Diagrammatic and geometric approaches to Schur-Weyl duality

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Polynomial representations of $\mathsf{GL}_n(\mathbb{C})$

$\mathsf{Rep}_{\mathrm{pol}}(\mathsf{GL}_n(\mathbb{C})) \subset \mathsf{Rep}(\mathsf{GL}_n(\mathbb{C}))$ cat of polynomial representations of $\mathsf{GL}_n(\mathbb{C})$

$$\{\text{poly irreps}\} \leftrightarrow \{\text{Young diagrams with} \leq n \text{ rows}\}$$

If V is an irrep, then $\lambda \mathrm{Id} \in Z(\mathsf{GL}_n(\mathbb{C})) \cong \mathbb{C}$ acts on V by λ^d for some $d \in \mathbb{Z}$ called the degree of V.
V is a poly irrep iff the degree is nonnegative.

$$\left\{\begin{matrix}\text{poly irreps}\\ \text{of degree } d\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Young diagrams with} \leq n \text{ rows}\\ \text{and } d \text{ boxes}\end{matrix}\right\}$$

We let $\mathsf{Rep}_{\mathrm{pol}}{}^d(\mathsf{GL}_n(\mathbb{C}))$ denote the subcategory generated by the irreps of degree d with $d \geq 0$.

Schur-Weyl duality

Irreducible representations of S_d are parametrized by Young diagrams with d boxes. Thus, if $n \geq d$,

$$\left\{\begin{matrix}\text{poly irreps}\\ \text{of degree } d\end{matrix}\right\} \longleftrightarrow \left\{\text{irreps of } S_d\right\}$$

More generally, there is a combinatorial identifcation between poly irreps of degree d and the irreps of S_d corresponding to partitions with at most n parts.

Explanation: The d-fold tensor product $\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n$ is endowed with actions of $\mathrm{GL}_n(\mathbb{C})$ and S_d which are full mutual centralizers of each other. This implies the decomposition

$$\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n = \sum_D V_D \otimes W_D,$$

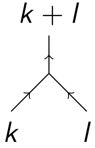
where the sum runs over Young diagrams with d boxes and $\leq n$ boxes and V_D (resp. W_D) is the irrep of S_d (resp. $\mathrm{GL}_n(\mathbb{C})$) corresponding to D.

The free spider category

$\mathbb{k}$ a field (or commutative ring)

Definition (Cautis, Kamnitzer, Morrison 2013)

The free spider category $\mathcal{FS}p(n, \mathbb{k})$ is the monoidal category with
Objects: finite sequences with $\mathbf{a} = (a_1, \ldots, a_s)$, $a_i \in \{1, \ldots, n\}$
Morphisms: $\mathbb{k}$-linear combinations of webs, trivalent graphs built out of the basic morphisms



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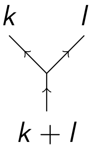

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Composition is given by vertical concatenation.
The monoidal structure is given by horizontal juxtaposition.

A presentation of $\mathrm{Rep}_{\mathrm{pol}}(\mathrm{GL}_n(\mathbb{C}))$ I

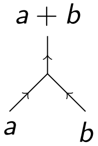
$\mathrm{Rep}_{\Lambda}(\mathrm{GL}_n(\Bbbk)) \subset \mathrm{Rep}_{\mathrm{pol}}(\mathrm{GL}_n(\Bbbk))$ full subcat, objects iso isomorphic to tensor prods of the fund reps $\Lambda^a(\Bbbk^n)$ for $a \in \{1, \dots, n\}$.

Distinguished morphisms: $S = \{j_1, \dots, j_s\}, e_S = e_{j_1} \wedge \cdots \wedge e_{j_s}$

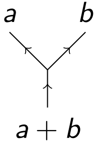
$$\Lambda^a \Bbbk^n \otimes \Lambda^b \Bbbk^n \to \Lambda^{a+b} \Bbbk^n \qquad e_S \otimes e_T \mapsto e_S \wedge e_T$$

$$\Lambda^{a+b} \Bbbk^n \to \Lambda^a \Bbbk^n \otimes \Lambda^b \Bbbk^n \qquad e_S \mapsto (-1)^{ab} \sum_{T \subset S} (-1)^{\ell(S \setminus T, T)} e_T \otimes e_{S \setminus T}$$

There is a natural functor $\Psi : \mathcal{FS}p(n, \Bbbk) \to \mathrm{Rep}_{\Lambda}(\mathrm{GL}_n(\Bbbk))$ with $\mathbf{a} \mapsto \Lambda^{\mathbf{a}}(\Bbbk^n) := \Lambda^{a_1}(\Bbbk^n) \otimes \cdots \otimes \Lambda^{a_s}(\Bbbk^n)$,



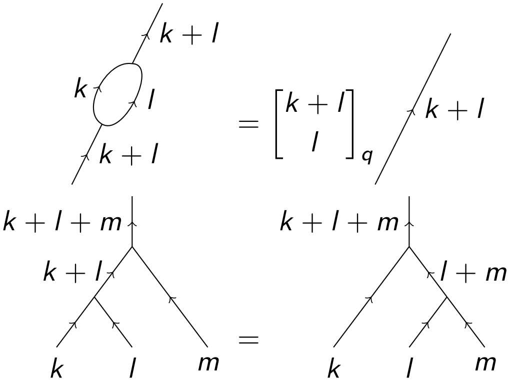
$$\mapsto \qquad \Lambda^a \Bbbk^n \otimes \Lambda^b \Bbbk^n \to \Lambda^{a+b} \Bbbk^n$$



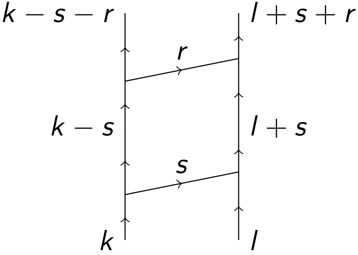
$$\mapsto \qquad \Lambda^{a+b} \Bbbk^n \to \Lambda^a \Bbbk^n \otimes \Lambda^b \Bbbk^n.$$

The spider category

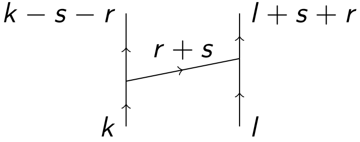
The spider category $\mathcal{S}p(n, \mathbb{k})$ is the quotient category of $\mathcal{FS}p(n, \mathbb{k})$ obtained by imposing the following relations and their mirror images on the webs:

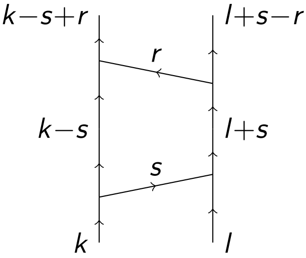


The spider category (cont.)

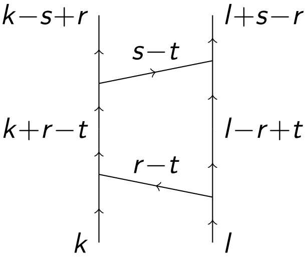


$$= \binom{r+s}{r}$$





$$= \sum_t \binom{k-l+r-s}{t}$$



A presentation of $\mathrm{Rep}_{\mathrm{pol}}(\mathsf{GL}_n(\mathbb{C}))$ II

For $\Bbbk = \mathbb{C}$, all polynomial irreps are direct summands of objects in $\mathrm{Rep}_{\Lambda}(\mathrm{GL}_n(\mathbb{C}))$, so $\mathrm{Rep}_{\mathrm{pol}}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{Kar}(\mathrm{Rep}_{\Lambda}(\mathrm{GL}_n(\mathbb{C})))$.

Theorem (CKM 2013)

1. *The functor Ψ induces a monoidal equivalence $\mathcal{S}p(n, \mathbb{C}) \to \mathrm{Rep}_{\Lambda}(\mathrm{GL}_n(\mathbb{C}))$.*
2. $\mathrm{Rep}_{\mathrm{pol}}(\mathrm{GL}_n(\mathbb{C})) \cong \mathrm{Kar}(\mathcal{S}p(n, \mathbb{C}))$.

The proof is quite indirect and uses skew-Howe duality, i.e., the commuting actions of $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_m(\mathbb{C})$ on $\Lambda(\mathbb{C}^n \otimes \mathbb{C}^m)$.

The infinite spider category

We have a diagrammatic presentation of $\mathrm{Rep}(\mathrm{GL}_n(\mathbb{C}))$. Now want a presentation for reps of symmetric groups.

Definition (Achar-S. 2019)

The free infinite spider category $\mathcal{FS}p(\infty, \Bbbk)$ is the category with objects finite sequences of natural numbers, morphisms as in the finite spider category but with no upper bound on the labels.
The infinite spider category $\mathcal{S}p(\infty, \Bbbk)$ is the quotient by similar relations to the finite case.
There are natural "truncation" functors $T^n : \mathcal{S}p(\infty, \Bbbk) \to \mathcal{S}p(n, \Bbbk)$ obtained by quotienting out objects and morphisms involving labels bigger than n.

Young modules

$S_{\mathbf{a}} = S_{a_1} \times \cdots \times S_{a_s}$
A Young module is an indecomposable summand of some permutation module $\mathrm{Ind}_{S_{\mathbf{a}}}^{S_{\sum a_i}} \mathbb{k}$.

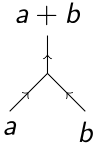
- $\mathcal{Y}(S_d, \mathbb{k})$ additive category generated by the Young modules
- $\mathcal{Y}_{\mathrm{tr}}(S_d, \mathbb{k})$ full subcategory with objects $\mathrm{Ind}_{S_{\mathbf{a}}}^{S_{\sum a_i}} \mathbb{k}$
- $\mathcal{Y}(\mathbb{k}) = \bigoplus_{d \geq 1} \mathcal{Y}(S_d, \mathbb{k})$
- $\mathcal{Y}_{\mathrm{tr}}(\mathbb{k}) = \bigoplus_{d \geq 1} \mathcal{Y}_{\mathrm{tr}}(S_d, \mathbb{k})$

Note $\mathcal{Y}(\mathbb{C}) = \bigoplus_{d \geq 1} \mathrm{Rep}(S_d, \mathbb{C})$.
Monoidal structure on $\mathcal{Y}_{\mathrm{tr}}(\mathbb{k})$ and $\mathcal{Y}(\mathbb{k})$: Given V and W appropriate reps of S_a and S_b, then $V * W = \mathrm{Ind}_{S_a \times S_b}^{S_{a+b}} (V \boxtimes W)$.

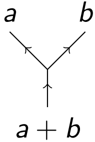
A presentation of the category of Young modules

There is a natural functor $\Theta : \mathcal{FS}p(\infty, \Bbbk) \to \mathcal{Y}_{\mathrm{tr}}(\Bbbk)$:

$$\mathbf{a} \mapsto \mathrm{Ind}_{S_{\mathbf{a}}}^{S_{\sum a_i}} \Bbbk$$



$$\mapsto \qquad \mathrm{Ind}_{S_a \times S_b}^{S_{a+b}} \Bbbk \to \mathrm{Ind}_{S_{a+b}}^{S_{a+b}} \Bbbk = \Bbbk$$



$$\mapsto \qquad \Bbbk = \mathrm{Ind}_{S_{a+b}}^{S_{a+b}} \Bbbk \to \mathrm{Ind}_{S_a \times S_b}^{S_{a+b}} \Bbbk$$

The distinguished morphisms are the counit and unit respectively of the $(\mathrm{Ind}, \mathrm{Res})$ and $(\mathrm{Res}, \mathrm{Ind})$ adjoint pairs.

A presentation of the category of Young modules (cont.)

Theorem (Achar-S. 2019)

1. *The functor* Θ *induces a monoidal equivalence* $\mathcal{S}p(\infty, \Bbbk) \to \mathcal{Y}_{\mathrm{tr}}(\Bbbk)$.
2. $\mathcal{Y}(\Bbbk)$ *is monoidally equivalent to* $\mathsf{Kar}(\mathcal{S}p(\infty, \Bbbk))$.

Key ingredient of the proof: To show that the functor is fully faithful, one needs to show that the dimensions of corresponding Hom-spaces are the same. We accomplish this by finding an explicit $\mathbb{Z}$-basis for the Hom-spaces in $\mathcal{S}p(\infty, \mathbb{Z})$.

A presentation of the category of tilting modules for $\mathrm{GL}_n(\Bbbk)$

What about a representation-theoretic interpretation of the finite spider categories for general $\Bbbk$?
The indecomposable summands of the objects of $\mathrm{Rep}_\Lambda(\mathrm{GL}_n(\Bbbk))$ are the indecomposable tilting modules., so
$\mathrm{Tilt}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{Kar}(\mathrm{Rep}_\Lambda(\mathrm{GL}_n(\Bbbk)))$.
Similar arguments to those in the infinite case give

Theorem (Achar-S. 2019)

1. *The functor Ψ induces a monoidal equivalence* $\mathcal{S}p(n, \Bbbk) \to \mathrm{Rep}_\Lambda(\mathrm{GL}_n(\Bbbk))$.
2. $\mathrm{Tilt}(\mathrm{GL}_n(\mathbb{C})) \cong \mathrm{Kar}(\mathcal{S}p(n, \Bbbk))$.

A version of modular Schur-Weyl duality

$$\{\text{indecomp tilting mods for } \mathrm{GL}_n(\Bbbk)\} \leftrightarrow \{\text{Young diagrams with} \leq n \text{ rows}\}$$

$$\{\text{Young modules for } S_d\} \leftrightarrow \{\text{Young diagrams with } d \text{ boxes}\}$$

Let $\mathrm{Tilt}^d(\mathrm{GL}_n(\Bbbk))$ (resp. $\mathcal{Y}^n(S_d, \Bbbk)$) be the full subcategories generated by the tilting modules (resp. Young modules) corresponding to partitions of d with at most n parts.

The truncation functor induces a functor from Young modules to tilting modules:

$\Phi^m : \mathcal{Y}(\Bbbk) \cong \mathrm{Kar}(\mathcal{S}p(\infty, \Bbbk))) \stackrel{T^m}{\to} \mathrm{Kar}(\mathcal{S}p(m, \Bbbk)) \cong \mathrm{Tilt}(\mathrm{GL}_m(\Bbbk))$

Theorem (Achar-S. 2019)

The functor Φ^m *induces an equivalence*
$\mathcal{Y}^m(S_d, \Bbbk) \cong \mathrm{Tilt}^d(\mathrm{GL}_m(\Bbbk))$

Over $\mathbb{C}$, this is the usual Schur-Weyl duality.

A nilpotent cone interpretation of Schur-Weyl duality

Let $\mathcal{N}_d$ denote the set of $d \times d$ nilpotent matrices.

By Jordan canonical form, each $\mathrm{GL}_d(\mathbb{C})$-orbit corresponds to a partition of d with kth part the number of $k \times k$ Jordan blocks.

Let $\mathcal{N}_d^n = \{x \in \mathfrak{gl}_d(\mathbb{C}) \mid x^n = 0\}$, the matrices with degree of nilpotency $\leq n$.

The simple objects in $\mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d^n)$ are parameterized by the $\mathrm{GL}_d(\mathbb{C})$ orbits:

$$\left\{ \mathrm{GL}_d(\mathbb{C})\text{-orbits in } \mathcal{N}_d^n \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Young diagrams with } d \text{ boxes} \\ \text{and} \leq n \text{ columns} \end{array} \right\}$$

Two tensor categories of perverse sheaves

- $\mathcal{P} = \bigoplus_{d \geq 1} \mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d)$
- $\mathcal{P}^n = \bigoplus_{d \geq 1} \mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d^n)$

The simple objects of the second category are parameterized by Young diagrams with at most n columns.
$\mathcal{P}$ is a tensor category via parabolic induction:
if $\mathcal{F}$ and $\mathcal{G}$ are perverse sheaves on $\mathcal{N}_a$ and $\mathcal{N}_b$, $\mathcal{F} * \mathcal{G}$ is induced from $\mathcal{F} \boxtimes \mathcal{G}$ via the block-diagonal embedding
$\mathcal{N}_a \times \mathcal{N}_b \hookrightarrow \mathcal{N}_{a+b} \subset \mathfrak{gl}_{a+b}(\mathbb{C})$.
Let $\hat{T}^n : \mathcal{P} \to \mathcal{P}^n$ be the truncation functor obtained by discarding perverse sheaves corresponding to orbits with $> n$ columns.

Theorem (Achar-S. 2017)

There exists a monoidal structure on $\mathcal{P}^n$ such that $\hat{T}^n : \mathcal{P} \to \mathcal{P}^n$ is monoidal.

A geometric version of Schur-Weyl duality

Let $\mathcal{P}_{\mathrm{sky}}$ be the subcategory of $\mathcal{P}$ consisting of sheaves isomorphic to those obtained by parabolic induction of the skyscraper sheaf at $0 \in \mathcal{N}_{a_1} \times \cdots \times \mathcal{N}_{a_s}$ for any $\mathbf{a}$.
There is a functor $\mathcal{FS}p(\infty, \Bbbk) \to \mathcal{P}(\Bbbk)$ with essential image $\mathcal{P}_{\mathrm{sky}}$.

Theorem (Achar-S. 2019)

1. *This functor induces an equivalence* $\mathcal{S}p(\infty, \mathbb{C}) \to \mathcal{P}_{\mathrm{sky}}$*, and hence a tensor equivalence* $\mathrm{Kar}(\mathcal{S}p(\infty, \mathbb{C})) \cong \mathcal{P}$.
2. *There is a monoidal equivalence* $\mathrm{Kar}(\mathcal{S}p(m, \mathbb{C})) \to \mathcal{P}^m$ *compatible with truncation of the equivalence of (1).*

Remark: We conjecture that the first equivalence holds for general $\Bbbk$ if perverse sheaves are replaced by parity sheaves.

A geometric version of Schur-Weyl duality (cont.)

Let $\mathrm{Perv}^n_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d, \mathbb{C})$ be the subcategory corresponding to diagrams with with at most n columns.

Theorem (Achar-S. 2019)

The truncation functor $\hat{T}^n$ induces an equivalence of categories $\mathrm{Perv}^n_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d, \mathbb{C}) \cong \mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}^n_d, \mathbb{C})$.

This is a geometric interpretation of Schur-Weyl duality.

Construction of product on $\mathcal{P}^n$

We first introduce some notation and some maps.
Fix d, d', and let $P_{d,d'} \subset \mathrm{GL}_{d+d'}(\mathbb{C})$ be the block-upper triangular parabolic subgroup with diagonal blocks of sizes d and d'.
Let $\mathfrak{p}_{d,d'} = \mathrm{Lie}(P_{d,d'})$.

$$Y_{d,d'} = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \mid x \in \mathcal{N}_d^n, y \in \mathcal{N}_{d'}^n, z \in \mathcal{M}_{d,d'}(\mathbb{C}) \right\} \subset \mathfrak{p}_{d,d'} \cap \mathcal{N}_{d+d'}$$

We have the following maps (all defined in the obvious way):

$$\begin{tikzcd}
Y_{d,d'} \arrow[r, hook] \arrow[d, "\pi"] & \mathrm{GL}_{d+d'}(\mathbb{C}) \times^{P_{d,d'}} Y_{d,d'} \arrow[r, "\mu"] & \mathcal{N}_{d+d'} \\
\mathcal{N}_d \times \mathcal{N}_{d'} & &
\end{tikzcd}$$

Construction of product on $\mathcal{P}^n$ (cont.)

The product $\circledast$ is now defined as the following composition:

$$
\begin{aligned}
\mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})}(\mathcal{N}_d^n) \times \mathrm{Perv}_{\mathrm{GL}_{d'}(\mathbb{C})}(\mathcal{N}_{d'}^n) &\overset{\boxtimes}{\to} \mathrm{Perv}_{\mathrm{GL}_d(\mathbb{C})\times\mathrm{GL}_{d'}(\mathbb{C})}(\mathcal{N}_d^n \times \mathcal{N}_{d'}^n) \\
&\overset{\pi^*}{\to} \mathrm{Perv}_P(Y) \\
&\overset{\simeq}{\to} \mathrm{Perv}_{\mathrm{GL}_{d+d'}(\mathbb{C})}(\mathrm{GL}_{d+d'}(\mathbb{C}) \times^P Y) \\
&\overset{\mu_*}{\to} \mathrm{Perv}_{\mathrm{GL}_{d+d'}(\mathbb{C})}(\mathcal{N}_{d+d'}) \\
&\overset{\text{discard}}{\to} \mathrm{Perv}_{\mathrm{GL}_{d+d'}(\mathbb{C})}(\mathcal{N}_{d+d'}^n).
\end{aligned}
$$

The last step throws away the simple objects corresponding to Young diagrams with more than n columns.