Math 2025, Test #2 . Fall 2002 Name: ______________________________

1[8P]) Apply the two dimensional Haar wavelet transform to the matrix $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$.

Answer: $\begin{pmatrix} 11/4 & -5/4 \\ -1/4 & 3/2 \end{pmatrix}$

2[12P]) Appy the two dimensional Haar wavelet transform to the matrix $\begin{pmatrix} 4 & -2 & 11 & -1 \\ 2 & 0 & 5 & -3 \\ 20 & -4 & 2 & -2 \\ 8 & 2 & -4 & -4 \end{pmatrix}$

Answer: $\begin{pmatrix} 17/8 & 13/8 & 2 & 5 \\ -1/8 & -21/8 & 15/2 & 1 \\ 0 & 2 & 1 & 1 \\ 3/2 & 2 & 9/2 & 1 \end{pmatrix}$.

3[8P]) Let $z = 2 + 3i$ and $w = \frac{1}{2+i}$. Evaluate the following:

a) $z \cdot w = \frac{7}{5} + \frac{4}{5}i$

Notice first that for a compex number $\frac{1}{x+iy}$ we have

$$\frac{1}{x+iy} = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i,$$

Hence $w = \frac{1}{5}(2 - i)$. Thus

$$\begin{aligned} z \cdot w &= \frac{1}{5}(2+3i) \cdot (2-i) \\ &= \frac{1}{5}(2 \cdot 2 - (3i) \cdot i) + \frac{1}{5}(3i \cdot 2 + 2 \cdot (-i)) \\ &= \frac{1}{5}(4+3) + \frac{i}{5}(6-2)) = \frac{7}{5} + \frac{4}{5}i \end{aligned}$$

b) $\bar{z} = 2 - 3i$: Recall that for a complex number $x + iy$ we have $\overline{x+iy} = x - iy$.

c) $z^2 = z \cdot z = (2+3i) \cdot (2+3i) = (4-9) + 2 \cdot 3i = -5 + 6i$.

d) $|w|^2 = \frac{1}{25}(4+1) = \frac{1}{5}$.

Recall that for any complex number $x + iy$ the number $|x + iy|^2$ is a nonnegative real number give by

$$\begin{aligned} |x+iy|^2 &= (x+iy) \cdot \overline{(x+iy)} \\ &= (x+iy) \cdot (x-iy) \\ &= x^2 + y^2 \ , \end{aligned}$$

4[8P]) Evaluate the following multiplication of matrices:

a) $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 4 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -9 & 10 \\ 1 & -16 & 5 \end{bmatrix}$

Recall first of all that we can only multiply $m \times n$ matrix by an $n \times q$ matrix and the outcome is always a $m \times q$ matrix. Furthermore if $A \cdot B = C$ then we have

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \ .$$

Thus

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 4 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} 1\cdot 2+2\cdot 1 & 1\cdot 1+2\cdot(-5) & 1\cdot 4+2\cdot 3 \\ (-1)\cdot 2+3\cdot 1 & (-1)\cdot 1+3\cdot(-5) & (-1)\cdot 4+3\cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -9 & 10 \\ 1 & -16 & 5 \end{bmatrix}$$

b) $\begin{bmatrix} 2 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ -1 & 2 \\ 4 & 3 \end{bmatrix} = [11 \quad 9].$

First notice that this is a product of a 1×4 matrix by a 4×2 matrix. The outcome should therefore by a 1×2 matrix (or row vector):

$$\begin{bmatrix} 2 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 2 \\ -1 & 2 \\ 4 & 3 \end{bmatrix} = [2\cdot 1+2\cdot 2+(-1)\cdot(-1)+1\cdot 4, 2\cdot 2+2\cdot 2+(-1)\cdot 2+1\cdot 3]$$
$$= [2+4+1+4, 4+4-2+3]$$
$$= [11, 9]$$

Before discussing the next problems let me recall few facts:

Definition: Let $\mathbb{F}$ be a field. A **vector space** V over $\mathbb{F}$ is a nonempty set with operations of vector addition, i.e., a map

$$V \times V \ni (u, v) \mapsto u + v \in V$$

and a scalar multiplication, i.e., a map

$$\mathbb{F} \times V \ni (r, v) \mapsto r \cdot v \in V$$

satisfying the following properties:

A1 (Commutativity of addition) For all vectors $u, v \in V$ we $u + v = v + u$;

A2 (Associativity for addition) For all $u, v, w \in V : u + (v + w) = (u + v) + w$;

A3) (Existense of additive identity) There exists an element, denote by $\mathbf{0} \in V$, such that for all $u \in V : u + \mathbf{0} = u$;

A4) (Existense of additive inverse) For every $u \in V$ there exists an element, denoted by $-u$, such that $u + (-u) = \mathbf{0}$;

A5) For all $u \in V : 1 \cdot u = u$;

A6) (Associativity of scalar multiplication) For all $r, s \in \mathbb{F}$ and $u \in V$ we have $(rs) \cdot u = r \cdot (s \cdot u)$;

A7) (First distributive property) For all $r \in \mathbb{F}$ and $u, v \in V$ we have $r \cdot (u + v) = (r \cdot u) + (r \cdot v)$;

A8) (Second distributive property) For all $r, s \in \mathbb{F}$ and all $u \in V : \ (r + s) \cdot u = (r \cdot u) + (s \cdot u)$;

The first thing the check is therefore always: Is the addition and multiplication defined, and do those operations always give an element in V!

From the axioms A1-A8 it follows that:

1. We have $0 \cdot u = \mathbf{0}$ for all $u \in V$.
2. The additive inverse is $-u = (-1) \cdot u$. That is, we take the vector u and multiply it by -1. This follows from

$$\begin{aligned} u + (-1)\cdot u &= 1 \cdot u + (-1) \cdot u \quad \text{(by A5)} \\ &= (1 + (-1)) \cdot u \quad \text{(by A8)} \\ &= 0 \cdot u \\ &= \mathbf{0} \quad \text{(by the remark just made)} \end{aligned}$$

Important examples of vector spaces:

1. The space $\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{F}\}$. Here the addition and scalar multiplication is given by

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$
$$r \cdot (x_1, \ldots, x_n) = (rx_1, \ldots, rx_n) \ .$$

   Notice that we can actually view $\mathbb{F}^n$ as the space of row vectors ($1 \times n$-matrices) or as the space of column vectors ($n \times 1$-matrices).

2. Let S be a set and let $V = \mathbb{F}^S =$ the space of functions from S to $\mathbb{F}$. Then we can define addition and scalar multiplication by

$$(f + g)(s) = f(s) + g(s)$$
$$(r \cdot f)(s) = rf(s) \ .$$

   We will not prove here that this gives us a vector space. Notice that in this example we can replace the target space $\mathbb{F}$ by any vector space over $\mathbb{F}$.

3. Let $M(n \times m, \mathbb{F})$ be the set of $n \times m$ matrices with coefficients in $\mathbb{F}$. Define addition and scalar multiplication by

$$[a_{ij}] + [b_{ij}] := [a_{ij} + b_{ij}]$$
$$r[a_{ij}] = [ra_{ij}] \ .$$

Often we construct vector spaces in the following way:

1. We have given a vector space V. In particular we know that all the axiomes A1-A8 are valid for elements in V.

2. Then we define a subset S of V by

$$S = \{v \in V \mid \text{some conditions holds for } v\}$$

   Thus S is in general not all of V but only those elements that satisfy the given condition. Here are some examples:

   (a) Let $\mathbb{F} = \mathbb{R}$, and let I be a nonempty interval in $\mathbb{R}$. Let V be the vector space of functions on I. According to above, we know that V is a vector space. Now let us consider the condition **continuous**. Thus we set

$$C(I) = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\}$$

   So a function on I is in the subset $C(I)$ if and only if f is continuous. Let for example $I = [0, 1)$ for a moment, then the function f defined by $f(x) = x^2$ is in $C([0, 1))$ but the function φ_1^2 is not in S.

   (b) If $\mathbb{F}$=$\mathbb{C}$ then we write $C(I, \mathbb{C})$ for the set of functions $f : I \to \mathbb{C}$ that are continuous.

   (c) Let $V = \mathbb{R}^3$ and consider $S = \{(x, y, z) \in \mathbb{R}^3 \mid 3x - y + 2z = 0\}$. Then only those elements in $\mathbb{R}^3$ that are solutions to the equation $3x - y + 2x = 0$ belonge to S. As an example the point/vector $(1, 3, 0)$ is in S $(3 \cdot 1 - 3 + 2 \cdot 0 = 0)$ whereas $(1, 1, 1)$ does not belonge to S $(3 \cdot 1 - 1 + 2 \cdot 1 = 4 \neq 0)$. One can show that S is the plane of points in $\mathbb{R}^3$ that are perpenticular to $(3, -1, 2)$.

   (d) Let $V = M(1 \times m, \mathbb{R})$ and let A be a $m \times k$ matrix. Consider $S = \{\mathbf{x} \in V \mid \mathbf{x}A = \mathbf{0}\}$. Thus S is the set of solutions of a system of k- equations with m unkowns.

   (e) Let $V = \mathbb{R}^I$, i.e., the space of functions on an interval I. Define $S = \{f \in V \mid f \text{ is piecewise continuous}\}$. Then all the continuous functions are in S as well as all the functions that are discontinuous at finitely many points. For example if $I = [0, 1)$ then all the functions φ_j^N and all the functions ψ_j^N are elements in S. Here the condition that has to be satisfied is that f is piecewise continuous.

(f) Let V be the space of all functions on the real line $\mathbb{R}$ (thus we are looking at the above example with $I = \mathbb{R}$). Let S be the set be the set of polynomials of degree $\leq n$ where n is some fixed nonnegative integer. Thus every element in S can be written in the form $p(x) = \sum_{j=0}^{n} a_j x^j$ where a_j are real numbers.

3. After defining a set S in this way we often need to know if S is a vector space or not. For that we again notice some simple facts:

   (a) If u and v are two elements in S then we can define $u + v \in V$ **because both u and v are elements of the vector space V and addition is defined in V**;

   (b) If u is in S and $r \in \mathbb{F}$ then - **again because V is a vector space** - the vector $r \cdot v \in V$ is defined.

   (c) As V is a vector space it follows that all the axioms A1-A8 are valid.

   (d) **What is missing is the first part in the definition: Are the vectors $u + v$ and $r \cdot u$ again in S?** If that is the case it follows that S is in fact a vector space.

4. We collect this in the following:

   Definition: Let V be a vector space and S a nonempty subset of V. Then S is said to be a **(vector) subspace** (of V) if S with the addition and scalar multiplication from V is a vector space.

   Theorem: Let V be a vector space and S a nonempty subset of V, then S is a subspace of V if for all $u, v \in S$ and $r \in \mathbb{F}$ we have

   (S is closed under addition): $u + v \in S$;

   (S is closed under scalar multiplication): $r \cdot u \in S$.

   Notice that this implies that $\mathbf{0} \in S$ by taking $r = 0$ and using that $0 \cdot u = \mathbf{0}$ for all $u \in V$. As S is supposed to be closed under scalar multiplication it follows that $\mathbf{0} \in S$. We can therefore conclude:

   Corollary: Suppose that S is a nonempty subset of V and $\mathbf{0} \notin S$, then S is **not** a vector subspace.

   Notice: This conclusion is only one way. From $\mathbf{0} \in S$ it **does not follows that S is a subspace. To show that a subset is a vector subspace, we have to show that it is closed under addition and scalar multiplication!**

   Notice: We can replace the two conditions $u + v \in S$ and $r \cdot u \in S$ by one condition: For all $u, v \in S$ and all $r, s \in \mathbb{F}$: $ru + sv \in V$.

5. It can now be shown that all the examples for (a)-(f) above are vector spaces.

Let now V and W be two vector spaces. Then we are mainly interested in special kind of maps from V to W. Those are the functions that **preserve the algebraic structure that we have.**
Definition: Let V and W be vector spaces. A map $T : V \to W$ is said to be linear if

$$T(ru + sv) = rT(u) + sT(v)$$

for all $r, s \in \mathbb{F}$ and all $u, v \in V$.
Notice that this one condition can also be split up in two condition: $T(u + v) = T(u) + T(v)$ and $T(ru) = rT(u)$ for all $u, v \in V$ and all $r \in \mathbb{F}$.
Lemma: Let $T : V \to W$ be linear. Then $T(\mathbf{0}_V) = \mathbf{0}_W$ where $\mathbf{0}_V$ is the zero element in V and $\mathbf{0}_W$ is the zero element in W.
Proof: Let $u \in V$ and take $r = 0$. Then

$$\begin{aligned} T(\mathbf{0}_V) &= T(r \cdot u) \qquad \text{(because } 0 \cdot u = \mathbf{0}_V) \\ &= rT(u) \qquad \text{(because } T \text{ is linear)} \\ &= 0_{\mathbf{W}} \ . \end{aligned}$$

Notice again, that this this is only a one way conclusion. $T(\mathbf{0}) = \mathbf{0}$ does not imply that T is linear!
Lemma: Let $T : V \to W$ be linear, then the set

$$S = \{u \in V \mid T(u) = \mathbf{0}\}$$

is a subspace of V. This subspace is denoted by $\mathrm{Ker}(T)$.
Proof: Let $u, v \in \mathrm{Ker}(T)$ and $r, s \in \mathbb{F}$. Then

$$T(ru + sv) = rT(u) + sT(v) = 0 \ .$$

Hence $ru + sv \in \mathrm{Ker}(T)$.
Lemma: Let $T : V \to W$ be linear, then the set

$$S = \{w \in W \mid \exists v \in V : w = T(v)\}$$

is a subspace of W. This space is denoted by $\mathrm{Im}(T)$.
Proof: Let $w, z \in \mathrm{Im}(T)$ and $r, s \in \mathbb{F}$. To show that $rw + sz \in \mathrm{Im}(T)$ we need to find a vector $a \in V$ such that $T(a) = rw + sz$. The only thing we know for sure is, that by definition there are vectors $u, v \in V$ such that $T(u) = w$ and $T(v) = z$. Let $a = ru + sv \in V$. Then

$$\begin{aligned} T(a) &= T(ru + sv) \\ &= rT(u) + sT(v) \\ &= rw + sz \ . \end{aligned}$$

Lemma: Let V, W be vector spaces, let $S, T : V \to W$ be linear maps and let $r, s \in \mathbb{F}$. Then the map $rR + sS : V \to W$, $u \mapsto rR(u) + sS(u)$, is linear.
Proof: Let $a, b \in \mathbb{F}$ and $u, v \in V$. Then the following holds:

$$\begin{aligned} (rR + sS)(au + bv) &= rR(au + bv) + sS(au + bv) \\ &= arR(u) + brR(v) + asS(u) + bsS(v) \qquad (R \text{ and } S \text{ are linear}) \\ &= a(rR + sS)(u) + b(rR + sS)(v) \ . \end{aligned}$$

Remark: What we have in fact shown is that the space of linear maps from V to W is a vector space!
Let us now take few examples:

1. $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ (both viewed as row vectors). Let $A = [a_{ij}]$ be a $n \times m$ matrix and define a map $T : V \to W$ by

   $$T([x_1, \ldots, x_n]) = [x_1, \ldots, x_n]A \ .$$

   Then T is linear. This follows from the rules of matrix multiplication: $[r\mathbf{x} + s\mathbf{y}]A = r(\mathbf{x}A) + s(\mathbf{y}A)$.

2. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then there exists a matrix A such that $T(\mathbf{x}) = \mathbf{x}A$. To find A we let $e_1 = [1, 0, \ldots, 0]$, $e_2 = [0, 1, 0, \ldots, 0]$,..., $e_n = [0, \ldots, 0, 1]$. Let

   $$\mathbf{a}_j = T(e_j).$$

   :Let

   $$A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_n \end{bmatrix} \ .$$

   We leave it out as an exercise to show that $T(\mathbf{x}) = \mathbf{x}A$.

5[28P]) Determine if the each of the following sets is a vector space or not, and **state why:**

a) The space of polynomials of degree ≤ 5, i.e., $V = \left\{\sum_{j=0}^{5} a_j x^j \mid \forall j \ : \ a_j \in \mathbb{R}\right\}$; **Answer:** This is a vector space.

Solution: As this is a subset of the **vector space** of all functions on the real line, we only have to show that V is closed under addition and scalar multiplication.

Closed under addition: Let $p(x) = \sum_{j=0}^{5} a_j x^j$ and $q(x) = \sum_{j=0}^{5} b_j x^j$ be elements in V. Then

$$\begin{aligned}(p+q)(x) &= \sum_{j=0}^{5} a_j x^j + \sum_{j=0}^{5} b_j x^j \\ &= \sum_{j=0}^{5} (a_j + b_j) x^j \in V\end{aligned}$$

Closed under scalar multiplication: Let $r \in \mathbb{R}$, then

$$\begin{aligned}(rp)(x) &= r \sum_{j=0}^{5} a_j x^j \\ &= \sum_{j=0}^{5} (r a_j) x^j \in V \ .\end{aligned}$$

b) $V = \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + x^2 - y + 2z = 0\right\}$. **Answer:** This is not a vector space.

Solution: Notice that $(0,0,0) \in V$ because $2 \cdot 0 + 0^2 - 0 + 2 \cdot 0 = 0$. We will now give different ways to show that this is not a vector space.

Solution$_1$: The vector $(2,0,0)$ is in V. But $2 \cdot (2,0,0) = (4,0,0) \notin V$ because

$$2 \cdot 4 - 4^2 = 8 - 16 = -8 \neq 0.$$

Thus V is not closed under scalar multiplication. (This shows how one can use concrete examples to show that a set is not closed under scalar multiplication).

Solution$_2$: V is not closed under scalar multiplication. Let $(x, y, z) \in V$ and $r \in \mathbb{R}$. Then $2x + x^2 - y + 2z = 0$. On the other hand we have $r \cdot (x, y, z) = (rx, ry, rz)$. We test now the condition:

$$\begin{aligned}2(rx) + (rx)^2 - (ry) + 2(rz) &= r(2x + rx^2 - y + 2z) \\ &= r(2x + x^2 - y + 2z) + (r-1)rx^2 \\ &= (r-1)rx^2 \ .\end{aligned}$$

Here I have used that $2x + x^2 - y + 2z = 0$. I also added and substracted rx^2 to get it into the correct form. So we see that the right hand is only zero if $r = 0$, $r = 1$, or $x = 0$. By taking (as above) element in S with $x \neq 0$ and take $r \neq 0, 1$ we se that V is not closed under scalar multiplication.

Solution$_3$: V is not closed under addition. Let $(x, y, z), (r, s, t) \in V$. Then we have to test if $(x + r, y + s, z + t) \in V$. For that we calculate:

$$\begin{aligned}2(x+r) + (x+r)^2 - (y+s) + 2(z+t) &= 2(x+r) + x^2 + r^2 + 2xr - (y+s) + 2(z+t) \\ &= (2x + x^2 - y + 2z) + (2r + r^2 - s + 2t) + 2xr \\ &= 2xr \ .\end{aligned}$$

The right hand side is only zero if $xr = 0$. So we take two elements in V with the first coordinate not equal to zero, i.e., $(2,0,0)$ in both cases.

c) $V = \left\{f \in C([-1,1]) \mid \int_{-1}^{1} f(t)\ dt = 0\right\}$; **Answer:** This is a vector space.

Solution: Let $f, g \in V$ and $r, s \in \mathbb{F}$ then

$$\int_{-1}^{1} rf + sg \; dt = r\int_{-1}^{1} f \; dt + s\int_{-1}^{1} g \; dt = 0 + 0 = 0 \; .$$

Hence $rf = sg \in V$.

d) The space V_3 of all functions on the interval $[0, 1)$ of the form $\sum_{j=0}^{7} a_j\psi_j^3$, with arbitrary real numbers $a_1, \ldots, a_7$. Here $\psi_j^3(t) = \psi(8t - j)$. **Answer:** This is a vector space.

Solution: Let $f = \sum_{j=0}^{7} a_j\psi_j^3$, $g = \sum_{j=0}^{7} b_j\psi_j^3 \in V$ and $r, s \in \mathbb{R}$. Then

$$\begin{aligned}(rf + sg) &= r\sum_{j=0}^{7} a_j\psi_j^3 + s\sum_{j=0}^{7} b_j\psi_j^3 \\ &= \sum_{j=0}^{7}(ra_j + sb_j)\psi_j^3 \in V \; .\end{aligned}$$

e) Let A be a $n \times m$ matrices and $V = \{\mathbf{x} = [x_1, \ldots, x_n] \in \mathbb{R}^n \mid \mathbf{x}A = \mathbf{0}\}$. **Answer:** This is a vector space.

Solution: Let $\mathbf{x}, \mathbf{y} \in V$ and $r, s \in \mathbb{R}$. Then

$$\begin{aligned}(r\mathbf{x} + r\mathbf{y})A &= r(\mathbf{x}A) + s(\mathbf{y}A) \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

f) $V = \{u \in U \mid T(u) = y\}$ where U and W are vector spaces, $T : U \to W$ is linear and $y \in W$, $y \neq 0$. **Anser:** This is not a vector space.

Solution: If T is a linear map, then $T(\mathbf{0}_U) = \mathbf{0}_W$, so $\mathbf{0} \notin V$.

g) The space of functions on the real line $\mathbb{R}$ that are solutions to the differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = 0$, i.e. $V = \{y \in C^\infty(\mathbb{R}) \mid y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0y = 0\}$. **Answer:** This is a vector space.

Solution: Let $f, g \in V$ and $r, s \in \mathbb{R}$. Then we have to show that

$$(rf + sg)^{(n)} + a_{n-1}(rf + sg)^{(n-1)} + \ldots + a_0(rf + sg) = 0$$

But

$$\begin{aligned}(rf + sg)^{(n)} + a_{n-1}(rf + sg)^{(n-1)} + \ldots + a_0(rf + sg) &= rf^{(n)} + a_{n-1}rf^{(n-1)} + \ldots + ra_0f + rg^{(n)} + a_{n-1}sg^{(n-1)} + \ldots + ra_0g \\ &= r\left(f^{(n)} + a_{n-1}f^{(n-1)} + \ldots + a_0f\right) + s\left(g^{(n)} + a_{n-1}g^{(n-1)} + \ldots + a_0g\right) \\ &= 0 \; .\end{aligned}$$

6[24P]) Determine if the following maps are linear or not, **state why:**

a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x, y, z) = (2x + y - z, xy)$. **Answer:** This map is not linear because of the factor xy. (Do the details!)

b) V the space of polynomials of degree ≤ 5 and W the space of polynomials of degree ≤ 4, $T(p)(x) = 2p'(x) + 3p''(x)$. **Answer:** This map is linear because differentiation is linear amd linear combination of linear maps is linear. (You can also show this directly by plugging a linear combination in the definition of T).

c) $T : \mathbb{R}^4 \to \mathbb{R}$; $T(x_1, x_2, x_3, x_4) = 2x_1 + x_2 - 3x_3 + 4x_4$. **Answer**: Linear.

Solution: Write T as

$$T(\mathbf{x}) = \mathbf{x}\begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix}$$

and use that any map of this form is linear (see above or notes from class).
d) Let $V_N = \left\{\sum_{j=0}^{2^N-1} s_j\varphi_j^N \mid \forall j = 0,\ldots,2^N-1 : a_j \in \mathbb{R}\right\}$ and $T : V_N \to V_{N-1}$ given by

$$T(\sum_{j=0}^{2^N-1} s_j\varphi_j^N) = \sum_{j=0}^{2^{N-1}-1} \frac{s_{2j}+s_{2j+1}}{2}\varphi_j^{N-1} .$$

Answer: This map is linear.
Solution: Let $f = \sum_{j=0}^{2^N-1} s_j\varphi_j^N$ and $g = \sum_{j=0}^{2^N-1} t_j\varphi_j^N$ be vectors in V_N. Then for $r, s \in \mathbb{R}$ we get:

$$rf = \sum_{j=0}^{2^N-1} rs_j\varphi_j^N$$

$$sg = \sum_{j=0}^{2^N-1} st_j\varphi_j^N$$

and

$$rf + sg = \sum_{j=0}^{2^N-1} (rs_j + st_j)\varphi_j^N$$

and hence

$$\begin{aligned} T(rf+sg) &= \sum_{j=0}^{2^{N-1}-1} \frac{rs_{2j}+rs_{2j+1}+st_{2j}+st_{2j+1}}{2}\varphi_j^{N-1} \\ &= \sum_{j=0}^{2^{N-1}-1} \left((r\frac{s_{2j}+s_{2j+1}}{2}) + s(\frac{t_{2j}+t_{2j+1}}{2})\right)\varphi_j^{N-1} \\ &= r\sum_{j=0}^{2^{N-1}-1} \frac{s_{2j}+s_{2j+1}}{2}\varphi_j^{N-1} + s\sum_{j=0}^{2^{N-1}-1} \frac{t_{2j}+t_{2j+1}}{2}\varphi_j^{N-1} \\ &= rT(f) + sT(g) \end{aligned}$$

e) $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(x,y,z) = (2x+y-3z, 3x+y+2, x-4y+z)$. **Answer:** Not linear.
Solution: By direct calculation we get

$$T(0,0,0) = (0,2,0) \neq (0,0,0) .$$

f) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $T(f) = f'' + f'\cdot f$. **Answer:** Not linear because of the factor $f'\cdot f$.
7[12P]) In the following problems, evaluate the given linear map T at the given point:
a) $T : \mathbb{R}^3 \to \mathbb{R}^2$, $T(x,y,z) = (2x+3y, -x+4y)$, $(x,y,z) = (2,-1,4)$. **Answer:** $(1,-6)$
b) $T : \mathbb{C}^3 \to \mathbb{C}^3$, $T(z_1,z_2,z_3) = ((1+i)z_1 + 2z_2 - iz_3, z_1 + (1-i)z_2, z_2 - \frac{1}{1+i}z_3)$, $(z_1,z_2,z_3) = (i, 1+i, 2+i)$. **Answer:** $(2+i, 2+i, -\frac{1}{2}+\frac{3}{2}i)$
c) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $T(f) = f'' + 4f$, $f = 2\cos(x) + \sin(x) + e^x$. **Answer:** $6\cos(x) + 3\sin(x) + 5e^x$
d) $T : C([-1,1]) \to \mathbb{R}$, $T(f) = \int_{-1}^1 f(t)\,dt$, $f(t) = t^2 + t + \cos(\pi t)$. **Answer:** $\frac{2}{3}$.