v}_1 = (1,2), \mathbf{v}_2 = (-1,1)$.

$\boxed{u_1 = (1,2),\quad u_2 = \frac{3}{5}(-2,1)}$

$$u_2 = v_2 - \frac{(v_1,v_2)}{\|v_1\|^2}v_1 = (-1,1) - \frac{1}{5}(+1,2)$$
$$= \left(-\frac{6}{5}, \frac{3}{5}\right) = \frac{3}{5}(-2,1)$$

2) $\mathbf{v}_1 = (1,0,1), \mathbf{v}_2 = (0,1,1), \mathbf{v}_3 = (1,0,-1)$.

$u_1 = (1,0,1)\qquad u_2 = \frac{1}{2}(-1,2,1)\qquad u_3 = \frac{2}{3}(1,1,-2)$

$$u_2 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2},1,\frac{1}{2}\right);\quad \|u_2\|^2 = \frac{1}{4}(1+4+1) = \frac{3}{2}$$
$$u_3 = (1,0,-1) - 0 - \frac{2}{3}\underbrace{\left(-\frac{1}{2}-\frac{1}{2}\right)}_{=-1}\cdot\frac{1}{2}(-1,2,1)$$
$$= (1,0,-1) + \left(-\frac{1}{3},\frac{2}{3},\frac{1}{3}\right) = \left(\frac{2}{3},\frac{2}{3},-\frac{2}{3}\right) = \frac{2}{3}(1,1,-2)$$

3) $f(x) = 1, g(x) = 2-3x^2$

$u_1(x) = 1;\quad u_2(x) = 1-3x^2$

$$(f,g) = \int_0^1 2-3x^2 = 2x - x^3\Big]_0^1 = 1$$
$$u_2(x) = 2-3x^2 - 1 = 1-3x^2$$

4) $f(x) = \chi_{[0,1)}, g(x) = 2\chi_{[0,1/2)}$

$u_1(x) = \chi_{[0,1)}(x);\quad u_2(x) = \chi_{[0,1/2)}(x) - \chi_{[\frac{1}{2},1)}(x)$.

$$(f,g) = 2\int_0^1 \chi_{[0,1)}(x)\chi_{[0,1/2)}(x)\,dx = 2\int_0^{1/2} dx = 1$$

$$u_2(x) = 2\chi_{[0,1/2)}(x) - \chi_{[0,1)}(x) = \chi_{[0,1/2)}(x) - \chi_{[\frac{1}{2},1)}(x)$$

Math 2025, Quiz #6 (Fall 2003) Name: ____________

1) Denote by W the plane $\{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$.

a) Find the formula for the orthogonal projection $P : \mathbb{R}^3 \to W$.

$P(x,y,z) =$ $\frac{1}{3}(2x-y-z, -x+2y-z, -x-y+2z)$

i) The vectors $u_1 = (1,1,-2)$ and $u_2 = (1,-1,0)$ form an orthogonal basis for W. The orthogonal projection is then $P(v) = \frac{(v,u_1)}{\|u_1\|^2}u_1 + \frac{(v,u_2)}{\|u_2\|^2}u_2$.

ii) $\|u_1\|^2 = 1+1+(-2)^2 = 6$, $\|u_2\|^2 = 1^2+(-1)^2 = 2$.

iii) $((x,y,z),(1,1,-2)) = x+y-2z$

$((x,y,z),(1,-1,0)) = x-y$.

iv) $P(x,y,z) = \frac{x+y-2z}{6}(1,1,-2) + \frac{x-y}{2}(1,-1,0)$ and

$\frac{x+y-2z}{6} + \frac{x-y}{2} = \frac{x+y-2z+3x-3y}{6} = \frac{4x-2y-2z}{6} = \frac{2x-y-z}{3}$

and then the same calculation for the other coordinates.

b) What is $P(1,0,-1) =$ $(1,0,-1)$

We insert into the formula we got in (a) or notice, that $(1,0,-1) \in W$.

2) Let W be the space of functions on $[0,1)$ generated by the step functions $\chi_{[0,1/2)}$ and $\chi_{[1/2,1)}$. Notice that those two functions form an orthogonal basis for W and $\|\chi_{[0,1/2)}\|^2 = \|\chi_{[1/2,1)}\|^2 = 1/2$. Find the orthogonal projection of the function $f(x) = 3x^2$ onto W.

$P(f)(x) =$ $\frac{1}{4}\chi_{[0,1/2)}(x) + \frac{7}{4}\chi_{[\frac{1}{2},1)}(x) \Longleftrightarrow [\frac{1}{4}, \frac{7}{4}]$

$P(3x^2)(x) =$ [Average value of $3x^2$ over $[0,1/2)$] $\chi_{[0,1/2)}(x)$

$+$ [Average value of $3x^2$ over $[\frac{1}{2},1)$] $\chi_{[0,\frac{1}{2})}(x)$

$= \left[2\int_0^{1/2} 3x^2\,dx\right]\chi_{[0,\frac{1}{2})}(x) + \left[2\int_{1/2}^1 3x^2\,dx\right]\chi_{[\frac{1}{2},1)}(x)$

$6\int_0^{1/2} x^2\,dx = 2\cdot x^3\Big]_0^{1/2} = \frac{1}{4}$

$6\int_{1/2}^1 x^2\,dx = 2x^3\Big]_{\frac{1}{2}}^1 = 2 - \frac{1}{4} = \frac{7}{4}$