Solutions to Sample Midterm One
Brendan Owens, 1552

[1]
(a) Write down, but do not simplify, the expression for the trapezoid approximation T_4 for the integral $\int_0^1 e^x\,dx$.

We have $a = 0$, $b = 1$, $f(x) = e^x$, $n = 4$. This gives $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$, and so

$$T_4 = \frac{1}{8}\left(e^0 + 2e^{\frac{1}{4}} + 2e^{\frac{1}{2}} + 2e^{\frac{3}{4}} + e^1\right).$$

(b)

How large do we have to choose n so that the approximation T_n to the integral in part (a) is accurate to within 0.00001?

Use the error bound formula

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Here K is an upper bound for $|f''(x)|$ on the interval $[0,1]$. Since $f''(x) = e^x$ we see that the maximum value of $|f''(x)|$ on the interval $[0,1]$ is e. We can work with $K = 3$ for convenience. We need to find n so that

$$\begin{aligned} \frac{K(b-a)^3}{12n^2} &\leq \frac{1}{100000} \\ \iff \frac{3}{12n^2} &\leq \frac{1}{100000} \\ \iff n^2 &\geq 25000 \\ \iff n &\geq 50\sqrt{10} \end{aligned}$$

Since $3 < \sqrt{10} < 4$, it will suffice to take $n = (50)(4) = 200$.

[2] Use integration by parts to find

$$\int e^x \cos x\,dx$$

$$\begin{aligned} u &= e^x & dv &= \cos x\,dx \\ du &= e^x dx & v &= \sin x \end{aligned}$$

$$\begin{aligned} \int u\,dv &= uv - \int v\,du \\ &= e^x \sin x - \int e^x \sin x\,dx \end{aligned}$$

Now use integration by parts again with

$$\begin{array}{ll} u = e^x & dv = \sin x\,dx \\ du = e^x dx & v = -\cos x \end{array}$$

to get

$$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x - \int e^x \cos x\,dx,$$

which yields

$$\begin{aligned} 2\int e^x \cos x\,dx &= e^x \sin x + e^x \cos x + C \\ \Longrightarrow \int e^x \cos x\,dx &= (e^x \sin x + e^x \cos x + C)/2. \end{aligned}$$

[**3**] Evaluate the following integrals.

(a)

$$\int_{-\pi/2}^{\pi/2} \sin^4 x \cos^3 x\,dx = \int_{-\pi/2}^{\pi/2} \sin^4 x(1 - \sin^2 x)\cos x\,dx$$

Substitute:

$$\begin{aligned} u &= \sin x \\ du &= \cos x\,dx \end{aligned}$$

Change limits:

x	u
$-\pi/2$	$\sin(-\pi/2) = -1$
$\pi/2$	$\sin(\pi/2) = 1$

The integral becomes

$$\begin{aligned} \int_{-1}^{1} u^4(1 - u^2)\,du &= \int_{-1}^{1} u^4 - u^6\,du \\ &= \left[\frac{u^5}{5} - \frac{u^7}{7}\right]_{-1}^{1} \\ &= \frac{2}{5} - \frac{2}{7} = \frac{4}{35}. \end{aligned}$$

(b)

$$\begin{aligned} \int \sec x\,dx &= \int \sec x \frac{\sec x + \tan x}{\sec x + \tan x}\,dx \\ &= \int \frac{\sec^2 x + \sec x \tan x\,dx}{\sec x + \tan x}. \end{aligned}$$

Substitute $u = \sec x + \tan x$, $du = (\sec^2 x + \sec x \tan x)\, dx$, to find

$$\begin{aligned} \int \sec x \, dx &= \int \frac{du}{u} \\ &= \ln |u| + C \\ &= \ln |\sec x + \tan x| + C. \end{aligned}$$

[**4**] Evaluate the integral.

$$\int_2^3 \frac{1}{x^2 - 1}\, dx$$

Find the partial fraction decomposition:

$$\begin{aligned} \frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} &= \frac{A}{x-1} + \frac{B}{x+1} \\ \text{Clear denominators:} \quad 1 &= A(x+1) + b(x-1) \\ x = 1: \quad 1 &= 2A \implies A = 1/2 \\ x = -1: \quad 1 &= -2B \implies B = -1/2 \end{aligned}$$

So

$$\begin{aligned} \int_2^3 \frac{1}{x^2-1}\, dx &= \frac{1}{2} \int_2^3 \frac{1}{x-1} - \frac{1}{x+1}\, dx \\ &= \frac{1}{2} \left[\ln|x-1| - \ln|x+1| \right]_2^3 \\ &= \frac{1}{2}(\ln 2 - \ln 1 - \ln 4 + \ln 3) = \frac{1}{2} \ln \left(\frac{3}{2} \right). \end{aligned}$$