Theorem 2.10 *Under the assumptions of Theorem 2.9 for each $f \in X$ the successive approximations*

$$\varphi_{n+1} := A\varphi_n + f, \quad n = 0, 1, 2, \dots, \tag{2.3}$$

with arbitrary $\varphi_0 \in X$ converge to the unique solution φ of $\varphi - A\varphi = f$.

Proof. By induction it can be seen that

$$\varphi_n = A^n\varphi_0 + \sum_{k=0}^{n-1} A^k f, \quad n = 1, 2, \dots,$$

whence

$$\lim_{n\to\infty} \varphi_n = \sum_{k=0}^{\infty} A^k f = (I - A)^{-1} f$$

follows. □

We explicitly want to state this result for integral equations.

Corollary 2.11 *Let K be a continuous kernel satisfying*

$$\max_{x\in G} \int_G |K(x,y)|\, dy < 1.$$

Then for each $f \in C(G)$ the integral equation of the second kind

$$\varphi(x) - \int_G K(x,y)\varphi(y)\, dy = f(x), \quad x \in G,$$

has a unique solution $\varphi \in C(G)$. The successive approximations

$$\varphi_{n+1}(x) := \int_G K(x,y)\varphi_n(y)\, dy + f(x), \quad n = 0, 1, 2, \dots,$$

with arbitrary $\varphi_0 \in C(G)$ converge uniformly to this solution.

The method of successive approximations has two drawbacks. First, the Neumann series ensures existence of solutions to integral equations of the second kind only for sufficiently small kernels, and second, in general, it cannot be summed in closed form. Later in the book we will have more to say about using successive approximations to obtain approximate solutions (see Section 10.5).

2.4 Compact Operators

To provide the tools for establishing the existence of solutions to a wider class of integral equations we now turn to the introduction and investigation of compact operators.

Definition 2.12 *A linear operator $A : X \to Y$ from a normed space X into a normed space Y is called* compact *if it maps each bounded set in X into a relatively compact set in Y.*

= precompact

Since by Definition 1.16 and Theorem 1.15 a subset U of a normed space Y is relatively compact if each sequence in U contains a subsequence that converges in Y, we have the following equivalent condition for an operator to be compact.

Theorem 2.13 *A linear operator $A : X \to Y$ is compact if and only if for each bounded sequence (φ_n) in X the sequence $(A\varphi_n)$ contains a convergent subsequence in Y.*

We proceed by establishing the basic properties of compact operators.

Theorem 2.14 *Compact linear operators are bounded.*

Proof. This is obvious, since relatively compact sets are bounded (see Theorem 1.15). □

Theorem 2.15 *Linear combinations of compact linear operators are compact.*

Proof. Let $A, B : X \to Y$ be compact linear operators and let $\alpha, \beta \in \mathbb{C}$. Then for each bounded sequence (φ_n) in X, since A and B are compact, we can select a subsequence $(\varphi_{n(k)})$ such that both sequences $(A\varphi_{n(k)})$ and $(B\varphi_{n(k)})$ converge. Hence $(\alpha A + \beta B)\varphi_{n(k)}$ converges, and therefore $\alpha A + \beta B$ is compact. □

Theorem 2.16 *Let $X, Y,$ and Z be normed spaces and let $A : X \to Y$ and $B : Y \to Z$ be bounded linear operators. Then the product $BA : X \to Z$ is compact if one of the two operators A or B is compact.*

Proof. Let (φ_n) be a bounded sequence in X. If A is compact, then there exists a subsequence $(\varphi_{n(k)})$ such that $A\varphi_{n(k)} \to \psi \in Y$, $k \to \infty$. Since B is bounded and therefore continuous, we have $B(A\varphi_{n(k)}) \to B\psi \in Z$, $k \to \infty$. Hence BA is compact. If A is bounded and B is compact, the sequence $(A\varphi_n)$ is bounded in Y, since bounded operators map bounded sets into bounded sets. Therefore, there exists a subsequence $(\varphi_{n(k)})$ such that $(BA)\varphi_{n(k)} = B(A\varphi_{n(k)}) \to \chi \in Z$, $k \to \infty$. Hence, again BA is compact. □

Theorem 2.17 *Let X be a normed space and Y be a Banach space. Let the sequence $A_n : X \to Y$ of compact linear operators be norm convergent to a linear operator $A : X \to Y$, i.e., $\|A_n - A\| \to 0$, $n \to \infty$. Then A is compact.*

Proof. Let (φ_m) be a bounded sequence in X, i.e., $\|\varphi_m\| \le C$ for all $m \in \mathbb{N}$ and some $C > 0$. Because the A_n are compact, by the standard diagonalization procedure (see the proof of Theorem 1.18), we can choose a subsequence $(\varphi_{m(k)})$ such that $(A_n\varphi_{m(k)})$ converges for every fixed n as $k \to \infty$. Given $\varepsilon > 0$, since $\|A_n - A\| \to 0$, $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\|A_{n_0} - A\| < \varepsilon/3C$. Because $(A_{n_0}\varphi_{m(k)})$ converges, there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\|A_{n_0}\varphi_{m(k)} - A_{n_0}\varphi_{m(l)}\| < \frac{\varepsilon}{3}$$

for all $k, l \ge N(\varepsilon)$. But then we have

$$\|A\varphi_{m(k)} - A\varphi_{m(l)}\| \le \|A\varphi_{m(k)} - A_{n_0}\varphi_{m(k)}\| + \|A_{n_0}\varphi_{m(k)} - A_{n_0}\varphi_{m(l)}\|$$

$$+\|A_{n_0}\varphi_{m(l)} - A\varphi_{m(l)}\| < \varepsilon.$$

Thus $(A\varphi_{m(k)})$ is a Cauchy sequence, and therefore it is convergent in the Banach space Y. □

Theorem 2.18 *Let $A : X \to Y$ be a bounded linear operator with finite-dimensional range $A(X)$. Then A is compact.*

Proof. Let $U \subset X$ be bounded. Then the bounded operator A maps U into the bounded set $A(U)$ contained in the finite-dimensional space $A(X)$. By the Bolzano–Weierstrass Theorem 1.17 the set $A(U)$ is relatively compact. Therefore A is compact. □

Lemma 2.19 (Riesz) *Let X be a normed space, $U \subset X$ a closed subspace with $U \ne X$, and $\alpha \in (0,1)$. Then there exists an element $\psi \in X$ with $\|\psi\| = 1$ such that $\|\psi - \varphi\| \ge \alpha$ for all $\varphi \in U$.*

Proof. Because $U \ne X$, there exists an element $f \in X$ with $f \notin U$, and because U is closed, we have

$$\beta := \inf_{\varphi \in U} \|f - \varphi\| > 0.$$

We can choose $g \in U$ such that

$$\beta \le \|f - g\| \le \frac{\beta}{\alpha}.$$

Now we define

$$\psi := \frac{f - g}{\|f - g\|}.$$

Then $\|\psi\| = 1$, and for all $\varphi \in U$ we have

$$\|\psi - \varphi\| = \frac{1}{\|f - g\|}\,\|f - \{g + \|f - g\|\,\varphi\}\| \ge \frac{\beta}{\|f - g\|} \ge \alpha,$$

since $g + \|f - g\|\,\varphi \in U$. □

Theorem 2.20 *The identity operator $I : X \to X$ is compact if and only if X has finite dimension.*

Proof. Assume that I is compact and X is not finite-dimensional. Choose an arbitrary $\varphi_1 \in X$ with $\|\varphi_1\| = 1$. Then $U_1 := \operatorname{span}\{\varphi_1\}$ is a finite-dimensional and therefore closed subspace of X. By Lemma 2.19 there exists $\varphi_2 \in X$ with $\|\varphi_2\| = 1$ and $\|\varphi_2 - \varphi_1\| \ge 1/2$. Now consider $U_2 := \operatorname{span}\{\varphi_1, \varphi_2\}$. Again by Lemma 2.19 there exists $\varphi_3 \in X$ with $\|\varphi_3\| = 1$ and $\|\varphi_3 - \varphi_1\| \ge 1/2$, $\|\varphi_3 - \varphi_2\| \ge 1/2$. Repeating this procedure, we obtain a sequence (φ_n) with the properties $\|\varphi_n\| = 1$ and $\|\varphi_n - \varphi_m\| \ge 1/2$, $n \ne m$. This implies that the bounded sequence (φ_n) does not contain a convergent subsequence. Hence we have a contradiction to the compactness of I. Therefore, if the identity operator is compact, X has finite dimension. The converse statement is an immediate consequence of Theorem 2.18. □

This theorem, in particular, implies that the converse of Theorem 2.14 is false. It also justifies the distinction between operator equations of the first and second kind, because obviously for a compact operator A the operators A and $I - A$ have different properties. Note that by Theorems 2.16 and 2.20 the compact operator A cannot have a bounded inverse unless its range has finite dimension.

Theorem 2.21 *The integral operator with continuous kernel is a compact operator on $C(G)$.*

Proof. Let $U \subset C(G)$ be bounded, i.e., $\|\varphi\|_\infty \le C$ for all $\varphi \in U$ and some $C > 0$. Then

$$|(A\varphi)(x)| \le C|G| \max_{x,y \in G} |K(x,y)|$$

for all $x \in G$ and all $\varphi \in U$, i.e., $A(U)$ is bounded. Since K is uniformly continuous on the compact set $G \times G$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|K(x,z) - K(y,z)| < \frac{\varepsilon}{C|G|}$$

for all $x, y, z \in G$ with $|x - y| < \delta$. Then

$$|(A\varphi)(x) - (A\varphi)(y)| < \varepsilon$$

for all $x, y \in G$ with $|x - y| < \delta$ and all $\varphi \in U$, i.e., $A(U)$ is equicontinuous. Hence A is compact by the Arzelà–Ascoli Theorem 1.18. □

We wish to mention that the compactness of the integral operator with continuous kernel also can be established by finite-dimensional approximations using Theorems 2.17 and 2.18 in the Banach space $C(G)$. In this context note that the proofs of Theorems 2.17 and 1.18 are similar in structure. The finite-dimensional operators can be obtained by approximating either the continuous kernel by polynomials through the Weierstrass approximation theorem or the integral through a finite sum (see [24]).

Now we extend our investigation to integral operators with a *weakly singular kernel*, i.e., the kernel K is defined and continuous for all $x, y \in G \subset \mathbb{R}^m$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m]$ such that

$$|K(x,y)| \leq M|x-y|^{\alpha-m}, \quad x, y \in G,\ x \neq y. \tag{2.4}$$

Theorem 2.22 *The integral operator with a weakly singular kernel is a compact operator on $C(G)$.*

Proof. The integral in (2.2) defining the operator A exists as an improper integral, since

$$|K(x,y)\varphi(y)| \leq M\|\varphi\|_\infty |x-y|^{\alpha-m}$$

and

$$\int_G |x-y|^{\alpha-m}\,dy \leq \omega_m \int_0^d \rho^{\alpha-m}\rho^{m-1}\,d\rho = \frac{\omega_m}{\alpha}\,d^\alpha,$$

where we have introduced polar coordinates with origin at x, d is the diameter of G, and ω_m denotes the surface area of the unit sphere in $\mathbb{R}^m$.

Now we choose a piecewise linear continuous function $h : [0,\infty) \to \mathbb{R}$ by setting

$$h(t) := \begin{cases} 0, & 0 \leq t \leq 1/2, \\ 2t-1, & 1/2 \leq t \leq 1, \\ 1, & 1 \leq t < \infty, \end{cases}$$

and for $n \in \mathbb{N}$ we define continuous kernels $K_n : G \times G \to \mathbb{C}$ by

$$K_n(x,y) := \begin{cases} h(n|x-y|)K(x,y), & x \neq y, \\ 0, & x = y. \end{cases}$$

The corresponding integral operators $A_n : C(G) \to C(G)$ are compact by Theorem 2.21. We have the estimate

$$|(A\varphi)(x) - (A_n\varphi)(x)| = \left| \int_{G\cap B[x;1/n]} \{1 - h(n|x-y)\}K(x,y)\varphi(y)\,dy \right|$$

$$\leq M\|\varphi\|_\infty \omega_m \int_0^{1/n} \rho^{\alpha-m}\rho^{m-1}\,d\rho$$

$$= M\|\varphi\|_\infty \frac{\omega_m}{\alpha n^\alpha}, \quad x \in G.$$

From this we observe that $A_n\varphi \to A\varphi$, $n \to \infty$, uniformly, and therefore $A\varphi \in C(G)$. Furthermore it follows that

$$\|A - A_n\|_\infty \leq M\,\frac{\omega_m}{\alpha n^\alpha} \to 0, \quad n \to \infty,$$

and thus A is compact by Theorem 2.17. □

Finally, we want to expand the analysis to integral operators defined on surfaces in $\mathbb{R}^m$. Having in mind applications to boundary value problems, we will confine our attention to surfaces that are boundaries of smooth domains in $\mathbb{R}^m$. A bounded open domain $D \subset \mathbb{R}^m$ with boundary ∂D is said to be of *class* C^n, $n \in \mathbb{N}$, if the closure $\bar{D}$ admits a finite open covering

$$\bar{D} \subset \bigcup_{q=1}^{p} V_q$$

such that for each of those V_q that intersect with the boundary ∂D we have the properties: The intersection $V_q \cap \bar{D}$ can be mapped bijectively onto the half-ball $H := \{x \in \mathbb{R}^m : |x| < 1,\ x_m \geq 0\}$ in $\mathbb{R}^m$, this mapping and its inverse are n times continuously differentiable, and the intersection $V_q \cap \partial D$ is mapped onto the disk $H \cap \{x \in \mathbb{R}^m : x_m = 0\}$.

In particular, this implies that the boundary ∂D can be represented locally by a *parametric representation*

$$x(u) = (x_1(u), \dots, x_m(u))$$

mapping an open parameter domain $U \subset \mathbb{R}^{m-1}$ bijectively onto a *surface patch* S of ∂D with the property that the vectors

$$\frac{\partial x}{\partial u_i}, \quad i = 1, \dots, m-1,$$

are linearly independent at each point x of S. Such a parameterization we call a *regular parametric representation*. The whole boundary ∂D is obtained by matching a finite number of such surface patches.

On occasion, we will express the property of a domain D to be of class C^n also by saying that its boundary ∂D is of class C^n.

The vectors $\partial x/\partial u_i$, $i = 1, \dots, m-1$, span the tangent plane to the surface at the point x. The unit *normal* ν is the unit vector orthogonal to the tangent plane. It is uniquely determined up to two opposite directions. The surface element at the point x is given by

$$ds = \sqrt{g}\,du_1 \cdots du_{m-1},$$

where g is the determinant of the positive definite matrix with entries

$$g_{ij} := \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j}, \quad i, j = 1, \dots, m-1.$$

In this book, for two vectors $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_m)$ in $\mathbb{R}^m$ (or $\mathbb{C}^m$) we denote by $a \cdot b = a_1 b_1 + \cdots + a_m b_m$ the *dot product.*

Assume that ∂D is the boundary of a bounded open domain of class C^1. In the Banach space $C(\partial D)$ of real- or complex-valued continuous functions defined on the surface ∂D and equipped with the maximum norm

$$\|\varphi\|_\infty := \max_{x \in \partial D} |\varphi(x)|,$$

we consider the integral operator $A : C(\partial D) \to C(\partial D)$ defined by

$$(A\varphi)(x) := \int_{\partial D} K(x,y)\varphi(y)\, ds(y), \quad x \in \partial D, \tag{2.5}$$

where K is a continuous or weakly singular kernel. According to the dimension of the surface ∂D, a kernel K is said to be *weakly singular* if it is defined and continuous for all $x, y \in \partial D$, $x \neq y$, and there exist positive constants M and $\alpha \in (0, m-1]$ such that

$$|K(x,y)| \le M|x-y|^{\alpha-m+1}, \quad x, y \in \partial D,\ x \neq y. \tag{2.6}$$

Analogously to Theorems 2.21 and 2.22 we can prove the following theorem.

Theorem 2.23 *The integral operator with continuous or weakly singular kernel is a compact operator on $C(\partial D)$ if ∂D is of class C^1.*

Proof. For continuous kernels the proof of Theorem 2.21 essentially remains unaltered. For weakly singular kernels the only major difference in the proof compared with the proof of Theorem 2.22 arises in the verification of the existence of the integral in (2.5). Since the surface ∂D is of class C^1, the normal vector ν is continuous on ∂D. Therefore, we can choose $R \in (0, 1]$ such that

$$\nu(x) \cdot \nu(y) \ge \frac{1}{2} \tag{2.7}$$

for all $x, y \in \partial D$ with $|x - y| \le R$. Furthermore, we can assume that R is small enough such that the set $S[x; R] := \{y \in \partial D : |y - x| \le R\}$ is connected for each $x \in \partial D$. Then the condition (2.7) implies that $S[x; R]$ can be projected bijectively onto the tangent plane to ∂D at the point x. By using polar coordinates in the tangent plane with origin in x, we now can estimate

$$\begin{aligned}\left| \int_{S[x;R]} K(x,y)\varphi(y)\, ds(y) \right| &\le M\|\varphi\|_\infty \int_{S[x;R]} |x-y|^{\alpha-m+1}\, ds(y)\\ &\le 2M\|\varphi\|_\infty \omega_{m-1} \int_0^R \rho^{\alpha-m+1}\rho^{m-2}\, d\rho\\ &= 2M\|\varphi\|_\infty \omega_{m-1} \frac{R^\alpha}{\alpha}.\end{aligned}$$

Here we have used the facts that $|x - y| \ge \rho$, that the surface element

$$ds(y) = \frac{\rho^{m-2} d\rho d\omega}{\nu(x) \cdot \nu(y)},$$

expressed in polar coordinates on the tangent plane, can be estimated with the aid of (2.7) by $ds(y) \le 2\rho^{m-2} d\rho d\omega$, and that the projection of $S[x; R]$ onto the tangent plane is contained in the interior of the disk of radius R and center x. Furthermore, we have

$$\begin{aligned}\left| \int_{\partial D \setminus S[x;R]} K(x,y)\varphi(y)\, ds(y) \right| &\le M\|\varphi\|_\infty \int_{\partial D \setminus S[x;R]} R^{\alpha-m+1}\, ds(y)\\ &\le M\|\varphi\|_\infty R^{\alpha-m+1} |\partial D|.\end{aligned}$$

Hence, for all $x \in \partial D$ the integral (2.5) exists as an improper integral. For the compactness of A, we now can adopt the proof of Theorem 2.22.

Problems

2.1 Let $A : X \to Y$ be a bounded linear operator from a normed space X into a normed space Y and let $\widetilde{X}$ and $\widetilde{Y}$ be the completions of X and Y, respectively. Then there exists a uniquely determined bounded linear operator $\widetilde{A} : \widetilde{X} \to \widetilde{Y}$ such that $\widetilde{A}\varphi = A\varphi$ for all $\varphi \in X$. Furthermore, $\|\widetilde{A}\| = \|A\|$. The operator A is called the *continuous extension* of A. (In the sense of Theorem 1.13 the space X is interpreted as a dense subspace of its completion $\widetilde{X}$.)
Hint: For $\varphi \in \widetilde{X}$ define $\widetilde{A}\varphi = \lim_{n\to\infty} A\varphi_n$, where (φ_n) is a sequence from X with $\varphi_n \to \varphi$, $n \to \infty$.

2.2 Show that Theorem 2.10 remains valid for operators satisfying $\|A^k\| < 1$ for some $k \in \mathbb{N}$.

2.3 Write the proofs for the compactness of the integral operator with continuous kernel in $C(G)$ using finite-dimensional approximations as mentioned after the proof of Theorem 2.21.

2.4 Show that the result of Theorem 2.8 for the norm of the integral operator remains valid for weakly singular kernels.
Hint: Use the approximations from the proof of Theorem 2.22.

2.5 For the integral operator A with continuous kernel use the Cauchy–Schwarz inequality to establish that each set $U \subset C(G)$ that is bounded with respect to the mean square norm is mapped into a set $A(U) \subset C(G)$ that is bounded with respect to the maximum norm and equicontinuous. From this, deduce that the integral operator with continuous kernel is compact with respect to the mean square norm. Use the same method and the approximations from the proof of Theorem 2.22 to extend this result to weakly singular kernels with $\alpha > m/2$.

3
Riesz Theory

We now present the basic theory for an operator equation

$$\varphi - A\varphi = f$$

of the second kind with a compact linear operator $A : X \to X$ on a normed space X. This theory was developed by Riesz [153] and initiated through Fredholm's [42] work on integral equations of the second kind.

3.1 Riesz Theory for Compact Operators

We define

$$L := I - A,$$

where I denotes the identity operator.

Theorem 3.1 (First Riesz Theorem) *The nullspace of the operator L, i.e.,*

$$N(L) := \{\varphi \in X : L\varphi = 0\},$$

is a finite-dimensional subspace.

Proof. The nullspace of the bounded linear operator L is a closed subspace of X, since for each sequence (φ_n) with $\varphi_n \to \varphi$, $n \to \infty$, and $L\varphi_n = 0$ we have that $L\varphi = 0$. Each $\varphi \in N(L)$ satisfies $A\varphi = \varphi$, and therefore the restriction of A to $N(L)$ coincides with the identity operator on $N(L)$. The operator A is compact on X and therefore also compact from $N(L)$ onto

$N(L)$, since $N(L)$ is closed. Hence $N(L)$ is finite-dimensional by Theorem 2.20. □

Theorem 3.2 (Second Riesz Theorem) *The range of the operator L, i.e.,*

$$L(X) := \{L\varphi : \varphi \in X\},$$

is a closed linear subspace.

Proof. The range of the linear operator L is a linear subspace. Let f be an element of the closure $\overline{L(X)}$. Then there exists a sequence (φ_n) in X such that $L\varphi_n \to f$, $n \to \infty$. By Theorem 1.24 to each φ_n we choose a best approximation χ_n with respect to $N(L)$, i.e.,

$$\|\varphi_n - \chi_n\| = \inf_{\chi \in N(L)} \|\varphi_n - \chi\|.$$

The sequence defined by

$$\widetilde{\varphi}_n := \varphi_n - \chi_n, \quad n \in \mathbb{N},$$

is bounded. We prove this indirectly, i.e., we assume that the sequence $(\widetilde{\varphi}_n)$ is not bounded. Then there exists a subsequence $(\widetilde{\varphi}_{n(k)})$ such that $\|\widetilde{\varphi}_{n(k)}\| \geq k$ for all $k \in \mathbb{N}$. Now we define

$$\psi_k := \frac{\widetilde{\varphi}_{n(k)}}{\|\widetilde{\varphi}_{n(k)}\|}, \quad k \in \mathbb{N}.$$

Since $\|\psi_k\| = 1$ and A is compact, there exists a subsequence $(\psi_{k(j)})$ such that

$$A\psi_{k(j)} \to \psi \in X, \quad j \to \infty.$$

Furthermore,

$$\|L\psi_k\| = \frac{\|L\widetilde{\varphi}_{n(k)}\|}{\|\widetilde{\varphi}_{n(k)}\|} \leq \frac{\|L\widetilde{\varphi}_{n(k)}\|}{k} \to 0, \quad k \to \infty,$$

since the sequence $(L\varphi_n)$ is convergent and therefore bounded. Hence

$$L\psi_{k(j)} \to 0, \quad j \to \infty.$$

Now we obtain

$$\psi_{k(j)} = L\psi_{k(j)} + A\psi_{k(j)} \to \psi, \quad j \to \infty,$$

and since L is bounded, from the two previous equations we conclude that $L\psi = 0$. But then, because $\chi_{n(k)} + \|\widetilde{\varphi}_{n(k)}\|\,\psi \in N(L)$ for all k in $\mathbb{N}$, we find

$$\|\psi_k - \psi\| = \frac{1}{\|\widetilde{\varphi}_{n(k)}\|}\,\|\varphi_{n(k)} - \{\chi_{n(k)} + \|\widetilde{\varphi}_{n(k)}\|\,\psi\}\|$$

$$\geq \frac{1}{\|\widetilde{\varphi}_{n(k)}\|}\,\inf_{\chi \in N(L)} \|\varphi_{n(k)} - \chi\| = \frac{1}{\|\widetilde{\varphi}_{n(k)}\|}\,\|\varphi_{n(k)} - \chi_{n(k)}\| = 1.$$

This contradicts the fact that $\psi_{k(j)} \to \psi$, $j \to \infty$.

Therefore the sequence $(\widetilde{\varphi}_n)$ is bounded, and since A is compact, we can select a subsequence $(\widetilde{\varphi}_{n(k)})$ such that $(A\widetilde{\varphi}_{n(k)})$ converges as $k \to \infty$. In view of $L\widetilde{\varphi}_{n(k)} \to f$, $k \to \infty$, from $\widetilde{\varphi}_{n(k)} = L\widetilde{\varphi}_{n(k)} + A\widetilde{\varphi}_{n(k)}$ we observe that $\widetilde{\varphi}_{n(k)} \to \varphi \in X$, $k \to \infty$. But then $L\widetilde{\varphi}_{n(k)} \to L\varphi \in X$, $k \to \infty$, and therefore $f = L\varphi \in L(X)$. Hence $\overline{L(X)} = L(X)$, and the proof is complete. □

For $n \geq 1$ the iterated operators L^n can be written in the form

$$L^n = (I - A)^n = I - A_n,$$

where

$$A_n = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} A^k$$

is compact by Theorems 2.15 and 2.16. Therefore by Theorem 3.1 the nullspaces $N(L^n)$ are finite-dimensional subspaces, and by Theorem 3.2 the ranges $L^n(X)$ are closed subspaces.

Theorem 3.3 (Third Riesz Theorem) *There exists a uniquely determined nonnegative integer r, called the* Riesz number *of the operator A, such that*

$$\{0\} = N(L^0) \subsetneqq N(L^1) \subsetneqq \cdots \subsetneqq N(L^r) = N(L^{r+1}) = \cdots, \tag{3.1}$$

and

$$X = L^0(X) \supsetneqq L^1(X) \supsetneqq \cdots \supsetneqq L^r(X) = L^{r+1}(X) = \cdots. \tag{3.2}$$

Furthermore, we have the direct sum

$$X = N(L^r) \oplus L^r(X),$$

i.e., for each $\varphi \in X$ there exist uniquely determined elements $\psi \in N(L^r)$ and $\chi \in L^r(X)$ such that $\varphi = \psi + \chi$.

Proof. Our proof consists of four steps:

1. Because each φ with $L^n\varphi = 0$ satisfies $L^{n+1}\varphi = 0$, we have

$$\{0\} = N(L^0) \subset N(L^1) \subset N(L^2) \subset \cdots.$$

Now assume that

$$\{0\} = N(L^0) \subsetneqq N(L^1) \subsetneqq N(L^2) \subsetneqq \cdots.$$

Since by Theorem 3.1 the nullspace $N(L^n)$ is finite-dimensional, the Riesz Lemma 2.19 implies that for each $n \in \mathbb{N}$ there exists $\varphi_n \in N(L^{n+1})$ such that $\|\varphi_n\| = 1$ and

$$\|\varphi_n - \varphi\| \geq \frac{1}{2}$$

for all $\varphi \in N(L^n)$. For $n > m$ we consider

$$A\varphi_n - A\varphi_m = \varphi_n - (\varphi_m + L\varphi_n - L\varphi_m).$$

Then $\varphi_m + L\varphi_n - L\varphi_m \in N(L^n)$, because

$$L^n(\varphi_m + L\varphi_n - L\varphi_m) = L^{n-m-1}L^{m+1}\varphi_m + L^{n+1}\varphi_n - L^{n-m}L^{m+1}\varphi_m = 0.$$

Hence

$$\|A\varphi_n - A\varphi_m\| \geq \frac{1}{2}$$

for $n > m$, and thus the sequence $(A\varphi_n)$ does not contain a convergent subsequence. This is a contradiction to the compactness of A.

Therefore in the sequence $N(L^n)$ there exist two consecutive nullspaces that are equal. Define

$$r := \min\{k : N(L^k) = N(L^{k+1})\}.$$

Now we prove by induction that

$$N(L^r) = N(L^{r+1}) = N(L^{r+2}) = \cdots.$$

Assume that we have proven $N(L^k) = N(L^{k+1})$ for some $k \geq r$. Then for each $\varphi \in N(L^{k+2})$ we have $L^{k+1}L\varphi = L^{k+2}\varphi = 0$. This implies that $L\varphi \in N(L^{k+1}) = N(L^k)$. Hence $L^{k+1}\varphi = L^kL\varphi = 0$, and consequently $\varphi \in N(L^{k+1})$. Therefore, $N(L^{k+2}) \subset N(L^{k+1})$, and we have established that

$$\{0\} = N(L^0) \subsetneqq N(L^1) \subsetneqq \cdots \subsetneqq N(L^r) = N(L^{r+1}) = \cdots.$$

2. Because for each $\psi = L^{n+1}\varphi \in L^{n+1}(X)$ we can write $\psi = L^nL\varphi$, we have

$$X = L^0(X) \supset L^1(X) \supset L^2(X) \supset \cdots.$$

Now assume that

$$X = L^0(X) \supsetneqq L^1(X) \supsetneqq L^2(X) \supsetneqq \cdots.$$

Since by Theorem 3.2 the range $L^n(X)$ is a closed subspace, the Riesz Lemma 2.19 implies that for each $n \in \mathbb{N}$ there exist $\psi_n \in L^n(X)$ such that $\|\psi_n\| = 1$ and

$$\|\psi_n - \psi\| \geq \frac{1}{2}$$

for all $\psi \in L^{n+1}(X)$. We write $\psi_n = L^n\varphi_n$ and for $m > n$ we consider

$$A\psi_n - A\psi_m = \psi_n - (\psi_m + L\psi_n - L\psi_m).$$

Then $\psi_m + L\psi_n - L\psi_m \in L^{n+1}(X)$, because

$$\psi_m + L\psi_n - L\psi_m = L^{n+1}(L^{m-n-1}\varphi_m + \varphi_n - L^{m-n}\varphi_m).$$

Hence

$$\|A\psi_n - A\psi_m\| \geq \frac{1}{2}$$

for $m > n$, and we can derive the same contradiction as before.

Therefore in the sequence $L^n(X)$ there exist two consecutive ranges that are equal. Define

$$q := \min\{k : L^k(X) = L^{k+1}(X)\}.$$

Now we prove by induction that

$$L^q(X) = L^{q+1}(X) = L^{q+2}(X) = \cdots.$$

Assume that we have proven $L^k(X) = L^{k+1}(X)$ for some $k \geq q$. Then for each $\psi = L^{k+1}\varphi \in L^{k+1}(X)$ we can write $L^k\varphi = L^{k+1}\widetilde{\varphi}$ for some $\widetilde{\varphi} \in X$, because $L^k(X) = L^{k+1}(X)$. Hence $\psi = L^{k+2}\widetilde{\varphi} \in L^{k+2}(X)$, and therefore $L^{k+1}(X) \subset L^{k+2}(X)$, i.e., we have proven that

$$X = L^0(X) \underset{\neq}{\supset} L^1(X) \underset{\neq}{\supset} \cdots \underset{\neq}{\supset} L^q(X) = L^{q+1}(X) = \cdots.$$

3. Now we show that $r = q$. Assume that $r > q$ and let $\varphi \in N(L^r)$. Then, because $L^{r-1}\varphi \in L^{r-1}(X) = L^r(X)$, we can write $L^{r-1}\varphi = L^r\widetilde{\varphi}$ for some $\widetilde{\varphi} \in X$. Since $L^{r+1}\widetilde{\varphi} = L^r\varphi = 0$, we have $\widetilde{\varphi} \in N(L^{r+1}) = N(L^r)$, i.e., $L^{r-1}\varphi = L^r\widetilde{\varphi} = 0$. Thus $\varphi \in N(L^{r-1})$, and hence $N(L^{r-1}) = N(L^r)$. This contradicts the definition of r.

On the other hand, assume that $r < q$ and let $\psi = L^{q-1}\varphi \in L^{q-1}(X)$. Because $L\psi = L^q\varphi \in L^q(X) = L^{q+1}(X)$, we can write $L\psi = L^{q+1}\widetilde{\varphi}$ for some $\widetilde{\varphi} \in X$. Therefore $L^q(\varphi - L\widetilde{\varphi}) = L\psi - L^{q+1}\widetilde{\varphi} = 0$, and from this we conclude that $L^{q-1}(\varphi - L\widetilde{\varphi}) = 0$, because $N(L^{q-1}) = N(L^q)$. Hence $\psi = L^q\widetilde{\varphi} \in L^q(X)$, and consequently $L^{q-1}(X) = L^q(X)$. This contradicts the definition of q.

4. Let $\psi \in N(L^r) \cap L^r(X)$. Then $\psi = L^r\varphi$ for some $\varphi \in X$ and $L^r\psi = 0$. Therefore $L^{2r}\varphi = 0$, whence $\varphi \in N(L^{2r}) = N(L^r)$ follows. This implies $\psi = L^r\varphi = 0$.

Let $\varphi \in X$ be arbitrary. Then $L^r\varphi \in L^r(X) = L^{2r}(X)$ and we can write $L^r\varphi = L^{2r}\widetilde{\varphi}$ for some $\widetilde{\varphi} \in X$. Now define $\psi := L^r\widetilde{\varphi} \in L^r(X)$ and $\chi := \varphi - \psi$. Then $L^r\chi = L^r\varphi - L^{2r}\widetilde{\varphi} = 0$, i.e., $\chi \in N(L^r)$. Therefore the decomposition $\varphi = \chi + \psi$ proves the direct sum $X = N(L^r) \oplus L^r(X)$. □

We are now ready to derive the following fundamental result of the Riesz theory.

Theorem 3.4 *Let $A : X \to X$ be a compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective (and therefore also bijective), then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.*

Proof. By (3.1) injectivity of $I - A$ is equivalent to $r = 0$, and by (3.2) surjectivity of $I - A$ is also equivalent to $r = 0$. Therefore injectivity of $I - A$ and surjectivity of $I - A$ are equivalent.

It remains to show that L^{-1} is bounded when $L = I - A$ is injective. Assume that L^{-1} is not bounded. Then there exists a sequence (f_n) in X with $\|f_n\| = 1$ such that $\|L^{-1}f_n\| \geq n$ for all $n \in \mathbb{N}$. Define

$$g_n := \frac{f_n}{\|L^{-1}f_n\|}, \quad \varphi_n := \frac{L^{-1}f_n}{\|L^{-1}f_n\|}, \quad n \in \mathbb{N}.$$

Then $g_n \to 0$, $n \to \infty$, and $\|\varphi_n\| = 1$ for all n. Since A is compact, we can select a subsequence $(\varphi_{n(k)})$ such that $A\varphi_{n(k)} \to \varphi \in X$, $k \to \infty$. Then, since

$$\varphi_n - A\varphi_n = g_n,$$

we observe that $\varphi_{n(k)} \to \varphi$, $k \to \infty$, and $\varphi \in N(L)$. Hence $\varphi = 0$, and this contradicts $\|\varphi_n\| = 1$ for all $n \in \mathbb{N}$. □

We can rewrite Theorems 3.1 and 3.4 in terms of the solvability of an operator equation of the second kind as follows.

Corollary 3.5 *Let $A : X \to X$ be a compact linear operator on a normed space X. If the homogeneous equation*

$$\varphi - A\varphi = 0 \tag{3.3}$$

only has the trivial solution $\varphi = 0$, then for each $f \in X$ the inhomogeneous equation

$$\varphi - A\varphi = f \tag{3.4}$$

has a unique solution $\varphi \in X$ and this solution depends continuously on f.

If the homogeneous equation (3.3) has a nontrivial solution, then it has only a finite number $m \in \mathbb{N}$ of linearly independent solutions $\varphi_1, \ldots, \varphi_m$ and the inhomogeneous equation (3.4) is either unsolvable or its general solution is of the form

$$\varphi = \widetilde{\varphi} + \sum_{k=1}^{m} \alpha_k \varphi_k,$$

where $\alpha_1, \ldots, \alpha_m$ are arbitrary complex numbers and $\widetilde{\varphi}$ denotes a particular solution of the inhomogeneous equation.

The main importance of the Riesz theory for compact operators lies in the fact that it reduces the problem of establishing the existence of a solution to (3.4) to the generally much simpler problem of showing that (3.3) has only the trivial solution $\varphi = 0$.

It is left to the reader to formulate Theorem 3.4 and its Corollary 3.5 for integral equations of the second kind with continuous or weakly singular kernels.

Corollary 3.6 *Theorem 3.4 and its Corollary 3.5 remain valid when $I-A$ is replaced by $S-A$, where $S : X \to Y$ is a bounded linear operator that has a bounded inverse $S^{-1} : Y \to X$ and $A : X \to Y$ is a compact linear operator from a normed space X into a normed space Y.*

Proof. This follows immediately from the fact that we can transform the equation

$$S\varphi - A\varphi = f$$

into the equivalent form

$$\varphi - S^{-1}A\varphi = S^{-1}f,$$

where $S^{-1}A : X \to X$ is compact by Theorem 2.16. □

The decomposition $X = N(L^r) \oplus L^r(X)$ of Theorem 3.3 generates an operator $P : X \to N(L^r)$ that maps $\varphi \in X$ onto $P\varphi := \psi$ defined by the unique decomposition $\varphi = \psi + \chi$ with $\psi \in N(L^r)$ and $\chi \in L^r(X)$. This operator is called a *projection operator*, because it satisfies $P^2 = P$ (see Chapter 13). We conclude this section with the following lemma on this projection operator.

Lemma 3.7 *The projection operator $P : X \to N(L^r)$ defined by the decomposition $X = N(L^r) \oplus L^r(X)$ is compact.*

Proof. Assume that P is not bounded. Then there exists a sequence (φ_n) in X with $\|\varphi_n\| = 1$ such that $\|P\varphi_n\| \geq n$ for all $n \in \mathbb{N}$. Define

$$\psi_n := \frac{\varphi_n}{\|P\varphi_n\|}, \quad n \in \mathbb{N}.$$

Then $\psi_n \to 0$, $n \to \infty$, and $\|P\psi_n\| = 1$ for all $n \in \mathbb{N}$. Since $N(L^r)$ is finite-dimensional and $(P\psi_n)$ is bounded, by Theorem 1.17 there exists a subsequence $(\psi_{n(k)})$ such that $P\psi_{n(k)} \to \chi \in N(L^r)$, $k \to \infty$. Because $\psi_n \to 0$, $n \to \infty$, we also have $P\psi_{n(k)} - \psi_{n(k)} \to \chi$, $k \to \infty$. This implies that $\chi \in L^r(X)$, since $P\psi_{n(k)} - \psi_{n(k)} \in L^r(X)$ for all k and $L^r(X)$ is closed. Hence $\chi \in N(L^r) \cap L^r(X)$, and therefore $\chi = 0$, i.e., $P\psi_{n(k)} \to 0$, $k \to \infty$. This contradicts $\|P\psi_n\| = 1$ for all $n \in \mathbb{N}$. Hence P is bounded, and because P has finite-dimensional range $P(X) = N(L^r)$, by Theorem 2.18 it is compact. □

3.2 Spectral Theory for Compact Operators

We continue by formulating the results of the Riesz theory in terms of *spectral analysis*.

Definition 3.8 *Let $A : X \to X$ be a bounded linear operator on a normed space X. A complex number λ is called an* eigenvalue *of A if there exists an element $\varphi \in X$, $\varphi \neq 0$, such that $A\varphi = \lambda\varphi$. The element φ is called an* eigenelement *of A. A complex number λ is called a* regular value *of A if $(\lambda I - A)^{-1} : X \to X$ exists and is bounded. The set of all regular values of A is called the* resolvent set *$\rho(A)$ and $R(\lambda; A) := (\lambda I - A)^{-1}$ is called the* resolvent *of A. The complement of $\rho(A)$ in $\mathbb{C}$ is called the* spectrum *$\sigma(A)$ and*

$$r(A) := \sup_{\lambda \in \sigma(A)} |\lambda|$$

is called the spectral radius *of A.*

For the spectrum of a compact operator we have the following properties.

Theorem 3.9 *Let $A : X \to X$ be a compact linear operator on an infinite-dimensional normed space X. Then $\lambda = 0$ belongs to the spectrum $\sigma(A)$ and $\sigma(A) \setminus \{0\}$ consists of at most a countable set of eigenvalues with no point of accumulation except, possibly, $\lambda = 0$.*

Proof. Suppose that $\lambda = 0$ is a regular value of A, i.e., A^{-1} exists and is bounded. Then $I = A^{-1}A$ is compact by Theorem 2.16, and by Theorem 2.20 we obtain the contradiction that X is finite-dimensional. Therefore $\lambda = 0$ belongs to the spectrum $\sigma(A)$.

For $\lambda \neq 0$ we can apply the Riesz theory to the operator $\lambda I - A$. Either $N(\lambda I - A) = \{0\}$ and $(\lambda I - A)^{-1}$ exists and is bounded by Corollary 3.6 or $N(\lambda I - A) \neq \{0\}$, i.e., λ is an eigenvalue. Thus each $\lambda \neq 0$ is either a regular value or an eigenvalue of A.

It remains to show that for each $R > 0$ there exist only a finite number of eigenvalues λ with $|\lambda| \geq R$. Assume, on the contrary, that we have a sequence (λ_n) of distinct eigenvalues satisfying $|\lambda_n| \geq R$. Choose eigenelements φ_n such that $A\varphi_n = \lambda_n\varphi_n$ for $n = 0, 1, \ldots$, and define finite-dimensional subspaces

$$U_n := \operatorname{span}\{\varphi_0, \ldots, \varphi_n\}.$$

It is readily verified that eigenelements corresponding to distinct eigenvalues are linearly independent. Hence, we have $U_{n-1} \subset U_n$ and $U_{n-1} \neq U_n$ for $n = 1, 2, \ldots$. Therefore, by the Riesz Lemma 2.19 we can choose a sequence (ψ_n) of elements $\psi_n \in U_n$ such that $\|\psi_n\| = 1$ and

$$\|\psi_n - \psi\| \geq \frac{1}{2}$$

for all $\psi \in U_{n-1}$ and $n = 1, 2, \ldots$. Writing

$$\psi_n = \sum_{k=0}^{n} \alpha_{nk}\varphi_k$$

we obtain

$$\lambda_n\psi_n - A\psi_n = \sum_{k=0}^{n-1}(\lambda_n - \lambda_k)\alpha_{nk}\varphi_k \in U_{n-1}.$$

Therefore, for $m < n$ we have

$$A\psi_n - A\psi_m = \lambda_n\psi_n - (\lambda_n\psi_n - A\psi_n + A\psi_m) = \lambda_n(\psi_n - \psi),$$

where $\psi := \lambda_n^{-1}(\lambda_n\psi_n - A\psi_n + A\psi_m) \in U_{n-1}$. Hence

$$\|A\psi_n - A\psi_m\| \geq \frac{|\lambda_n|}{2} \geq \frac{R}{2}$$

for $m < n$, and the sequence $(A\psi_n)$ does not contain a convergent subsequence. This contradicts the compactness of A. □

3.3 Volterra Integral Equations

Integral equations of the form

$$\int_a^x K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b],$$

and

$$\varphi(x) - \int_a^x K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b],$$

with variable limits of integration are called *Volterra integral equations* of the *first* and *second kind*, respectively. Equations of this type were first investigated by Volterra [180]. One can view Volterra equations as special cases of Fredholm equations with $K(x,y) = 0$ for $y > x$, but they have some special properties. In particular, Volterra integral equations of the second kind are always uniquely solvable.

Theorem 3.10 *For each right-hand side $f \in C[a,b]$ the Volterra integral equation of the second kind*

$$\varphi(x) - \int_a^x K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b],$$

with continuous kernel K has a unique solution $\varphi \in C[a,b]$.

Proof. We extend the kernel onto $[a,b] \times [a,b]$ by setting $K(x,y) := 0$ for $y > x$. Then K is continuous for $x \neq y$ and

$$|K(x,y)| \leq M := \max_{a \leq y \leq x \leq b} |K(x,y)|$$

for all $x \neq y$. Hence, K is weakly singular with $\alpha = 1$.

Now let $\varphi \in C[a,b]$ be a solution to the homogeneous equation

$$\varphi(x) - \int_a^x K(x,y)\varphi(y)\,dy = 0, \quad x \in [a,b].$$

By induction we show that

$$|\varphi(x)| \leq \|\varphi\|_\infty \frac{M^n(x-a)^n}{n!}, \quad x \in [a,b], \tag{3.5}$$

for $n = 0, 1, 2, \ldots$. This certainly is true for $n = 0$. Assume that the inequality (3.5) is proven for some $n \geq 0$. Then

$$|\varphi(x)| = \left|\int_a^x K(x,y)\varphi(y)\,dy\right| \leq \|\varphi\|_\infty \frac{M^{n+1}(x-a)^{n+1}}{(n+1)!}.$$

Passing to the limit $n \to \infty$ in (3.5) yields $\varphi(x) = 0$ for all $x \in [a,b]$. The statement of the theorem now follows from Theorems 2.22 and 3.4. □

In terms of spectral theory we can formulate the last result as follows: A Volterra integral operator with continuous kernel has no spectral values different from zero.

Despite the fact that, in general, integral equations of the first kind are more delicate than integral equations of the second kind, in some cases Volterra integral equations of the first kind can be treated by reducing them to equations of the second kind. Consider

$$\int_a^x K(x,y)\varphi(y)\,dy = f(x), \quad x \in [a,b], \tag{3.6}$$

and assume that the derivatives $K_x = \partial K/\partial x$ and f' exist and are continuous and that $K(x,x) \neq 0$ for all $x \in [a,b]$. Then differentiating with respect to x reduces (3.6) to

$$\varphi(x) + \int_a^x \frac{K_x(x,y)}{K(x,x)}\,\varphi(y)\,dy = \frac{f'(x)}{K(x,x)}, \quad x \in [a,b]. \tag{3.7}$$

Equations (3.6) and (3.7) are equivalent if $f(a) = 0$. If $K_y = \partial K/\partial y$ exists and is continuous and again $K(x,x) \neq 0$ for all $x \in [a,b]$, then there is a second method to reduce the equation of the first kind to one of the second kind. In this case, setting

$$\psi(x) := \int_a^x \varphi(y)\,dy, \quad x \in [a,b],$$

and performing an integration by parts in (3.6) yields

$$\psi(x) - \int_a^x \frac{K_y(x,y)}{K(x,x)}\,\psi(y)\,dy = \frac{f(x)}{K(x,x)}, \quad x \in [a,b]. \tag{3.8}$$

We leave it as an exercise to extend this short discussion of Volterra integral equations to the case of Volterra integral equations for functions of more than one independent variable.

Problems

3.1 Let $A : X \to Y$ be a compact linear operator from a normed space X into a normed space Y. The continuous extension $\tilde{A} : \tilde{X} \to \tilde{Y}$ of A is compact with $\tilde{A}(\tilde{X}) \subset Y$ (see Problem 2.1).

3.2 Let X be a linear space, let $A, B : X \to X$ be linear operators satisfying $AB = BA$, and let AB have an inverse $(AB)^{-1} : X \to X$. Then A and B have inverse operators $A^{-1} = B(AB)^{-1}$ and $B^{-1} = A(AB)^{-1}$.

3.3 Prove Theorem 3.4 under the assumption that A^n is compact for some $n \geq 1$.
Hint: Use Problem 3.2 to prove that the set $(\sigma(A))^n := \{\lambda^n : \lambda \in \sigma(A)\}$ is contained in the spectrum $\sigma(A^n)$. Then use Theorem 3.9 to show that there exists an integer $m \geq n$ such that each of the operators

$$L_k := \exp\frac{2\pi ik}{m}\,I - A, \quad k = 1, \ldots, m-1,$$

has a bounded inverse. Then the equations $R(I - A)\varphi = Rf$ and $(I - A)\varphi = f$, where $R := \prod_{k=1}^{m-1} L_k$, are equivalent.

3.4 Let X_i, $i = 1, \ldots, n$, be normed spaces. Show that the Cartesian product $X := X_1 \times \cdots \times X_n$ of n-tuples $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a normed space with the maximum norm

$$\|\varphi\|_\infty := \max_{i=1,\ldots,n} \|\varphi_i\|.$$

Let $A_{ik} : X_k \to X_i$, $i, k = 1, \ldots, n$, be linear operators. Show that the matrix operator $A : X \to X$ defined by

$$(A\varphi)_i := \sum_{k=1}^{n} A_{ik}\varphi_k$$

is bounded or compact if and only if each of its components $A_{ik} : X_k \to X_i$ is bounded or compact, respectively. Formulate Theorem 3.4 for systems of operator and integral equations of the second kind.

3.5 Show that the integral operator with continuous kernel

$$K(x, y) := \sum_{k=0}^{\infty} \frac{1}{(k+1)^2}\{\cos(k+1)x \sin ky - \sin(k+1)x \cos ky\}$$

on the interval $[0, 2\pi]$ has no eigenvalues.

4
Dual Systems and Fredholm Alternative

In the case when the homogeneous equation has nontrivial solutions, the Riesz theory, i.e., Theorem 3.4 gives no answer to the question of whether the inhomogeneous equation for a given inhomogeneity is solvable. This question is settled by the Fredholm alternative, which we shall develop in this chapter. Rather than presenting it in the context of the Riesz–Schauder theory for the adjoint operator in the dual space we will consider the Fredholm theory for compact adjoint operators in dual systems generated by nondegenerate bilinear or sesquilinear forms. This symmetric version is more elementary and better suited for applications to integral equations.

4.1 Dual Systems via Bilinear Forms

Throughout this chapter we tacitly assume that all linear spaces under consideration are complex linear spaces; the case of real linear spaces can be treated analogously.

Definition 4.1 *Let X, Y be linear spaces. A mapping $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{C}$ is called a* bilinear form *if*

$$\langle \alpha_1\varphi_1 + \alpha_2\varphi_2, \psi\rangle = \alpha_1\langle\varphi_1, \psi\rangle + \alpha_2\langle\varphi_2, \psi\rangle,$$

$$\langle \varphi, \beta_1\psi_1 + \beta_2\psi_2\rangle = \beta_1\langle\varphi, \psi_1\rangle + \beta_2\langle\varphi, \psi_2\rangle$$

for all $\varphi_1, \varphi_2, \varphi \in X$, $\psi_1, \psi_2, \psi \in Y$, and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The bilinear form is called nondegenerate *if for every $\varphi \in X$ with $\varphi \neq 0$ there exists*