Math 2057-5, Test # 2. Fall 2005 **Name:** ______________________

$$D = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2 .$$

1) Let $f(x,y) = \ln(x^2+y^2)$.

a[10P]) Find the directional derivative of f at the point $P(2,1)$ in the direction of the vector $\mathbf{v} =< 3,4 >$. **Solution:** $4/5$

The directional derivative of f in the direction of an **unit** vector **u** is given by

$$D_{\mathbf{u}}f(x) = \nabla f(\mathbf{x}) \cdot \mathbf{u} .$$

It is **inportant** to note that we must have $|\mathbf{u}| = 1$.
We have $| < 3,4 > | = \sqrt{9+16} = 5$. Hence $\mathbf{u} = \frac{1}{5} < 3,4 >$. The gradient of f at the point $P(2,1)$ is given by:

$$\begin{aligned} \frac{\partial f}{\partial x}(2,1) &= \left.\frac{2x}{x^2+y^2}\right|_{(x,y)=(2,1)} = \frac{4}{5} \\ \frac{\partial f}{\partial y}(2,1) &= \left.\frac{2y}{x^2+y^2}\right|_{(x,y)=(2,1)} = \frac{2}{5} . \end{aligned}$$

Hence

$$\nabla f(2,1) = \frac{2}{5} < 2,1 > .$$

We therefore get that the directional derivative is:

$$< \frac{4}{5}, \frac{2}{5} > \cdot < \frac{3}{5}, \frac{4}{5} >= \frac{2}{25}(6+4) = 4/5 .$$

b[7P]) Find the directional derivative of f at the point $P(2,1)$ in the direction of $Q(1,3)$. **Solution:** 0.

The direction is $< 1,3 > - < 2,1 >=< -1,2 >$ which is perpenticular to the gradient. Hence the directional derivative in this direction is zero.

c[6P]) What is the maximal rate of change of f at the point $P(2,1)$? **Solution:** $2/\sqrt{5}$

This maximal rate of change is always in the direction of the gradient, and the change in that direction is $|\nabla f(\mathbf{x})$. We have $\mathbf{x} = (2,1)$

$$|\nabla f(2,1)| = | < 4/5, 2/5 > | = 2/\sqrt{5} .$$

2[11P]) Find the equation of the tangent plane to the surface $x^2 + 2y^2 + 3z^3 = 21$ at the point $(4,-1,1)$. **Solution:** The equation is: $8x - 4y + 9z = 45$.

The equation of the tangent plane of the surface $F(x,y,z) = c$ at the point $P(a,b,c)$ is given by

$$F_x(a,b,c)(x-a) + F_y(a,b,c)(y-b) + F_z(a,b,c)(z-c) = \nabla F(a,b,c)\cdot < x-a, y-b, z-c >= 0 .$$

We have in this case $F(x,y,z) = x^2 + 2y^2 + 3z^3 = 21$ and hence

$$\begin{aligned} F_x(4,-1,1) &= 2\cdot 4 = 8 \\ F_y(4,-1,1) &= 4\cdot(-1) = -4 \\ F_z(4,-1,1) &= 9\cdot 1 = 9\ ,. \end{aligned}$$

The equation is therefore

$$8(x-4) - 4(y+1) + 9(z-1) = 8x - 4y + 9z - 45 = 0$$

which can be written as $8x - 4y + 9z = 45$.

3[17P]) Find - if any - the local maximum value and local minimum values and saddle point(s) of the function $f(x,y) = 1+xy^2-2x^2-4y^2$. **Solution:**There is no local minimum. The local maximum is 0 at the point $(0,0)$. There are two saddle points, one at $(4,4)$ and the other at $(4,-4)$.

We have to start by solving the equation $\nabla f(x,y) =< 0, 0 >$. We get the two equations:

$$\begin{aligned} f_x(x,y) &= y^2 - 4x = 0 \\ f_y(x,y) &= 2xy - 8y = 2y(x-4) = 0\,. \end{aligned}$$

The first equation shows that $y = \pm 2\sqrt{x}$ and the second equation shows that either $y = 0$ or $x = 4$. If $y = 0$ then $x = 0$. If $x = 4$ then $y = \pm 4$. Hence there are three possilbe points $(0,0), (4,4)$ and $(4,-4)$. To decide if those are local max/min/saddle points we use the second derivative test with

$$D = f_{xx}f_{yy} - (f_{xy})^2 = -8(x-4) - 4y^2 = -4(2x - 8 + y)\,.$$

We get

$$\begin{aligned} D(0,0) &= 32\ \ f_{xx}(0,0) = -4 < 0 \quad \text{local max}\,. \\ D(4,\pm 4) &= -64 \quad \text{saddle point}\,. \end{aligned}$$

There is no local minimum. The question is about the local max/min. For that we have to evaluate the function at $(0,0)$ and get $f(0,0) = 0$ which is the local maximum.

4[12P]) Find the point on the plain $2x - y - 2z = 9$ closest to the point $(1,2,0)$. **Solution:** The point is: $P(3,1,-2)$.

There are more than one ways to solve this problem. We use Lagrange multipliers. The function that we want to minimize is $f(x,y,z) = (x-1)^2 + (y-2)^2 + z^2$ given the constrain $g(x,y,z) = 2x - y - 2z - 9 = 0$. We get the three equations:

$$\begin{aligned} x &= \lambda + 1 \\ y &= -\frac{\lambda}{2} + 2 \\ z &= -\lambda\,. \end{aligned}$$

Thus

$$2x = 2\lambda + 2, \quad y = -\frac{\lambda}{2} + 2 \quad 2z = -2\lambda\,.$$

Inserting this into the equation $2x - y - 2z - 9 = 0$ gives

$$2\lambda + 2 + \frac{\lambda}{2} - 2 + 2\lambda - 9 == \frac{9}{2}\lambda - 9 = 0\,.$$

Thus $\lambda = 2$. Inserting that into the above equations for x, y and z gives $x = 3$, $y = 1$ and $z = -2$.

5[17P]) Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = 10x + 6y + 2z$ subject to the constrain $x^2 + y^2 + z^2 = 35$.
Solution: The maximum value is:70 and the minimum value is: -70

We get the following equations:

$$\begin{aligned} 5 &= \lambda x \quad \text{or} \quad x = 5/\lambda \\ 3 &= \lambda y \quad \text{or} \quad y = 3/\lambda \\ 1 &= \lambda z \quad \text{or} \quad z = 1/\lambda \end{aligned}$$

Inserting this into the equation $x^2 + y^2 + z^2 = 35$ gives

$$\frac{25}{\lambda^2} + \frac{9}{\lambda^2} + \frac{1}{\lambda^2} = \frac{35}{\lambda^2} = 35\,.$$

Thus

$$\lambda = \pm 1\,.$$

The corresponding points are $\pm(5, 3, 1)$. Inserting this into the function gives the answer.

6[21P]) Evaluate the following integrals:

a) $\int_0^2 \int_0^1 \frac{xy}{\sqrt{1+x^2+y^2}}\, dydx = \frac{1}{6}\left(6^{3/2} = 2^{3/2} - 5^{3/2} + 1\right)$

b) $R = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 1\}$, $\displaystyle\iint_R 3x^2y^3 - 5y^4\, dA = 0$.

If we integrate first with respect to y then the first integral is:

$$\begin{aligned} \int_0^1 3x^2y^3 - 5y^4\, dy &= \left.\frac{3}{4}x^2y^4 - y^5\right]_0^1 \\ &= \frac{3}{4}x^2 - 1\,. \end{aligned}$$

The second integral is then

$$\int_0^2 \frac{3}{4}x^2 - 1\, dx = \left.\frac{x^3}{4} - x\right]_0^2 = 0\,.$$

c) $R = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$, $\displaystyle\iint_R xe^{xy}\, dA = e - 2$

It is best to integrate first with respect to y. The first integral is then

$$\int_0^1 xe^{xy}\, dy = e^{xy}|_0^1 = e^x - 1\,.$$

The next integral is

$$\int_0^1 e^x - 1\, dx = e^x - x]_0^2 = e - 1 - 1 = e - 2\,.$$