- U/K U compact, $K = (G^\theta)_0$ for an involution θ on U (curvature >0)
- $\mathbb{R}^n$ (curvature $=0$)
- G/K where G is a semisimple Lie group and K a maximal compact subgroup.

<u>Def</u>. Let $G \subseteq GL(n,\mathbb{C})$. Then

1) G is <u>reductive</u> if there exists a $g \in GL(n,\mathbb{C})$ such that gGg^{-1} is invariant under the anti-homomorphism

$$X \mapsto X^*$$

2) G is semisimple if $\dim G > 1$ and the center of G is discrete.

<u>Ex</u> $GL(n,\mathbb{R})$ is reductive but not semisimple because the group

$$\mathbb{R}^+ I \subseteq Z(G(n,\mathbb{R}))$$

On the other hand the groups $SL(n,\mathbb{R})$ and $SO(n,\mathbb{R})$ are semisimple. ☒

~~Let G~~

From now on we will assume that $G^* = G$ if G is reductive (semisimple). Define $\theta(g) = (g^*)^{-1}$. Note that $\theta|_{U(n)} = \mathrm{id}$, and we always have $\dot{\theta}(X) = X^*$.

<u>Lemma</u> If G is reductive, then $\mathfrak{g}^* = \mathfrak{g}$. ☒

<u>Lemma</u> Let $K \subseteq GL(n,\mathbb{C})$. Then there exist a $g \in GL(n,\mathbb{C})$, g positive definite, such that

$$gKg^{-1} \subseteq U(n).$$

<u>Proof.</u> Let $(\cdot,\cdot)$ denote the standard inner product on $\mathbb{C}^n$. Denote by dk a normalized Haar measure on K ($\int_K dk = 1$). Define a new inner product on $\mathbb{C}^n$ by

$$(v,w)^\sim = \int_K (kv, kw)\,dk$$

Then $(\ ,\ )^\sim$ is K-invariant, i.e. for all $\tilde{k} \in K$, $v,w \in \mathbb{C}^n$ we have

$$(\tilde{k}v, \tilde{k}w)^\sim = \int_K (k\tilde{k}v, k\tilde{k}w)\,dk$$

$$= \int_K (kv, kw)\,dk$$

$$= (v,w)^\sim.$$

As $(\ ,\ )^\sim$ is an inner product, there exist a positive definite A such that

$$(Av, w) = (v,w)^\sim$$

for all $v,w \in \mathbb{C}^n$. Let $k \in K$. Then

$$(Av, w) = (v, w)$$

$$= (kv, kw)$$

$$= (Akv, kw)$$

$$= (k^* Akv, w)$$

$\Rightarrow$ $A = k^* A k$ or $k^* = A k^{-1} A^{-1}$. Let $g = \sqrt{A}$.

Then $(gkg^{-1})^* = (g^{-1})^* k^* g^*$

$$= g^{-1} A k^{-1} A^{-1} g$$

$$= g k^{-1} g^{-1}$$

$$= (gkg^{-1})^{-1}$$

or $gKg^{-1} \subseteq U(n)$ ∎

G-vector bundles

Let $M = G/K$ where $K \leq G$ is a symmetric subgroup (or even more generally, only a closed subgroup). Let $\mathfrak{p}$ be the -1 eigenspace of $\dot{\theta}$. Then $\mathfrak{p} \cong T_{x_0}M$. Let $\pi : TM \to M$ be the canonical projection $T_m M \to m \in M$. The group G acts on TM by $(g, v) \longmapsto (d\ell_g)_{\pi(v)} v$ and the diagram

$$\begin{array}{ccc} TM & \xrightarrow{d\ell_g} & TM \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{\ell_g} & M \end{array}$$

commutes. Note ~~that $(d\ell_k)_{x_0}$~~: that

1) If $g \in G$, then $(d\ell_g)_x : T_x M \to T_{g \cdot x} M$ is linear.

2) If $k \in K$, then $\sigma(k) = (d\ell_k)_{x_0} : T_{x_0} M \to T_{x_0} M$ is linear and $\sigma(k_1 k_2) = \sigma(k_1)\sigma(k_2)$. Identifying

$$\mathfrak{p} \simeq T_{x_0} M$$

we get for $X \in \mathfrak{p}$, $k \in K$ and f smooth

$$\begin{aligned} &(d\ell_k)_{x_0}(X) f \\ &= X_{x_0} f \circ \ell_k \\ &= \frac{d}{dt} f(k e^{tX} \cdot x_0)\Big|_{t=0} \\ &= \frac{d}{ds} f(e^{s\, Ad(k) X} \cdot x_0)\Big|_{t=0}. \end{aligned}$$

Thus $\sigma(k)X = \mathrm{Ad}(k)X$. To understand the situation better let us consider the general case of G-vector bundles over M. Recall first: (with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

<u>Def.</u> A $\mathbb{F}$-vector bundle over M is a topological space V and a continuous, surjective map $\pi : V \to M$ such that

1) If $m \in M$, then $\pi^{-1}(m) \subseteq V$ is a $\mathbb{F}$-vector space
2) If $m \in M \Rightarrow \exists\, U \ni m$ open and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{F}^n$ for some $n \in \mathbb{N}$ such that for each $u \in U$ the map

$$pr_2 \circ \Phi\big|_{\pi^{-1}(u)} : \pi^{-1}(u) \to \mathbb{K}^n$$

is linear. Here $pr_2(u, v) = v$ denotes the projection onto the second factor.

We will assume that the standard/basic facts about vectorbundles are known (i.e. definition of isomorphisms etc.) Examples are

- TM the tangent bundle
- T^*M the cotangent bundle
- $T^{p}_{q} M = \underbrace{(TM \otimes \cdots \otimes TM)}_{p\text{-times}} \otimes \underbrace{(T^*M \otimes \cdots \otimes T^*M)}_{q\text{-times}}$

<u>Def.</u> A vectorbundle V over M is called homogeneous (or a G-bundle) if G acts on V such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{g} & V \\ \pi \downarrow & & \downarrow \pi \\ M & \xrightarrow{g} & M \end{array}$$

commutes and $g : V_m \to V_{g \cdot m}$ is linear.

We will now discuss how to construct all G-bundle and then describe their sections. Let V be a finite dimensional K-module. Denote the corresponding K-representation by σ. Let $\tilde{\mathcal{V}} = G \times V$. K acts on $\tilde{\mathcal{V}}$ from the right by

$$(g,v)\cdot k = (gk, \sigma(k)^{-1}v)$$

Let $\mathcal{V} = (G\times V)/K$. Note that the equivalence relation is given by

$$(g,v) \sim (h,w) \iff \exists k \in K : h = gk,\ w = \sigma(k)^{-1}v .$$

We denote the equivalence class of (g,v) by $[g,v]$.
The group G acts on $\tilde{\mathcal{V}}$ on the left by $g\cdot(h,v) = (gh,v)$.
This action commutes with the K-action and defines a G-action on $\mathcal{V}$, $g\cdot[h,v] = [gh,v]$.
~~We denote the~~
We set

$$\mathcal{V} = G \times_K V \left(= G \times_\sigma V\right),$$ with the quotient structure.

Define $\pi : \mathcal{V} \to G/K$ by

$$\pi([g,v]) = g\cdot x_0$$

Then π is well defined. Let $U \ni 0$, $U \subseteq \mathfrak{q}$ be open such that $\mathrm{Exp} : U \to V = \mathrm{Exp}(U)$ is a diffeomorphism. Then

$$\pi^{-1}(V) \simeq U \times V \quad \text{(local triviality)}.$$

It follows that $\mathcal{V}$ is a G-bundle.
Let $W \overset{\pi}{\to} M = G/K$ be a G-bundle. Let

$$V = W_{x_0} = \pi^{-1}(x_0).$$

As $k\cdot x_0 = x_0$ for each $k \in K$ and $(k_1k_2)\cdot x = k_1(k_2\cdot x)$ it follows that K acts linearly on V defining a representation of K.

Lemma $\mathbb{W}$ is isomorphic to $G\times_G V$.

Proof. Define $G\times V \to \mathbb{W}$ by
$$\tilde{\Phi}(g,v) = g\cdot v \in \mathbb{W}_{g\cdot x_0}$$
Then $\tilde{\phi}(gk, \sigma(k)^{-1}v) = \tilde{\phi}(g,v)$. Hence $\Phi : G\times_K V \to \mathbb{W}$, $\phi([g,v]) = \tilde{\Phi}(g,v)$ is well defined and smooth. It is obviously linear on each fiber. Assume that $\phi([g,v]) = \Phi([h,w])$. Then
$$g\cdot v = h\cdot w \quad\text{or}\quad (h^{-1}g)\cdot v = w$$
As $v, w \in \mathbb{W}_{x_0}$ it follows that $h^{-1}g\cdot x_0 = x_0$ or $h^{-1}g \in K$. Let $k = h^{-1}g$, then $hk = g$ and $\sigma(k)^{-1}w = v$ or $(g,v) \sim (h,w)$. Let $w \in \mathbb{W}$. Let $x = gx_0$ be such that $w \in \mathbb{W}_x$. Then $g^{-1}\cdot w \in V = \mathbb{W}_{x_0}$ and $\phi([g, g^{-1}w]) = w$. Hence ϕ is surjective. ■

Denote by $\Gamma^\infty(\mathcal{V})$ the smooth sections of $\mathcal{V}$, i.e. the vector space of smooth functions $s: G/K \to \mathcal{V}$ such that $s(x) \in \mathcal{V}_x$ for all x. Let $\mathcal{V} = G\times_G V$. Denote by $\mathrm{Ind}_K^G V$ the space of smooth functions $f: G \to V$ such that for all $g\in G$ and all $k \in K$:
$$f(gk) = \sigma(k)^{-1}f(g).$$
Let G act on $\mathrm{Ind}_K^G V$ by $(\mathrm{Ind}_K^G\sigma)(g)f(x) = f(g^{-1}x)$.

Lemma The map $\psi : \mathrm{Ind}_K^G V \to \Gamma^\infty(\mathcal{V})$,
$$\psi(f)(gK) = [g, f(g)] = s_f(gK)$$
is a G-isomorphism.

<u>Remark</u>: The group G acts on $\Gamma^\infty(V)$ by

$$(g\cdot s)(x) = g\cdot s(g^{-1}x)$$

Note that $s(g^{-1}x) \in V_{g^{-1}x}$ and $g\cdot$ sends it back to V_x. ▨

<u>Idea of the proof</u>: First we note that s_f is well defined: If $g\cdot x_0 = h\cdot x_0$, then $gk = h$ for some $k \in K$. We have $f(gk) = \sigma(k)^{-1} f(g)$ and hence $(g, f(g)) \sim (h, f(h))$. Let $g, h \in G$. Then

$$\begin{aligned} s_{g\cdot f}(h\cdot x_0) &= [h, f(g^{-1}h)] \\ &= g\cdot [g^{-1}h, f(g^{-1}h)] \\ &= g\cdot (s_f(g^{-1}hx_0)) \\ &= (g\cdot s_f)(hx_0). \end{aligned}$$

Hence $f \mapsto s_f$ is a G-map.

Assume now that $s \in \Gamma^\infty$. Define f by

$$f(g) = g^{-1}\cdot(s(gK)) = (g^{-1}\cdot s)(x_0)$$

(identifying $V \simeq W_{x_0}$). Obviously

$$f(gk) = \sigma(k)^{-1} f(g).$$

and $s_f = s$ ▨

<u>Example</u>: We have identified $\mathfrak{q}$ with $T_{x_0}G/K$. Let $X \in \mathfrak{q}$ and $k \in K$. Then

$$X_{x_0} f(x_0) = \frac{d}{dt} f(e^{tX})\Big|_{t=0}.$$

For $k \in K$ we have:

$$(dl_k)_{x_0}(X)f = X(f\circ l_k)(x_0)$$
$$= \frac{d}{dt} f(ke^{tX}\cdot x_0)\big|_{t=0}$$
$$= \frac{d}{dt} f(e^{t\,Ad(k)X}\cdot x_0)\big|_{t=0}$$
$$= (Ad(k)X)f(x_0)$$

where $Ad(k)X = kXk^{-1}$. It follows that the K-action on $\mathfrak{q} \simeq T_{x_0}M$ is the natural one given by Ad.

In particular

$$TG/K \simeq G \times_{Ad} \mathfrak{q}.$$

Furthermore, the vector field, i.e. $\Gamma^\infty(TG/K)$ corresponds to functions $F: G \to \mathfrak{q}$ such that

$$F(gk) = k^{-1}F(g)k$$
$$= Ad(k^{-1})F(g).$$

The group K acts on $\mathfrak{q}^*$, the dual of $\mathfrak{q}$, by

$$\langle k\cdot\nu, X\rangle = \langle \nu, k^{-1}\cdot X\rangle$$
$$= \langle \nu, k^{-1}Xk\rangle$$
$$= \langle \nu, Ad(k^{-1})X\rangle$$
$$= \langle Ad(k^{-1})^t\nu, X\rangle$$

The 1-form corresponds then to maps

$$\omega: G \to \mathfrak{q}^*$$
$$\omega(gk) = Ad(k)^t\omega(g).$$

Assume that β is a K-invariant bilinear form on $\mathfrak{p}$. Then β defines a G-invariant 2-tensor on G/K by

$$\beta_{g\cdot x_0}([g,v],[g,w]) = \beta(v,w)$$

This is well defined because if $h = gk$ then $[gk, k^{-1}v] = [h, k^{-1}v]$ and $[h, k^{-1}w] = [g, w]$.

$$\beta_{h\cdot x_0}([h,k^{-1}v],[h,k^{-1}w]) = \beta(k^{-1}v, k^{-1}w) = \beta(v,w).$$

Let $a, g \in G$. Then

$$\beta_{a(g\cdot x_0)}((d\ell_a)_{g x_0}[g,v], (d\ell_a)_{g\cdot x_0}[g,w])$$

$$= \beta_{(ag)\cdot x_0}([ag,v],[ag,w])$$

$$= \beta(v,w)$$

$$= \beta_{g\cdot x_0}([g,v],[g,w]).$$

<u>Example</u> Let $S^n = SO(n+1)/SO(n)$. The tangent bundle of S^n is

$$TS^n \simeq \{(w,v) \in \mathbb{R}^{n+1}\times\mathbb{R}^{n+1} \mid w \in S^n, v \perp w\}$$

and $SO(n+1)$ acts on TS^n by

$$g\cdot(w,v) = (gw, gv).$$

The elements in $\mathfrak{p}$ are given by

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & -\vec{x}^t \\ \vec{x} & 0 \end{pmatrix} \,\middle|\, \vec{x} \in \mathbb{R}^n \right\}.$$

Let $A \in SO(n)$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & -\vec{x}^t \\ \vec{x} & 0 \end{pmatrix} \cdot {}^{-1} = \begin{pmatrix} 0 & -(A\vec{x})^t \\ A\vec{x} & 0 \end{pmatrix}$$

Hence the map

$$\mathfrak{p} \to \mathbb{R}^n, \quad \begin{pmatrix} 0 & -\vec{x}^t \\ \vec{x} & \end{pmatrix} \longmapsto \vec{x}$$

intertwines the adjoint action with the standard action on $\mathbb{R}^n$ and

$$TS^n \simeq SO(n+1) \times_{SO(n)} \mathbb{R}^n.$$

The isomorphism is

$$[g, v] \longmapsto (ge_1, g\binom{0}{v}).$$

Using this we note that the vector fields corresponds to maps $F: SO(n+1) \to \mathbb{R}^n$ such that

$$F(gk) = k^{-1} F(g)$$

and the differential 1-forms correspond to functions $F: SO(n+1) \to \mathbb{R}^n$

$$F(gk) = k^t F(g) = k^{-1} F(g).$$

Hence, there is a natural isomorphism between the space of vector fields and 1-forms.

If $\mathfrak{p}$ (and/or $\mathfrak{g}$) is invariant under $X \mapsto X^*$ then there is a natural $U(n) \supset K$ invariant inner product on $\mathfrak{p}$ given by

$$(X,Y) = +\mathrm{Tr}\,(XY^*).$$

In the case of $SO(n+1)/SO(n)$ this form is

$$+\mathrm{Tr}\begin{pmatrix} 0 & -\vec{x}^t \\ \vec{x} & 0 \end{pmatrix}\begin{pmatrix} 0 & \vec{x}^t \\ -\vec{x} & \end{pmatrix} = \mathrm{Tr}\begin{pmatrix} \|X\|^2 & * \\ * & 0 \end{pmatrix} = \|X\|^2.$$

Thus our definition of Riemannian metric on $T(G/K)$ is the natural one.

Semisimple and reductive Lie groups

From now on we will always assume that G is connected.

Definition A linear Lie group is reductive if there exist a $g \in GL(n,\mathbb{C})$ such that

$$(gGg^{-1})^* = gGg^{-1}.$$

G is semisimple if G is reductive and $Z(G)$, the center of G, is finite.

Lemma If K is compact (not necessarily connected) then K is reductive.

Proof. Denote by $(\cdot,\cdot)$ the standard inner product on $\mathbb{C}^n$ (or $\mathbb{R}^n$ if $K = SL(n,\mathbb{R})$). Let dk be a normalized Haar measure on K. Thus dk is a Radon measure on K with the following two properties:

- If $g \in K$ and $f \in C(K)$ then
$$\int_K f(gk)\,dk = \int f(k)\,dk$$
[left invariance].

- $\int_K dk = 1$ [normalization].

Define a new innerproduct by
$$\langle v, w \rangle = \int_K (kv, kw)\,dk.$$

Note, if $v = w$ then $\int_K (kv, kw)\,dk = \int \|kv\|^2\,dk \geq 0$ and $= 0 \iff v = 0$. Then, there exists a $A \in GL(n,\mathbb{C})$ A positive definite such that for all v, w:
$$\langle v, w \rangle = (Av, w)$$
Let $g \in K$. Then by the invariance of dk we get that $\langle gv, g \cdot w \rangle = \langle v, w \rangle$. Hence
$$\begin{aligned} \langle v, w \rangle &= (Av, w) \\ &= \langle g \cdot v, g \cdot w \rangle \\ &= (Agv, gw) \\ &= (g^* A g v, w) \end{aligned}$$
As this holds for all v, w it follows that
$$g^* A g = A$$
or
$$g^* = A g^{-1} A^{-1}.$$
Define
$$B = \sqrt{A}.$$
Then

$$(BgB^{-1})^* = B^{-1}g^*B = B^{-1}Ag^{-1}A^{-1}B = Bg^{-1}B^{-1} = (BgB^{-1})^{-1}$$

Hence the claim. (In fact we have shown that $BKB^{-1} \subseteq U(n)$.) ▨

From now on G is connected and $G^* = G$. Then $\mathfrak{g}^* = \mathfrak{g}$. We will also assume that G is noncompact. Then

$$\theta : G \to G, \quad g \longmapsto (g^{-1})^*$$

is a nontrivial involution on G. Its differential is given by

$$\dot\theta(X) = -X^*,$$

and

$$G^\theta = G \cap U(n)$$

compact. Define a G-invariant bilinear form on $\mathfrak{g}$ by

$$B(X,Y) = \mathrm{Tr}(X\cancel{Y^*}) = \mathrm{Tr}(XY)$$

Then B defines by restriction a K-invariant bilinear form on $\mathfrak{g}$. Furthermore

$$(X,Y) = \mathrm{Tr}(XY^*) = -B(X,\theta Y)$$

is an innerproduct on $\mathfrak{g}$ such that

$$\mathrm{Ad}(g)^* = \mathrm{Ad}(g^*).$$

$$\mathrm{ad}(X)^* = \mathrm{ad}(X^*)$$

where $\mathrm{ad}(X)Y = XY - YX = [X,Y]$.

As a consequence we get

1) $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{g}^\perp \oplus \mathfrak{g}$ where

$$V^\perp = \{X \mid \forall Y \in V : B(X,Y) = 0\}$$
$$= \{X \mid (\forall Y \in V) : (X,Y) = 0\}.$$

(same for $\mathfrak{gl}(n,\mathbb{R})$ if $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{R})$). Furthermore $[\mathfrak{g}, \mathfrak{g}^\perp] \subseteq \mathfrak{g}^\perp$.

2) If $\mathfrak{z} \subseteq \mathfrak{g}$ is G-invariant (and hence an ideal) then

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^\perp$$

and $\mathfrak{z}^\perp$ is an ideal.

3) There exist simple ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_m$ (i.e. can not be decomposed further) such that

$$\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m .$$

Let $\mathfrak{z}(\mathfrak{g}) = \bigoplus_{\dim \mathfrak{g}_j = 1} \mathfrak{g}_j$. Then $\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. For that we note the following:

- $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j$

  Hence $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

- If $\dim \mathfrak{g}_i = 1$, then $[\mathfrak{g}_i, \mathfrak{g}_i] = 0$. Hence

  $[\mathfrak{g}_i, \mathfrak{g}] = 0$.

- If $\dim \mathfrak{g}_i > 1$, then $[\mathfrak{g}_i, \mathfrak{g}_i] = [\mathfrak{g}, \mathfrak{g}_i] = \mathfrak{g}_i$. For that note that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_i$ is an ideal. As $\mathfrak{g}_i$ is minimal the assumption $[\mathfrak{g}, \mathfrak{g}_i] \neq \mathfrak{g}_i$ would imply that $\mathfrak{g}_i$ is abelian and hence one dimensional.

The Cartan decomposition

Let for the moment $G = SL(n, \mathbb{R})$. Let $g \in G$. Then we can write

$$g = e^X k$$

where $X \in \operatorname{sym}(n, \mathbb{R})$ and $k \in SO(n)$. We have

$$g g^t = e^{2X}$$

so $X = \log \sqrt{g g^t}$. One way to take log of a positive definite matrix B is to take $u \in SO(n)$ such that

$$B = u \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} u^{-1}$$

As B is positive definit $\Rightarrow \lambda_j > 0$. Let

$$\log B = u \begin{pmatrix} \log\lambda_1 & & 0 \\ & \ddots & \\ 0 & & \log\lambda_n \end{pmatrix} u^{-1}.$$

Then let $k = e^{-X} g$. It follows that

$$k k^t = e^{-X} g g^t e^{-X} = 1$$

Hence, as $\det k = \det g \det e^{-X} = 1$, $k \in SO(n)$.

It is clear that ← denoting $Tr(X) = 0$.

$$\operatorname{sym}(n, \mathbb{R})^0 \times K \longrightarrow SL(n, \mathbb{R})$$
$$(X, k) \longmapsto e^X k$$

is differentiable. By construction, the inverse is also differentiable.

Let $P = e^{\mathrm{Sym}(n,\mathbb{R})^0}$, the set of positive definite matrices, with determinant 1. Then $SL(n,\mathbb{R}) \simeq P \times K$

<u>Theorem</u> (Cartan decomposition) Let G be semisimple. Then the map

$$\mathfrak{q} \times K \to G, \quad X, k \mapsto e^X k$$

is a diffeomorphism. In particular K is connected.

<u>Proof</u>. Define - as before - $X = \log\sqrt{gg^t}$. We only have to show that $X \in \mathfrak{q}$. Let $B = \sqrt{gg^t}$. Then B and X commutes, and hence have a joint eigenspace decomposition. It follows then easily (as $\mathfrak{q}$ is B invariant) that $\mathrm{ad}(X): \mathfrak{q} \to \mathfrak{q}$. Write

acting on $\mathfrak{q}$ }

$$X = X_1 + X_2$$

with $X_1 \in \mathfrak{q}$, $X_2 \in \mathfrak{q}^\perp$. Let $Y \in \mathfrak{q}$. Then

$$\mathfrak{q} \ni [X_1, Y] = [X_1, Y] + [X_2, Y]$$

$\Rightarrow [X_2, Y] \in \mathfrak{q}$. But $[\mathfrak{q}, \mathfrak{q}^\perp] \subseteq \mathfrak{q}^\perp$ and hence $[X_2, \mathfrak{q}] = 0$. In particular it follows that $X_1 X_2 = X_2 X_1$ and hence

$$e^X = e^{X_1} e^{X_2} = e^{X_2} e^{X_1}.$$

As G is connected it follows that e^{nX_2} commutes with G. If $X_2 \neq 0$ we would have

$$e^{X_2} = e^{-X_1} gg^* \in G$$

and $e^{nX_2} \in Z(G)$ for all n. As $e^{nX_2} \neq e^{mX_2}$ $(n \neq m)$ it would follow that $Z(G)$ is infinite, contradicting the assumption that G is semisimple ∎

Theorem Let $G = SL(n,\mathbb{R})$, $N = N_n = \left\{ \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ the group of upper triangular matrices, $A = A_n = \{ d(a_{11}, \dots, a_{nn}) \mid a_{11} \cdots a_{nn} = 1 \}$ the group of diagonal matrices with determinant one, and $K = K_n = SO(n)$. Then the map

$$N \times A \times K \xrightarrow{\varphi} G, \; n, a, k \mapsto nak$$

is a diffeomorphism.

Proof. It is clear that φ is differentiable (in fact a polynomial in the matrix elements). Assume that

$$nak = n_1 a_1 k_1$$

$\Rightarrow a_1^{-1} n_1^{-1} n a = (a_1^{-1} n_1^{-1} n a_1)(a_1^{-1} a) = k_1 k^{-1}$ is both orthogonal and upper triangular and hence the identity. Thus $k = k_1$, $a_1 = a$ and $n_1 = n$. Now let $g \in G$. write

$$g^{-1} = (\vec{x}_1, \dots, \vec{x}_n), \text{ i.e. } \vec{x}_j = g^{-1} e_j$$

Apply the Gram-Schmidt orthogonalization to the vectors $\vec{x}_1, \dots, \vec{x}_n$; ~~thus~~ resulting in a set of orthonormal vectors

$$v_1 = \|\vec{x}_1\|^{-1} \vec{x}_1 = a_{11} \vec{x}_1$$

$$v_2 = \|\vec{y}_1\|^{-1} (\vec{x}_2 - (\vec{x}_2, v_1) \vec{v}_1)$$

$$= a_{22} \vec{x}_2 + a_{12} x_1$$

$$\vdots$$

$$v_j = a_{jj} \vec{x}_j + a_{12} \vec{x}_2 + \dots + a_{j-1,j} x_{j-1}$$

$$\vdots$$

$$v_n = a_{nn} \vec{x}_n + \sum_{j=1}^{n-1} a_{jn} x_j$$

with $a_{ii} > 0$. ~~let $a = d(a_{11}, \dots, a_{nn})$, $h = \begin{pmatrix} 1 & & a_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$, $i < j$~~.

Let

$$a = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \in A$$

$$n = \begin{pmatrix} 1 & \frac{a_{12}}{a_{22}} & \cdots \\ & \ddots & \\ 0 & & 1 \end{pmatrix} = (n_{ij}) \in N$$

with $n_{ij} = 0$ if $i > j$, $n_{ii} = 1$, $n_{ij} = \dfrac{a_{ij}}{a_{jj}}$, $i > j$.

Then $g^{-1} n a e_j = v_j$. Let $k \in O(n)$ be such that $k^{-1} e_j = v_j$. Then

$$g^{-1}(nak) v_j = v_j \quad , j = 1, \dots, n$$

and hence $g = nak$. By construction

$$g \mapsto (n, a, k)$$

is smooth. ▨

We write $g = n(g)a(g)k(g)$ if we need to indicate the dependence on g.

It is a little more work to generalize this to semisimple Lie groups. First we need some new notation.

Let $\mathfrak{a} \subseteq \mathfrak{p}$ be maximal abelian. Then $\mathrm{ad}(\mathfrak{a}) \subseteq \mathfrak{gl}(\mathfrak{g})$ is a commuting family of selfadjoint operators. We can therefore simultaniously diaganalize them all. For that let for $\lambda \in \mathfrak{a}^*$

$$\mathfrak{g}^\lambda = \{ X \in \mathfrak{g} \mid (\forall H \in \mathfrak{a}) \ [H, X] = \lambda(H) X \}$$

Let $\Delta = \{\alpha \in \mathfrak{a}^* \mid \alpha \neq 0, \mathfrak{g}^\alpha \neq \{0\}\}$. The elements of Δ are called roots or restricted roots. If $\alpha \notin \Delta$ then $\mathfrak{g}^\alpha = \{0\}$.

<u>Lemma</u> If $\alpha, \beta \in \Delta \cup \{0\}$ then the following holds
1) $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$

2) $-\alpha \in \Delta \cup \{0\}$ and $\theta\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$.

<u>Proof</u> 1) follows from the Jacobi identity
$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$
Thus for $H \in \mathfrak{a}$, $X_\alpha \in \mathfrak{g}^\alpha$ and $X_\beta \in \mathfrak{g}^\beta$:

$$\begin{aligned}[H,[X_\alpha, X_\beta]] &= [X_\alpha, [H, X_\beta]] + [X_\beta, [H, X_\alpha]] \\ &= \beta(H)[X_\alpha, X_\beta] + \alpha(H)[X_\alpha, X_\beta] \\ &= (\alpha+\beta)(H)[X_\alpha, X_\beta].\end{aligned}$$

2) $$\begin{aligned}[H, \theta X_\alpha] &= \theta[\theta H, X_\alpha] \\ &= \theta[-H, X_\alpha] = -\alpha(H)\theta X_\alpha \quad . \blacksquare\end{aligned}$$

From this we get:

<u>Lemma</u> $\mathfrak{g}^0 = \mathfrak{z}(\mathfrak{a})$ is a θ-stable subalgebra. Let $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$. Then
$$\mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}.$$

<u>Proof</u> $\mathfrak{g}^0$ is θ-stable. Hence
$$\mathfrak{g}^0 = \mathfrak{g}^0 \cap \mathfrak{k} \oplus \mathfrak{g}^0 \cap \mathfrak{p}.$$
Now $\mathfrak{g}^0 \cap \mathfrak{p}$ commutes with $\mathfrak{a}$. Assume $\mathfrak{g}^0 \cap \mathfrak{p} \neq \mathfrak{a}$. Let $X \in \mathfrak{g}^0 \cap \mathfrak{p} \setminus \mathfrak{a}$. Then $\mathfrak{a} \oplus \mathbb{R}X$ is a commutative subalgebra in $\mathfrak{p}$ strictly bigger than $\mathfrak{a}$, contradicting the

maximality of $\mathfrak{a}$. ☒

As $\dim \mathfrak{g} < \infty$ it follows that Δ is finite. For $\alpha \in \Delta$ let

$$\alpha^{\perp} = \{ H \in \mathfrak{a} \mid \alpha(H) = 0 \} = \ker(\alpha)$$

Then

$$\mathfrak{a}_{reg} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Delta} \alpha^{\perp} \quad \text{(the set of regular elements)}$$

$$= \{ H \in \mathfrak{a} \mid (\forall \alpha \in \Delta)\ \alpha(H) \neq 0 \}$$

is open and dense. Let $H \in \mathfrak{a}_{reg}$ and define

$$\Delta^{+} = \{ \alpha \in \Delta \mid \alpha(H) > 0 \}$$

(Note that Δ^{+} does only depend on the connected component containing H.)

<u>Lemma</u> 1) If $\alpha, \beta \in \Delta^{+}$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in \Delta^{+}$.
2) $\Delta = \Delta^{+} \cup -\Delta^{+}$ and $-\Delta^{+} = \theta \Delta^{+}$
3) $\Delta^{+} \cap -\Delta^{+} = \emptyset$. ~~~~

<u>Proof.</u> This is clear.

Define

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^{+}} \mathfrak{g}^{\alpha}$$

then $\mathfrak{n}$ is a subalgebra, $[\mathfrak{g}^{0}, \mathfrak{n}] \subseteq \mathfrak{n}$ and hence $\mathfrak{p} = \mathfrak{g}^{0} \oplus \mathfrak{n}$ is an algebra. Finally $\mathfrak{n}$ is nilpotent, i.e. $ad(X)$ is nilpotent for each $X \in \mathfrak{n}$.

<u>Lemma</u> Let $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Then

$$\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$$
$$= \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{(Iwasawa decomposition)}$$

<u>Proof</u>: The first part is just the decomposition into eigenspaces. For the other one let $X \in \mathfrak{g}$. Then we can write

$$X = \sum X_{-\alpha} + X_m + X_a + \sum X_\alpha$$
$$= \sum (X_{-\alpha} + \theta X_{-\alpha}) + X_m + X_a + \sum (X_\alpha - \theta X_{-\alpha})$$

We have $X_{-\alpha} + \theta X_{-\alpha} \in \mathfrak{k}$ and $\theta X_{-\alpha} \in \mathfrak{n}$. Hence $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$. Assume that $X \in \mathfrak{k}$, $Y \in \mathfrak{a}$ and $Z \in \mathfrak{n}$ is such that $X + Y + Z = 0$. But $\mathrm{ad}(X)$ is skew-symmetric, $\mathrm{ad}\,Y$ is diagonal and $\mathrm{ad}\,Z$ is uppertriangular. Hence $\mathrm{ad}(X) = \mathrm{ad}(Y) = \mathrm{ad}(Z) = 0$. As ad is injective, it follows that $X = Y = Z = 0$ ∎

<u>We make the following</u>:

<u>Lemma</u> Let $H \in \mathfrak{a}_{reg}$. Then

$$\mathfrak{z}_{\mathfrak{g}}(H) = \{X \in \mathfrak{g} \mid [H, X] = 0\}$$
$$= \mathfrak{g}^0$$

<u>Proof</u>. It is clear that $\mathfrak{g}^0 \subseteq \mathfrak{z}_{\mathfrak{g}}(H)$. Let $X \in \mathfrak{z}_{\mathfrak{g}}(H)$. Write

$$X = \sum X_\alpha + X_0$$

with $X_0 \in \mathfrak{g}^0$. Then

$$0 = [H, X] = \sum \alpha(H) X_\alpha$$

As the root vectors X_α are linearly independent, $\alpha(H) \neq 0$, it follows that $X = X_0$ ∎