Solutions to Midterm One
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[**1**] Evaluate the following integrals.

(a)

$$\int_0^{\pi/2} \cos x \sin^5 x \, dx$$

Substitute:

$$\begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}$$

Change limits:

x	u
0	$\sin(0) = 0$
$\pi/2$	$\sin(\pi/2) = 1$

The integral becomes

$$\int_0^1 u^5 \, du = \left[\frac{u^6}{6}\right]_0^1 = \frac{1}{6}.$$

(b)

$$\int (\ln x)^2 \, dx$$

Use integration by parts:

$$u = (\ln x)^2 \qquad dv = dx$$
$$du = 2 \ln x \left(\frac{1}{x}\right) \, dx \qquad v = x$$

$$\begin{aligned} \int u \, dv &= uv - \int v \, du \\ &= x(\ln x)^2 - 2 \int \ln x \, dx \end{aligned}$$

Now use integration by parts again with

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x} \, dx \qquad v = x$$

to get

$$\begin{aligned}\int (\ln x)^2\,dx &= x(\ln x)^2 - 2\left(x\ln x - \int dx\right)\\ &= x(\ln x)^2 - 2x\ln x + 2x + C.\end{aligned}$$

[**2**] How many subintervals are required so that the trapezoid rule approximation T_n to the integral

$$\int_1^3 \frac{1}{x^2}\,dx$$

is accurate to within 0.01?

Use the error bound formula

$$Error(T_n) \le \frac{K(b-a)^3}{12n^2}.$$

Here K is an upper bound for $|f''(x)|$ on the interval $[1,3]$. We have

$$\begin{aligned} f(x) &= \frac{1}{x^2} = x^{-2}\\ f'(x) &= -2x^{-3}\\ f''(x) &= 6x^{-4} = \frac{6}{x^4}.\end{aligned}$$

Since $f''(x)$ is positive and decreasing on the interval $[1,3]$, we can take $K = f''(1) = 6$.

We need to find n so that

$$\begin{aligned} \frac{K(b-a)^3}{12n^2} &\le \frac{1}{100}\\ \iff \frac{6(3-1)^3}{12n^2} &\le \frac{1}{100}\\ \iff \frac{4}{n^2} &\le \frac{1}{100}\\ \iff n^2 &\ge 400\\ \iff n &\ge 20.\end{aligned}$$

So it will suffice to take $n = 20$.

[**3**] Evaluate the integral.

$$\begin{aligned}
\int \cos^4 y\,dy &= \int (\cos^2 y)(\cos^2 y)\,dy \\
&= \frac{1}{4}\int (1+\cos(2y))(1+\cos(2y))\,dy \\
&= \frac{1}{4}\int (1+2\cos(2y)+\cos^2(2y)\,dy \\
&= \frac{1}{4}\int (1+2\cos(2y)+\frac{1}{2}(1+\cos(4y))\,dy \\
&= \int \frac{1}{4}+\frac{1}{2}\cos(2y)+\frac{1}{8}+\frac{1}{8}\cos(4y)\,dy \\
&= \frac{y}{4}+\frac{1}{4}\sin(2y)+\frac{y}{8}+\frac{1}{32}\sin(4y)+C.
\end{aligned}$$

[**4**] Find the partial fraction decomposition of

$$\frac{x^2}{(x+1)(x^2+1)}.$$

$$\begin{aligned}
\frac{x^2}{(x+1)(x^2+1)} &= \frac{A}{x+1}+\frac{Bx+C}{x^2+1} \\
\text{Clear denominators:}\quad x^2 &= A(x^2+1)+(Bx+C)(x+1) \\
x=-1:\quad 1 &= 2A \implies A=\frac{1}{2} \\
\text{Plug in:}\quad x^2 &= \frac{1}{2}(x^2+1)+(Bx+C)(x+1) \\
&= \frac{1}{2}x^2+\frac{1}{2}+Bx^2+(B+C)x+C \\
x^2\text{ coeff:}\quad 1 &= \frac{1}{2}+B \implies B=\frac{1}{2} \\
\text{constant coeff:}\quad 0 &= \frac{1}{2}+C \implies C=-\frac{1}{2}.
\end{aligned}$$

So

$$\frac{x^2}{(x+1)(x^2+1)} = \frac{1}{2}\left(\frac{1}{x+1}+\frac{x-1}{x^2+1}\right).$$

[**5**] Evaluate the integral.

$$\int \frac{dx}{x^2\sqrt{x^2-25}}.$$

Use the trig substitution:

$$\begin{aligned}
x &= 5\sec\theta \\
dx &= 5\sec\theta\tan\theta\,d\theta \\
\sqrt{x^2-25} &= 5\tan\theta
\end{aligned}$$

Then

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2-25}} &= \int \frac{5\sec\theta\tan\theta\,d\theta}{25\sec^2\theta\,5\tan\theta} \\ &= \int \frac{d\theta}{5\sec\theta} \\ &= \frac{1}{5}\int \cos\theta\,d\theta = \frac{1}{5}\sin\theta + C = \frac{1}{5}\frac{\sqrt{x^2-25}}{x} + C,\end{aligned}$$

where the expression for $\sin\theta$ in terms of x is obtained from a right triangle with adjacent side 5 and hypoteneuse x (since $\cos\theta = \frac{1}{\sec\theta} = \frac{5}{x}$.)