Math 2025, Quiz #4 **Name:** ____________________

Recall the following facts:

I) Suppose that V is a vector space, then a subset $W \subset V$ is a vector subspace if and only if the following hods:

(1) $u + w \in W$ for all $u, w \in W$

(2) $ru \in W$ for all $u \in W$ and all $r \in \mathbb{F}$.

Notice that (1) and (2) say that we can define addition and scalar multiplication for elements in W. **All the axioms for vector spaces follows then because they are valid in the vector space** W. Notice also that (2) implies (by taking $r = 0$) that $\mathbf{0} \in W$. This can be used to show that a subset of V is **not** a vector subspace. **If** $\mathbf{0} \notin W$ **then** W **is not a vector space**. But notice that the other implication **does not hold**, i.e., it does **not** follows form $\mathbf{0} \in W$ that W is a vector subspace. Notice can collect (1) and (2) into one condition:

$$ru + su \in W \quad \text{for all } u, v \in W,\ r, s \in \mathbb{F}\ .$$

(Try to prove that!)

II) A linear map $T : V \to W$ between two vector spaces V and W is linear if $T(ru + sv) = rT(u) + sT(v)$ for all $u, v \in V$ and all $r, s \in \mathbb{F}$. Taking $r = 0$ and $v = \mathbf{0}$ (and u and s arbitrary) we get by using that $0 \cdot u = \mathbf{0}$ and $\mathbf{0} + \mathbf{0} = \mathbf{0}$:

$$T(\mathbf{0}) = T(0 \cdot u) = 0 \cdot T(u) = \mathbf{0}\ .$$

Hence a linear map will always map the zero vector in V to the zero vector in W. But again, this does not work the other way, there are lots of maps that map the zero vector in V to the zero vector in W but are **not** linear (see probelm 2, part d).

Recall now that we can write any vector in $\mathbb{R}^n$ as a column vector or as a row vector. If we write the vector $\mathbf{x} \in \mathbb{R}^n$ as a row vector $\mathbf{x} = (x_1, \ldots, x_n)$ and if $A = [a_{ij}]$ is a $n \times m$ matrix then we can define a linear map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ by matrix multiplication

$$\begin{aligned} T_A(x_1, \ldots, x_n) &= [x_1, \ldots, x_n]\,[a_{ij}] \\ &= [\sum_{j=1}^n a_{j1} x_j, \sum_{j=1}^n a_{j2} x_j, \ldots, \sum_{j=1}^n a_{jm} x_j] \end{aligned}$$

It is a fact that any linear map $\mathbb{R}^n \to \mathbb{R}^m$ can be obtained in this way. For that let $e_k = (0, \ldots, 0, \underset{k^{th}-place}{1}, \ldots, 0)$. Then we have

$$T_A(e_j) = (a_{k1}, \ldots, a_{km})$$

which show us that we have to take A to be the matrix with row-vectors $T(e_j)$.

We can also view $\mathbb{R}^n$ as the space of column vectors. Then every linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by a $m \times n$ matrix (notice the difference in the order of m and n). This time the connection between matrices and linear maps is given by

$$[a_{ij}] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{1j} x_j \\ \vdots \\ \sum_{j=1}^n a_{nj} x_j \end{bmatrix}$$

Notice that this time we are summing over the **rows** of A and not the column. **Question**: Given a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ how can you now find the corresponding matrix?

We have the following two facts:

Lemma 0.1. *Let $T : V \to W$ be a linear map between the vector spaces V and W. Then the set*

$$\text{Ker}(T) = \{u \in V \mid T(u) = \mathbf{0}\}$$

is a vector subspace of V.

Proof. Let $u, v \in \text{Ker}(T)$ and $r, s \in \mathbb{F}$. Then

$$\begin{aligned} T(ru + sv) &= rT(u) + sT(v) \\ &= r \cdot \mathbf{0} + s \cdot \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

Hence $ru + sv \in \text{Ker}(T)$. ☐

Lemma 0.2. *Let $V \to W$ be a linear map between the vector spaces V and W. Then the set*

$$\mathrm{Im}(T) = \{w \in W \mid \exists u \in V \ : \ T(u) = w\}$$

is a vector subspace of W.

Proof. Obviously $\mathbf{0} = T(\mathbf{0}) \in \mathrm{Im}(T)$. Let $w_1, w_2 \in \mathrm{Im}(T)$ and $r, s \in \mathbb{F}$. Then there are $u_1, u_2 \in V$ such that $T(u_1) = w_1$ and $T(u_2) = w_2$. Using that T is linear we get

$$\begin{aligned} T(ru_1 + su_2) &= rT(u_1) + sT(u_2) \\ &= rw_1 + sw_2 \ . \end{aligned}$$

Hence $rw_1 + sw_2 \in \mathrm{Im}(T)$. □

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1) Determine which of the following sets is **not** a vector space. Explain your answer:

a) $\{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$.

b) $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^{10} \mid x_1 + x_2 - x_3 + 5x_4 = 4\}$;

c) $\{f \in C^\infty(\mathbb{R}) \mid f' + f = 0\}$;

d) $\{x \in V \mid T(x) = y\}$ where V and W are vector spaces, $T : V \to W$ is linear and $y \in W$.

Solution:

a) Let us do this in two different ways:

a$_1$) The direct way: Let $(x_1, y_1, z_1), (x_2, y_2, z_z)$ by such that $x_j + y_j - z_j = 0$ $(j = 1, 2)$. As

$$r(x_1, y_1, z_1) + s(x_2, y_2, z_z) = (rx_1 + sx_2, ry_1 + sy_2, rz_1 + sz_2)$$

we have to show that

$$(rx_1 + sx_2) + (ry_1 + sy_2) - (rz_1 + sz_2) = 0 \ .$$

But

$$\begin{aligned} (rx_1 + sx_2) + (ry_1 + sy_2) - (rz_1 + sz_2) &= (rx_1 + ry_1 - rz_1) + (sx_2 + sy_2 - sz_2) \\ &= r(x_1 + y_1 - z_1) + s(x_2 + y_2 - z_2) \\ &= 0 \ . \end{aligned}$$

a$_2$) Indirect: Define a linear map $T : \mathbb{R}^3 \to \mathbb{R}$ by

$$T(x, y, z) = x + y - z = [x, y, z] \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \ .$$

Then the set in (a) is exactly the kernel of T which is a vector space.

b) This is not a vector space because $0 + 0 - 0 + 5 \cdot 0 = 0 \neq 4$ and hence $\mathbf{0}$ is not in this set.

c$_1$) The direct way: Let $f, g \in C^\infty(\mathbb{R})$ and $r, s \in \mathbb{R}$ then

$$\begin{aligned} (rf + sg) &= (rf + sg)' + (rf + sg) \\ &= rf' + sg' + rf + sg \\ &= r(f' + f) + s(g' + g) \\ &= r0 + s0 \\ &= 0 \end{aligned}$$

Hence the set is a vector space.

c$_2$) The indirect way: Define a linear map $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ by $T(f) = f' + f$. Then the set is just the kernel of this map and hence a vector space.

d) This is a vector space if y is the zero vector (then it is just the kernel) but **not** a vector space if $y \neq \mathbf{0}$ because in that case the zero vector is not in there.

2) Which of the following maps is **not** linear. Explain your answer:

a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (x + 2y, x - 3y)$;

b) $T : \mathbb{R}^3 \to \mathbb{R}$, $T(x, y, z) = 2x + 3y + z + 1$;

c) $T : C([a, b]) \to \mathbb{R}$, $T(f) = \int_a^b f(t)\ dt$

d) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $T(f) = f' \cdot f$.

Solution:

a) This is a linear map given by

$$T(x, y, z) = [x, y, z] \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix} .$$

You can also show this directly!

b) This is not a linear map because $T(0, 0, 0) = 1 \neq 0$.

c) The integral is linear, so this is a linear map ($\int_a^b rf(t) + sg(t)\ dt = r \int_a^b f(t)dt + s \int_a^b g(t)\ dt$)

d) This map is not linear. We have $T(rf) = (rf)'(rf) = r^2 f'f \neq rf'f$ if $f'f \neq 0$ and $r \neq 0$, $r \neq 1$. You can for example take $f(x) = x^2$ and $r = 2$. Then

$$T(rf) = 4 \cdot (2x) \cdot x^2 = 8x^3$$

and

$$rT(f) = 2 \cdot (2x) \cdot x^2 = 4x^2 \ .$$

Notice that $T(0) = 0$!

3) Evaluate the given linear map at the given vector

a) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $T(f) = 2f' + 3f$, $f(x) = x^2$;

b) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} 2 & 1 \\ -1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$;

c) $T : C([0.1]) \to \mathbb{R}$, $T(f) = \int_0^1 f(x)dx$, $f(x) = \cos(2\pi x)$.

d) $T : \mathbb{C}^3 \to \mathbb{C}$, $T(z_1, z_2, z_3) = z_1 + 2z_2 + (3 + i)z_3$, $(z_1, z_2, z_3) = (1 + i, 1 - i, 2 + 3i)$.

Solution:

a) $T(x^2) = 2 \cdot 2x + 3 \cdot x^2 = 4x + 3x^2$.

b)

$$\begin{aligned} T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) &= \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 + 1 \\ -2 + 2 \\ 4 - 2 \end{bmatrix} \\ & \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix} . \end{aligned}$$

c) We have

$$\begin{aligned} T(1+i, 1-i, 2+3i) &= (1+i) + 2(1-i) + (3+i)(2+3i) \\ &= 1 + i + 2 - 2i + 6 - 3 + 2i + 9i \\ &= 6 + 10i \end{aligned}$$