Math 4035, Fall 2013, Midterm

Name: SOLUTIONS

The first 14 problems count each 4 points, leading to 56 points. Each of the three "proof" problems at the end counts 15 points leading to 45 points for that part. To total numbers of points therefore 101. Your final grades is therefore $\{yournumberofpoints\} \times 100/101$ rounded to the closest integer. In the first 14 give a short argument, a counter example, or refer to a Theorem/Lemma.

1) True or False : The set $\{(x,y) \in \mathbb{E}^2 \mid x^2 = 1\}$ is connected.

Not connected $= \{(x,y) \mid x<0\} \cup \{(x,y) \mid x>0\}$

$\{(-1,y) \mid y \in \mathbb{R}\}$ $\{(1,x) \mid x \in \mathbb{R}\}$

2) True or False : The function defined on $\mathbb{E}^2 \setminus \{0\}$ by $f(x,y) = \dfrac{xy^2}{x^4+y^2}$ is continuous.

Quotient of rational functions and $x^4+y^2 \neq 0$ if $(x,y) \neq (0,0)$

3) True or False : $< \mathbf{x}, \mathbf{y} >= x_1y_1 - x_2y_2$ defines an inner product on $\mathbb{E}^2$.

$<(1,1),(1,1)> = 0$.

4) True or False : The dimension of $L(\mathbb{E}^3, \mathbb{E}^4)$ is 12.

$\dim(L(\mathbb{E}^n, \mathbb{E}^m)) = nm$

5) True or False : If $E_1 \supseteq E_2 \supseteq \ldots \supseteq E_j \supseteq E_{j+1} \supseteq \ldots$ is a decreasing sequence of non-empty open sets in $\mathbb{E}^n$ then $\bigcap E_j \neq \emptyset$.

$E_j = (0, 1/j),\ j = 1, 2, \ldots$

6) True or False : If $D \subset \mathbb{E}^n$ is compact and $f \in \mathcal{C}(D, \mathbb{E}^m)$ then $f(D)$ is compact.

Thm. 9.3.1

7) True or False : Let $D \subseteq \mathbb{E}^n$ be open and $f = (f_1, \ldots, f_m) : D \to \mathbb{E}^m$. Let $\mathbf{p} \in D$. Then f is differentiable at $\mathbf{p}$ if and only if the all the partial derivatives $\frac{\partial f_i}{\partial x_j}(\mathbf{p})$ exists.

One has to assume that $\partial f_i / \partial x_j$ are continuous

Thm 10.2.1

Prove three out of the following six statements. Here D stands for a non-empty subset of $\mathbb{E}^n$

15) Suppose $f \in C(D, \mathbb{E}^m)$. If D is connected then $f(D)$ is connected. Assume not. Let U_1, U_2 be open, $U_1 \cup U_2 \supseteq f(D)$, $f(D) \cap U_1 \cap U_2 = \emptyset$, $f(D) \cap U_1 \neq \emptyset$ and $f(D) \cap U_2 \neq \emptyset$. Then $f^{-1}(U_j)$ is open in D, $D = f^{-1}(U_1) \dot{\cup} f^{-1}(U_2)$ and $f^{-1}(U_j) \neq \emptyset$, contradicting D connected.

16) Suppose $D \subseteq \mathbb{E}^n$ is open and that $f : D \to \mathbb{E}^m$ is differentiable on D. Then f is continuous on D. We have $f(x) - f(y) = Df(y)(x-y) + \mathcal{E}(x-y)$ when $\frac{\|\mathcal{E}(x-y)\|}{\|x-y\|} \to 0$ as $\|x-y\| \to 0$. Thus $\|f(x) - f(y)\| \leq \|Df(y)\| \|x-y\| + 1 \cdot \|x-y\|$ $\leq (\|Df(y)\| + 1)(\|x-y\|) \to 0$

So $\exists \delta$: $\|\mathcal{E}(x-y)\| \leq \|x-y\|$ if $\|x-y\| < \delta$.

17) Let $T \in L(\mathbb{E}^n, \mathbb{E}^n)$. Assume that $\|T\| < 1$. Then $T_K = \sum_{j=0}^{K} T^j$ is Cauchy and $T_K \to (\mathrm{I} - T)^{-1}$.

Using induction we see that $\|T_K - T_N\| \leq \sum_{j=k+1}^{N} \|T\|^j$, $K > N$. As $\|T\| < 1$ and $\sum x^j = \frac{1}{1-x}$ converges it follows that $\{T_k\}$ is Cauchy. We have $T_K(1-T) = 1 - T^{K+1} \to I$.

18) Assume that D is compact and $f \in C(D, \mathbb{E}^m)$. Then f is uniformly continuous. Let $\varepsilon > 0$. If $x \in D$ then there exist δ s.t. $\|y - x\| < \delta \Rightarrow \|f(y) - f(x)\| < \varepsilon/2$. We have $D \subset \bigcup_{x \in D} B_{\delta/2}(x)$. D compact $\Rightarrow \exists x_1, \ldots, x_k$ s.t. $D \subseteq \bigcup_{j=1}^{k} B(x_j)$. Assume $\|x-y\| < \delta/2$. First there is j s.t. $x \in B_{\delta/2}(x_j) \Rightarrow \|f(x) - f(x_j)\| < \varepsilon/2$. We also have $\|y - x_j\| \leq \|y - x\| + \|x - x_j\| < \delta \Rightarrow \|f(y) - f(x_j)\| < \varepsilon/2$. Hence $\|f(x) - f(y)\| \leq \|f(x) - f(x_j)\| + \|f(x_j) - f(y)\| < \varepsilon$.

19) Let $< \cdot, \cdot >$ on the vector space V and $\|x\| = \sqrt{< x, x >}$ the corresponding nom-function. Let $\mathbf{x}, \mathbf{y} \in V$. Then $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $< \mathbf{x}, \mathbf{y} > = 0$.

$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x, x \rangle + \langle y, y \rangle + 2\langle x, y \rangle =$

$= \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle$ or $\frac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) = \langle x, y \rangle$.

20) Let U_j $j \in J$ be a collection of open sets in $\mathbb{E}^n$. Then the union $\cup_{j \in J} U_j$ is open.

Let $x \in \bigcup U_j \Rightarrow \exists j : x \in U_j$. U_j is open $\Rightarrow \exists \delta > 0$ s.t. $B_\delta(x) \subseteq U_j \subseteq \bigcup U_j$.

8) Given an example of a bounded set that is not compact.

(0,1). Not closed.

9) Let $D = \{\mathbf{x} \in \mathbb{E}^n \mid 1 \leq \|x\| \leq 2\}$. Give an example of a set $E \subseteq D$ that is open in D but closed in $\mathbb{E}^n$.

D

10) State if the following sets are (a) open (b) closed (c) neither:

(1) $\{(x, y) \mid x^2 + y^2 > 1\}$. (a)
(2) $\{(x, y) \mid x + y > 1, x \geq 0\}$ (c)
(3) $\{(x, y) \mid xy = 0\}$. (b)
(4) $\{(x, y) \mid xy > 1\}$. (a)
(5) $\{(x, y) \mid x + 2y = 1\}$. (b)

11) Define $f : \mathbb{E}^2 \setminus \{0\}$ by $f(x, y) = \dfrac{x^2y^3}{x^2 + y^2}$. Find $\lim_{\mathbf{x} \to 0} f(\mathbf{x})$. $= 0$

$(x,y) = r(\cos\theta, \sin\theta)$

$f(r(\cos\theta, \sin\theta)) = r^3 \cos^2\theta \sin^3\theta \to 0$

12) Let $f : \mathbb{E}^3 \to \mathbb{E}^3$ be given by $f(x_1, x_2, x_3) = (\cos(2\pi x_1), x_2 \sin(x_3), x_1x_2)$. Find $\dfrac{\partial f_2}{\partial x_1}(1, 1, 1)$. $= 0$

f_2 does not depend on x_1.

13) Assume that $f : \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 2\} \to \mathbb{E}^3$ is differentiable and $Df(1, 1, 0) = \begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 0 \end{pmatrix}$. Find the directional derivative $D_{(1,-1,3)}f(1, 1, 0)$.

$$\begin{pmatrix} 1 & -1 & 2 \\ -3 & 4 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 + 1 + 6 \\ -3 - 4 + 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \end{pmatrix}$$

14) State the definition of a compact set in $\mathbb{E}^n$.

Any open covering has a finite subcover.