Math 2057-5, Test # 3. Fall 2005 Name: ______________________

1[38]) Evaluate the following double integrals. You can use rectangular coordinates or polar coordinates. (a) counts for 8 points and the other for 10 points. Make sure to use the best possible order of integration!

a) $\int_1^2 \int_0^1 xy\, dxdy =$ $\underline{3/4}$

$\int_0^1 xy\,dx = \frac{1}{2} y x^2 \Big]_0^1 = \frac{1}{2} y$

$\int_1^2 \frac{1}{2} y\,dy = \frac{1}{4} y^2 \Big]_1^2 = \frac{1}{4}(4-1) = 3/4$

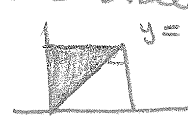
b) $\iint_D \frac{2y}{1+x^2}\, dA =$ $\underline{\frac{1}{2}\ln(2)}$ where $D = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$.

The integral is $\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{1+x^2}\, dydx$.

- $\int_0^{\sqrt{x}} \frac{2y}{1+x^2}\, dy = \frac{y^2}{1+x^2}\Big]_0^{\sqrt{x}} = \frac{x}{1+x^2}$.
- $\int_0^1 \frac{x}{1+x^2}\, dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{\ln(2)}{2}$

c) $\int_0^1 \int_x^1 \sin(y^2)\, dydx =$ $\underline{\frac{1}{2}(1-\cos 1)}$

In this problem it is best to interchange the order of the integration. The domain looks like



Thus

$$\int_0^1 \int_x^1 \sin(y^2)\,dydx = \int_0^1 \int_0^y \sin(y^2)\,dxdy$$

$$= \int_0^1 y \sin(y^2)\,dy$$

$$= -\frac{1}{2}\cos(y^2)\Big]_0^1 = \frac{1-\cos(1)}{2}$$

d) $\iint_R \sqrt{x^2+y^2}\,dA = \frac{26}{3}\pi$ where $R = \{(x,y) \mid 1 \le x^2+y^2 \le 9,\ 0 \le y\}$.

Here we use polar-coordinates $x = r\cos\theta,\ y = r\sin\theta,\ 1 \le r \le 3,\ 0 \le \theta \le \pi$.
Then the integral is

$$\int_0^{\pi}\int_1^3 r^2\,dr\,d\theta = \frac{\pi}{3} r^3\Big]_1^3 = \frac{\pi}{3}[27-1] = \frac{26}{3}\pi$$

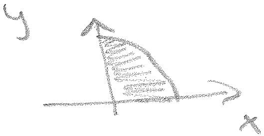
2[48P]) Evaluate the following triple integrals. You can use rectangular coordinates, cylindrical coordinates, or spherical coordinates. Part (a) counts for 8 points, all the other problems are 10 point each.

a) $\int_0^1 \int_0^z \int_0^{x+z} xz\,dy\,dx\,dz = 1/6$

- $\int_0^{x+z} xz\,dy = xzy\Big]_0^{(x+z)} = x^2z + xz^2$.
- $\int_0^z x^2z + xz^2\,dx = \frac{z}{3}x^3 + \frac{z^2}{2}x^2\Big]_0^z = \frac{z^4}{3} + \frac{z^4}{2} = \frac{5}{6}z^4$
- $\frac{5}{6}\int_0^1 z^4\,dz = \frac{z^5}{6}\Big]_0^1 = 1/6$

b) $\iiint_E 2x\,dV = 4$ where $E = \{(x,y,z) \mid 0 \le y \le 2,\ 0 \le x \le \sqrt{4-y^2},\ 0 \le z \le y\}$.

using cylindrical coordinates the integral is:



$$\int_0^{\pi/2}\int_0^2\int_0^{r\sin\theta} 2r^2\cos\theta\,dz\,dr\,d\theta = 2\int_0^{\pi/2}\int_0^2 r^3\cos\theta\sin\theta\,dr\,d\theta$$

$$= 2\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \int_0^2 r^3\,dr = [\sin^2\theta]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^2 = 4$$

c) $\iiint_E dV = 1/3$, where E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,1)$.

$x + \frac{1}{2}y + z = 1$: $y = 2 - 2x - 2z$; $z = 1 - x$:

$$\int_0^1\int_0^{1-x}\int_0^{2-2x-2z} dy\,dz\,dx = 2\int_0^1\int_0^{1-x} 1 - x - z\,dz\,dx = \int_0^1 1 - 2x + x^2\,dx = \left[x - x^2 + \frac{x^3}{3}\right]_0^1 = 1/3$$

$$\left[\ \int_0^{1-x} (1-x) - z\,dz = (1-x)z - \frac{1}{2}z^2\Big]_0^{1-x} = (1-x)^2 - \frac{1}{2}(1-x)^2 = \frac{1}{2}(1-x)^2 = \frac{1}{2}(1-2x+x^2)\ \right]$$

d) $\iiint_E dV = \underline{\ \pi/2\ }$ where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and beneath the paraboloid $z = 1 - x^2 - y^2$.

Use cylindrical coordinates

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta = 2\pi \int_0^1 r - r^3\,dr \qquad \text{(integrating first } d\theta$$

$$= 2\pi\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{\pi}{2}$$

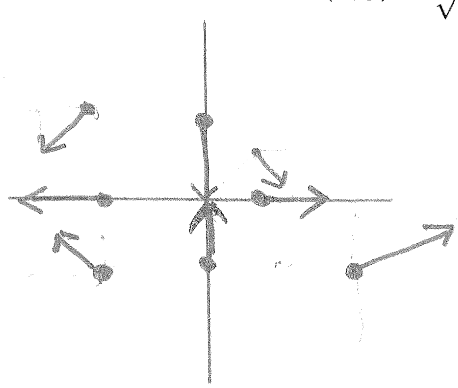
e) $\iiint_E (x^2 + y^2 + z^2)\,dV = \underline{\ 4\pi/5\ }$ where E is the unit ball $x^2 + y^2 + z^2 \le 1$.

use spherical coordinates:

$$\int_0^{2\pi}\int_0^1\int_0^{\pi} r^4 \sin\phi\,d\phi\,dr\,d\theta = [\theta]_0^{2\pi}\left[\frac{r^5}{5}\right]_0^1[-\sin\phi]_0^{\pi}$$

$$= 2\pi \cdot \frac{1}{5} \cdot 2 = \frac{4\pi}{5}$$

3[6P]) Shetch the vector field $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2+y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2+y^2}}\mathbf{j}$.



$F(1,1) = \frac{1}{\sqrt{2}}(1,-1)$

$F(1,0) = (1,0)$

$F(0,-1) = (0,1)$

$F(2,3) = \frac{1}{\sqrt{13}}(4,-9)$

always length 1

4[8P]) Find the gradient vector field of $r = \sqrt{x^2 + y^2 + z^2}$.

$$\frac{\partial}{\partial x} r = \frac{\frac{1}{2} \cdot 2x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$ and similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$

Thus

$$\nabla r = \frac{1}{r}\vec{x} = \frac{1}{\sqrt{x^2+y^2+z^2}}\langle x, y, z\rangle$$