Math 7311, Analysis 1, Homework #8.
Due Monday, Oct, 22, at 11:30 in Class

In the following $(X, \mathcal{A}, \mu)$ will always stand for a measurable space. $\mathcal{L}(X) = \mathcal{L}(X, \mu)$ stands for the space of integrable functions on X.

1) (# 5.8, p. 80 in the book.) Prove that the Lebesgue integral on $\mathbb{R}^n$ is invariant under reflections through the origin. That is, if $f \in \mathcal{L}(\mathbb{R}^n, \lambda)$, then

$$\int_{\mathbb{R}^n} f(-x)\,dx = \int_{\mathbb{R}^n} f(x)\,dx\,.$$

2) (# 5.9. p. 80) Let ν be the counting measure on $\mathbb{N}$. Show that a function $f : \mathbb{N} \to \mathbb{R}$ is integrable on $\mathbb{N}$ if and only if the sequence $\{f(n)\}_{n\in\mathbb{N}}$ is absolutely summable. Prove that if f is integrable with respect to ν then

$$\int_{\mathbb{N}} f\,d\nu = \sum_{n\in\mathbb{N}} f(n)\,.$$

3) Let $f \in \mathcal{L}(X)$ and $g : X \to \mathbb{R}$ bounded and measurable. Then $gf \in \mathcal{L}(X)$.

4) Let $f \in \mathcal{L}(X)$ and $\epsilon > 0$. Then there exists a simple function $\varphi = \sum_{j=1}^{N} \alpha_j \mathbf{1}_{A_j}$ such that

$$\int_X |f - \varphi|\,d\mu < \epsilon\,.$$

(Hint: Think first about positive functions.)

Solutions, Homework # 8

1) Assume first that $f = 1_{[a_1,b_1)} \times \cdots \times 1_{[a_n,b_n)}$. Then $f(-x) = 1_{(-b_1,-a_1]} \times \cdots \times 1_{(-b_n,-a_n]}$. It follows that

$$\int f(-x) = \prod\big((-a_j)-(-b_j)\big) = \prod (b_j - a_j) = \int f.$$

Assume that $f \geq 0$. As $\mathbb{R}^n$ is σ-finite it follows that

$$\int f = \sup_{\substack{0\leq \varphi \leq 1 \\ \varphi \text{ simple}}} \int \varphi = \lim_{\substack{0\leq \varphi \nearrow f \\ \varphi \text{ simple}}} \int \varphi$$

As $\varphi(x) \leq f(x) \iff \varphi(-x) \leq f(-x)$ we get

$$\int f = \sup_{0\leq\varphi\leq f} \int \varphi = \sup_{0\leq\varphi\leq f} \int \varphi(-x) = \sup_{0\leq\varphi(x)\leq f(-x)} \int \varphi$$
$$= \int f(-x)\, d\lambda(x).$$

For general $f \in \mathcal{L}$ apply this to f^+ and f^-.

2) Recall that functions $f: \mathbb{N} \to \mathbb{R}$ are exactly the same as sequence $\{f_n\}_{n=1}^{\infty}$ where the correspondence is given by $f_n = f(n)$. Let $f = \sum a_j 1_{A_j}$ be a simple functions. Then

$$\int f d\gamma = \sum a_j |A_j| \quad \Big(|A_j| = \text{number of elements in } A_j \Big)$$
$$= \sum f(n)$$

because $a_j |A_j| = \sum_{n\in A_j} a_j = \sum_{n \in A_j} f(n)$. Define for $f \geq 0$

$$F_N(n) = \begin{cases} f(n) & \text{if } n \leq N, \\ 0 & \text{if } n > N. \end{cases}$$

Then $F_N \nearrow f$. Hence by the MCT

$$\int f d\gamma = \lim_{N\to\infty} \int F_N d\gamma = \lim_{N\to\infty} \sum_{n=1}^{N} f(n) = \sum_{n=1}^{\infty} f(n).$$

8-2

For $f: \mathbb{N} \to \mathbb{R}$ let as usually f^+ respectively f^- be the positive respectively negative part. Then

$$f \in \mathcal{L} \iff f^+, f^- \in \mathcal{L}^+ \iff \sum f^+(n), \sum f^-(n) < \infty.$$

$$\iff \sum f^+(n) + \sum f^-(n) = \sum |f(n)| = \int |f| \, d\nu < \infty$$

(Here I have used that $\sum |f(n)| < \infty$ if and only if you can sum in any order you want.) It follows that $f \in \mathcal{L} \iff \{f(n)\}$ is absolutely summable and in that case

$$\int f \, d\nu = \sum f(n).$$

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You can also do it shorter: It was shown in the lecture that $f \in \mathcal{L} \iff |f| \in \mathcal{L}^+$. Now use the first part to show

$$\int |f| \, d\nu = \sum |f(n)|$$

so $\sum |f(n)| < \infty \iff \int |f| \, d\nu$. Now the general case follows by applying the first part to f^+ and f^-.

3) $|g(x) f(x)| \leq \|g\|_\infty |f(x)|$ a.e. As $\|g\|_\infty |f| \in \mathcal{L}^+$ it follows that $gf \in \mathcal{L}$

4) Assume ~~first that $f \geq 0$~~ Let $\varepsilon > 0$. Then there exists a $M > 0$ and $A \in \mathcal{A}$, $\mu(A) < \infty$ such that

$$\left| \int_X f - \int f_A^M \right| = \int f - f_A^M \, d\mu = \int |f - f_A^M| \, d\mu < \frac{\varepsilon}{2}$$

Here we have used that $f_A^M \leq f$ where f_A^M is defined by

$$f_A^M(x) = \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \leq M \\ M & \text{if } x \in A \text{ and } f(x) > M \\ 0 & \text{otherwise.} \end{cases}$$

8-3

As the carrier of f is σ-finite we know from the lecture that there exist a simple function φ, $0 \le \varphi \le f_A^M$ such that $\int f - \int \varphi = \int f - \varphi < \frac{\varepsilon}{2}$.

Now $\int |f - \varphi| d\mu = \int f - \varphi \, d\mu$

$$= \int f - f_A^M \, d\mu + \int f_A^M - \varphi \, d\mu$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Let $f = f^+ - f^- \in \mathcal{L}$. Let φ_1, φ_2 be simple functions such that

$0 \le \varphi_1 \le f^+$, $\int f^+ - \varphi_1 \, d\mu < \varepsilon/2$

$0 \le \varphi_2 \le f^-$, $\int f^- - \varphi_2 \, d\mu < \varepsilon/2$

Then $\varphi_1 - \varphi_2$ is a simple function and

$$\int |f - (\varphi_1 - \varphi_2)| d\mu = \int |(f^+ - \varphi_1) - (f^- - \varphi_2)| \, d\mu$$

$$\le \int |f^+ - \varphi_1| d\mu + \int |f^- - \varphi_2| d\mu < \varepsilon.$$

Math 7311, Analysis 1, Homework #9.
Due Monday, Oct, 29, at 11:30 in Class

All homework this time are from the book. You can find them on pages 86 and 87:

1) 5.17

2) 5.19

3) 5.20

4) 5.21

5) 5.22

Homework # 9 - solutions

1) Let $f_n(x) = \frac{x}{n} 1_{[-n,n]}(x)$. Find the pointwise limit $\lim_{n\to\infty} f_n(x) = f(x)$. Prove that $\int_{\mathbb{R}} f_n \, d\lambda \to \int_{\mathbb{R}} f \, d\lambda$.

Does $\{f_n\}$ satisfy the hypothesis of the LDCT? Explain.

Solution:

i) If x is fixed, then $\frac{x}{n} \to 0$. Hence $f_n(x) \to 0$.
So $f(x) = 0$ for all x.

ii) Fix n. Then (as $f_n|_{[-n,n]}$ is continuous and hence Riemann integrable)

$$\int_{\mathbb{R}} f_n \, d\lambda = \frac{1}{n} \int_{-n}^{n} x \, dx = \frac{1}{2n} x^2 \Big]_{-n}^{n} = 0.$$

(You could also use that f_n is odd.)

Hence

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} 0 = 0 = \int 0 \, d\lambda.$$

iii) There is no integrable function g such that $|f_n| \le g$ for all n. For that we note that for $x \in [\frac{n}{2}, n]$ we have $f_n(x) \geq \frac{1}{2}$.

Thus, if $g(x) \geq |f(x)|$, then for a fixed n

$$g(x) \geq \frac{1}{2} \sum_{n=1}^{\infty} 1_{[n-\frac{1}{2}, n]}(x)$$

so

$$\int g \geq \frac{1}{2} \sum_{n=1}^{\infty} n - (n - \tfrac{1}{2}) = \infty.$$

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2) Let f be integrable. Let $A_n \in \mathcal{A}$ be a sequence of disjoint measurable sets. Show, using the LDCT that

$$\int_{\dot{\cup} A_n} f \, d\mu = \sum_{n=0}^{\infty} \int f_n \, d\mu$$

<u>Proof</u> (as $f \in \mathcal{L}$ we can assume that X is σ-finite)

Let $B_N := \dot{\bigcup}_{n=1}^{N} A_n$ and $B = \dot{\bigcup}_{n=1}^{\infty} A_n$. Let

$$g_N = f \cdot 1_{B_N} = \sum_{n=1}^{\infty} f 1_{A_n}$$

and $g = f 1_B$.

Hence $g \in \mathcal{L}$. Furthermore

$$|g_N(x)| \leq \sum_{n=1}^{N} |f(x)| \, 1_{A_n}(x) \leq |f(x)|$$

as $x \in A_n$ for at most one n. It follows from LDCT that $g \in \mathcal{L}$ and

$$\int g \, d\mu = \lim_{N \to \infty} \int g_N \, d\mu$$

$$\int_{\dot{\cup} A_n} f \, d\mu = \lim_{N \to \infty} \sum_{n=1}^{N} \int_{A_n} f_n(x) \, d\mu = \sum_{n=1}^{\infty} \int_{A_n} f \, d\mu.$$

3) Let $f \in \mathcal{L}[1,\infty)$ with respect to Lebesgue measure. Prove or disprove:

a) $\int_{[1,b)} f \, d\lambda \to 0$ as $b \searrow 1$

b) $\int_{[b,\infty)} f \, d\lambda \to 0$ as $b \nearrow \infty$.

c) Assume $f \in \mathcal{L}(X, \mathcal{A}, \mu)$. Prove that

i) Let $\varepsilon > 0 \Rightarrow \exists A \in \mathcal{A}$, $\mu(A) < \infty$, such that

$$\int_{A^c} |f| < \varepsilon$$

ii) $\int_A f \, d\mu \to 0$ as $\mu(A) \to 0$.

9-3

<u>Solution:</u> I will only do (c) because (a) and (b) follows from (c). As $|\int_B f d\mu| \leq \int_B |f| d\mu$ for all $B \in \mathcal{A}$ we can assume that $f \geq 0$. But then

$$A \longmapsto \int_A f d\mu = \mu_f(A)$$

is a finite measure. Hence by previous homeworks. As $\{x \mid f(x) \neq 0\}$ is σ-finite we can find a increasing sequence $A_n \in \mathcal{A}$ such that $\mu(A_n) < \infty$ and $\bigcup A_n = \{x \mid f(x) \neq 0\}$. It follows that

$$\mu_f(X) = \mu_f(\{x \mid f(x) \neq 0\}) = \lim_{n \to \infty} \mu_f(A_n)$$

Let N be so that $\mu_f(X) - \mu_f(A_m) < \varepsilon$ for all $m \geq N$. But

$$\mu_f(X) - \mu_f(A_m) = \mu_f(X \setminus A_m) = \int_{A_m^c} f d\mu < \varepsilon.$$

Let A_n be a seq. in $\mathcal{A}$ such that $\mu(A_n) \to 0$.

As $\{x \mid f(x) \neq 0\}$ is σ-finite, we can find a simple function $\varphi = \sum_{j=1}^{N} a_j 1_{B_j}$ such that B_j disjoint, $0 \leq \varphi \leq f$ and

$$\int f - \varphi \, d\mu < \frac{\varepsilon}{2},$$

Let M be so that $\mu(A_n) \leq \dfrac{\varepsilon}{2((\max a_j)+1)N}$ for all $m \geq M$. Then for $m \geq M$

$$\int_{A_n} f = \int_{A_n} (f - \varphi) + \int_{A_n} \varphi$$

$$< \frac{\varepsilon}{2} + \sum a_j \mu(A_n \cap B_j)$$

$$\leq \frac{\varepsilon}{2} + \sum a_j \mu(A_n) \leq \varepsilon.$$

#9-4

4) Let $f_n(x) = \begin{cases} n(1-|x|) & \text{if } |x| \le \frac{1}{n} \\ 0 & \text{if } |x| > \frac{1}{n} \end{cases}$

Prove that $f_n \to 0$ pointwise a.e. Is

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n \quad ?$$

Solution. If $x=0$, then $f_n(x) = n$ for all n, so $\lim_{n\to\infty} f_n(0) = \infty$. If $|x|>0$, then there exist n s.t. $|x| > \frac{1}{n}$. Hence for all $m \geq n$ we have $f_m(x)=0$, so $\lim_{n\to\infty} f_n(x) = 0$.

As $n(1-|x|)1_{[-\frac{1}{n},\frac{1}{n}]}$ is Riemann integrable it follows that (as $x \mapsto |x|$ is even)

$$\int f_n \, d\lambda = 2n \int_0^{1/n} (1-x)\,dx =$$

$$= 2n\left(x - \frac{x^2}{2}\right)_0^{\frac{1}{n}}$$

$$= 2n\left(\frac{1}{n} - \frac{1}{2n^2}\right) = 2 - \frac{1}{n} \to 2$$

Hence

$$\lim_{n\to\infty} \int f_n \neq \int \lim_{n\to\infty} f_n.$$

#9-5

5) Let $f \in \mathcal{L}(\mathbb{R}, \lambda)$. Find $\lim_{n\to\infty} \int_{\mathbb{R}} e^{-nx^2} f(x)\, d\lambda(x)$.

<u>Solution</u>. Let $g_n(x) = e^{-nx^2} f(x)$. Then

$$|g_n(x)| \le |f(x)|$$

and $|f| \in \mathcal{L}^+$. It follows that

$$\lim \int g_n = \int \lim g_n$$

But $\lim_{n\to\infty} e^{-nx^2} f(x) = \begin{cases} f(0) & x = 0 \\ 0 & x \ne 0 \end{cases}$

Hence $\int \lim g_n = 0$. Thus

$$\lim_{n\to\infty} \int g_n(x)\, d\lambda = 0 \qquad \blacksquare$$

Math 7311, Analysis 1, Homework #10.
Due Monday, Nov 5 at 11:30 in Class

The first three problems are, with minor changes, from the comprehensive exam August 2012.

1) Let $r < 1$.

a) Show that the function $x \mapsto x^{-r}$ in in $L^1[0,1]$. (Hint: Find a monotone sequence that converges $f_n \to f$ and such that f_n is Riemann integrable.)

b) Let

$$a_n = \int_{[0,1]} \frac{1}{\frac{1}{n} + x^r}\, d\lambda(x) = \int_{[0,1]} \frac{x^{-r}}{1 + x^{-r}/n}\, d\lambda(x)\,.$$

Compute $\lim_{n\to\infty} a_n$.

2) Suppose that $f \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} |xf(x)| d\lambda(x) < \infty$. Define the Fourier sine transform F of f by

$$F(\xi) = \int_{\mathbb{R}} f(x)\sin(\xi x) d\lambda(x)\,.$$

for all $\xi \in \mathbb{R}$. Prove that F is differentiable and find its derivative.

3) Prove that if f_n is Lebesgue integrable on $[0,1]$ for each $n \in \mathbb{N}$ and

$$\sum_{n\in\mathbb{N}} \int_{[0,1]} |f_n(x)|\, d\lambda(x) < \infty$$

then $\sum_{n\in\mathbb{N}} f_n(x)$ is convergent almost everywhere, and

$$\int_{[0,1]} \sum_{n\in\mathbb{N}} f_n(x)\, d\lambda(x) = \sum_{n\in\mathbb{N}} \int_{[0,1]} f_n(x)\, d\lambda(x)\,.$$

(Hint: Use that an integrable function is finite a.e..)

4) Problem 5.34, page 92 in the book.

Homework #10-1

1) a) Define $f_n(x) = x^{-r} 1_{[\frac{1}{m}, 1]}$. Then $f_m \nearrow x^{-r}$ a.e.

Furthermore, as f_n is Riemann integrable,

$$\int f_n \, d\lambda = \int_{1/m}^{1} x^{-r} \, dx$$

$$= \frac{1}{1-r} x^{1-r} \Big]_{\frac{1}{m}}^{1}$$

$$= \frac{1}{1-r} - \frac{1}{1-r}\left(\frac{1}{m}\right)^{1-r}$$

As $r<1$ it follows that $1-r>0$ and $\left(\frac{1}{n}\right)^{1-r} \to 0$ as $n \to \infty$. It follows that $\lim_{n\to\infty} f_n \in \mathcal{L}$. As $f = \lim_{n\to\infty} f_n$ a.e. it follows that $f \in \mathcal{L}$ and

$$\int f \, d\lambda = \frac{1}{1-r}.$$

b) Let $g_m(x) = \dfrac{x^{-r}}{1 + x^{-r}/n} = \dfrac{1}{1+\frac{x^{-r}}{m}} f(x)$.

Then $|g_m(x)| \le f(x)$, $g_n \to f$, and $f \in \mathcal{L}$. It follows by LDCT that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \int g_n = \int \lim_{n\to\infty} g_n = \int f = \frac{1}{1-r} \quad \square$$

2) Let $\tilde{F}(x,\xi) = f(x)\sin(\xi x)$. Then, as $|\sin(\xi x)|, |\cos(\xi x)| \le 1$ and $\frac{\partial}{\partial \xi}\tilde{F}(x,\xi) = x f(x)\cos(\xi x)$. It follows that $x \mapsto \frac{\partial}{\partial \xi}\tilde{F}(x,\xi)$ and $x \mapsto \tilde{F}(x,\xi)$, are integrable for all ξ, and

$$\left|\frac{\partial}{\partial \xi}\tilde{F}(x,\xi)\right| \le |x f(x)| \in \mathcal{L}$$

It follows that F is differentiable and

$$F'(\xi) = \int x f(x)\cos(\xi x)\, dx.$$

#10-2

3) As $\sum_{n=1}^{\infty} \int_{[0,1]} |f_n(x)| d\lambda = \int_{[0,1]} \sum |f_n(x)| d\lambda(x)$ (Why?) it follows from the assumption $\sum \int |f_n(x)| d\lambda < \infty$, that $\sum_{n=1}^{\infty} |f_n| \in \mathcal{L}[0,1]$. Hence $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ almost all $x \in [0,1]$, i.e. $\exists N \in \mathcal{N}$ $(\mu(N) = 0)$ s.t. $\sum_{n=1}^{\infty} |f_n(x)| < \infty$ for all $x \in [0,1] \setminus \mathcal{N}$. Thus $\sum f_n(x)$ is absolute convergent for all $x \notin \mathcal{N}$. This implies that $\sum_{n=1}^{\infty} f_n(x)$ converges for $x \notin \mathcal{N}$. We also have

$$\left|\sum_{n=1}^{N} f_n\right| \le \sum_{n=1}^{\infty} |f_n| \in \mathcal{L} \quad \text{a.e.}$$

By LDCT $\lim_N \sum_{n=1}^{N} f_n \in \mathcal{L}[0,1]$ and

$$\int \sum_{n=1}^{\infty} f_n \, d\lambda = \sum_{n=1}^{\infty} \int f_n \, d\lambda.$$

4) Let

$$f_n(x) = n \, 1_{(0,\frac{1}{n}]}$$

Then $f_n \to 0$ everywhere. But, as f_n is Riemann integrable

$$\int_{[0,1]} f_n = n \int_0^{1/n} dx = 1$$

It follows that $\lim_{n\to\infty} \int f_n = \underline{\lim} \int f_n = 1 \neq \int \lim f_n = 0$.

Math 7311, Analysis 1, Homework #11.
Due Monday, Nov 12 at 11:30 in Class

This time all the problems are, with some minor changes, from previous comprehensive exams.

1) (January 2012) Consider the sequence of functions

$$f_n(x) := \mathbf{1}_{[-n,n]}(x) \sin\left(\frac{\pi x}{n}\right) \quad \text{for all } x \in \mathbb{R}\,.$$

(a) Determine $f(x) = \lim_{n\to\infty} f_n(x)$ and show that $f \in L^1$ and that the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on compact subsets of $\mathbb{R}$. Does the sequence converge uniformly on $\mathbb{R}$?

(b) Show that $\int_{\mathbb{R}} f(x)\, d\lambda(x) = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)\, d\lambda(x)$. Are the assumptions of Lebesgue's dominated convergence theorem satisfied?

2) (August 2011) Compute the following limit and justify your calculations:

$$\lim_{n\to\infty} \int_{\mathbb{R}^+} \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)\, d\lambda(x)\,.$$

(Hint: What is $\lim_n (1 + x/n)^n$?)

3) (August 2011, with some additions.) Let $(X, \mathcal{A}, \mu)$ be a measurspace such that $\mu(X) < \infty$. Show that $L^q(X) \subset L^p(X)$ for all $1 \le p \le q \le \infty$. Show that the inclusion map $L^q(X) \hookrightarrow L^p(X)$, $f \mapsto f$, is bounded. (Hint: You can assume that $p < q$. Let $a = q/p$ and let b be so that $\frac{1}{a} + \frac{1}{b} = 1$.)

4) (January 2010) Let $f \in L^4[0,1]$. Show that

$$\int_{[0,1]} \frac{f(x)}{x^{1/4}}\, d\lambda(x)$$

is finite.

#11-1

1) a) Let $K \subseteq \mathbb{R}$ be compact. Then there exists $R > 0$ such that $|x| \leq R$ for all $x \in K$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ s.t. $|\sin(t)| < \varepsilon$ for all $|x| < \delta$. Let $N \in \mathbb{N}$ be so that $|\frac{\pi R}{N}| < \delta$. Then for all $x \in K$ and $m \geq N$ we have $|\frac{\pi x}{m}| < \delta$. It follows that $|f_n(x)| < \varepsilon$. Hence $f_n(x) \to 0$ uniformly on compact sets. In particular $f = \lim f_n \in L^1$.

The sequence does not converge uniformly to zero. Assume that is the case. Then there exist $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}$ and $m \geq N$ we have

$$\left| 1_{[-n,n]}(x) \sin\left(\frac{\pi x}{m}\right)\right| < \frac{1}{4} < \frac{1}{\sqrt{2}}$$

Let $m \geq N$. As $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ we can find $x \in [-n,n]$ close to $\frac{m}{4}$ such that

$$\frac{1}{4} < \left| 1_{[-n,n]}(x) \sin\left(\frac{\pi x}{4}\right)\right|,$$

a contradiction.

b) As $\sin\left(\frac{\pi x}{m}\right)$ is Riemann integrable over $[-4,4]$ and $x \longmapsto \sin\left(\frac{\pi x}{m}\right)$ is odd, we get

$$\int f_m \, d\lambda = \int_{-m}^{m} \sin\left(\frac{\pi x}{m}\right) dx = 0$$

So

$$\lim \int f_m = \int \lim f_n = 0 .$$

Assume there exists $g \in \mathcal{L}^+$ s.t. $|f_n(x)| \leq g(x)$ for (almost) all x. Let $0 < \delta < \frac{\pi}{4}$ be so that

$$\frac{1}{4} < \sin(x) \leq \frac{1}{\sqrt{2}}$$

for all $x \in [\delta, \frac{\pi}{4}]$.

#11 - 2

As $\frac{m}{n+1} \to 1$ there exists $N \in \mathbb{N}$ s.t. $\delta \le \frac{\pi}{4}\frac{m}{n+1} < \frac{\pi}{4}$ for all $m \geq N$. It follows that

$$\frac{1}{4}\sum_{n=N}^{\infty} 1_{[2\pi+\frac{\pi}{4}(\frac{m}{n+1}),\, 2\pi+\frac{\pi}{4}]} \le g.$$

Thus

$$\int g \geq \frac{1}{4}\sum_{n=N}^{\infty}\int 1_{[2n+\frac{\pi}{4}(\frac{m}{n+1}),\, 2\pi+\frac{\pi}{4}]}\, d\lambda$$

$$= \frac{1}{4}\sum_{n=N}^{\infty}\frac{\pi}{4}\left(1-\frac{m}{n+1}\right) = \frac{\pi}{16}\sum_{n=N}^{\infty}\frac{1}{n+1} = \infty.$$

2) We show first that $f(x) = e^{-x}$ is integrable. For that let

$$f_n(x) = e^{-x} 1_{[0,m]}(x)$$

which is Riemann integrable on $[0,m]$. Thus

$$\int f_n\, d\lambda = \int_0^m f_n\, d\lambda = \int_0^m e^{-x}\, dx = -e^{-x}\Big]_0^n = 1 - e^{-n} \nearrow 1$$

It follows that

$$f = \lim f_n \in \mathcal{L} \text{ and } \int f = 1.$$

Let

$$F_x(y) = \left(1+\frac{x}{y}\right)^{-y} e^{x} > 0$$

Then

$$G_x(y) = \log F_x(y) = -y\log\left(1+\frac{x}{y}\right) + x$$

$$G_x'(y) = \frac{F_x'(y)}{F_x(y)} = -\log\left(1+\frac{x}{y}\right) + \frac{x}{y}\,\frac{1}{1+\frac{x}{y}}$$

$$= -\log(1+t) + \frac{t}{1+t}, \quad t = \frac{x}{y} \geq 0.$$

$$= h(t)$$

We have

$$h'(t) = -\frac{1}{1+t} + \frac{1}{1+t} - \frac{t}{(1+t)^2} = -\frac{t}{(1+t)^2} \le 0.$$

Thus h is decreasing, $h(t) \leq h(0) = 0$ and h is strictly decreasing if $t > 0$ (or $y < \infty$ for $x \neq 0$). It follows that

$$F_X(y) \nearrow \lim_{n\to\infty} F_X(n) = 1$$

In particular

$$0 \leq \left|(1+\tfrac{x}{m})^{-m}\cos(\tfrac{x}{m})\right| \leq e^{-x}$$

It follows by LDCT that

$$\lim_{n\to\infty} \int (1+\tfrac{x}{m})^{-m}\cos(\tfrac{x}{m})\,d\lambda = \int \lim_{n\to\infty} (1+\tfrac{x}{m})^{-m}\cos(\tfrac{x}{m})\,d\lambda$$

$$= \int e^{-x}\,dx = 1$$

Here we have used that for $x \in \mathbb{R}$, $\cos(\frac{x}{m}) \to \cos(0) = 1$ as $n \to \infty$.

3) Let $a = q/p$. Then $|g|^p \in L^a(X)$. There is nothing to show if $p = q$, so we can assume that $q > p$. Hence $a > 1$. Let $b = \frac{a}{a-1}$ (so $\frac{1}{a} + \frac{1}{b} = 1$). Then $1_X \in L^b(X)$ and by Hölder's inequality:

$$\|g\|_p^p = \int_X |g|^p = \int_X |g|^p 1_X \leq \||g|^p\|_a \cdot \mu(X)^{1/b}$$

But $p \cdot a = q$ so

$$\||g|^p\|_a = \|g\|_q^{\frac{q}{a}} = \|g\|_q^p.$$

It follows that

$$\|g\|_p \leq \|g\|_q \left(\mu(X)^{\frac{1}{pb}}\right)$$

so $g \in L^p$ and the inclusion has norm $\leq \mu(X)^{1/pb}$.

11-4

4) We have $\frac{4}{4-1} = \frac{4}{3}$ and $\left(x^{-\frac{1}{4}}\right)^{4/3} =$
$= x^{-1/3}$. By a previous homework $x^{-\frac{1}{4}} \in L^{\frac{4}{3}}$.
By Hölder's inequality we ~~get~~

$$\left| \int_{[0,1]} \frac{f(x)}{x^{1/4}} \, d\lambda \right| \leq \|f\|_4 \, \|x^{-\frac{1}{4}}\|_{4/3} < \infty.$$